Evolution of interfaces and expansion in width

H. Arodź and R. Pełka
Institute of Physics, Jagellonian University, Reymonta 4, 30-059 Cracow, Poland
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Interfaces in a model with a single, real nonconserved order parameter and purely dissipative evolution equation are considered. We show that a systematic perturbative approach, called the expansion in width and developed for curved domain walls, can be generalized to the interfaces. A procedure for calculating curvature corrections is described. We also derive formulas for local velocity and local surface tension of the interface. As an example, evolution of spherical interfaces is discussed, including an estimate of the critical size of small droplets.

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I. INTRODUCTION

An important aspect of the dynamics of phase transitions in condensed matter is time evolution of an interface separating a retreating phase from the new one. Studied in the framework of Ginzburg-Landau type effective macroscopic models, the interface can be regarded as a kind of smooth, asymmetric domain wall subject to a transverse force. The asymmetry and the force are due to a difference in potential energy across the interface. Pertinent evolution equations for order parameters typically are nonlinear partial differential equations. In general they imply rather nontrivial phase ordering dynamics; see, e.g., review article [1]. A relativistic version of the problem, not considered here, is also interesting because of its connection with field-theoretical cosmology [2].

Recently, evolution of ordinary domain walls has been studied with the help of the Hilbert-Chapman-Enskog method applied in a suitably chosen comoving coordinate system [3–5]; a systematic and consistent perturbative scheme has been developed. It yields the relevant solutions of the evolution equations in the form of expansion in a parameter l_0 that can be regarded as a measure of width of a static planar domain wall. Consecutive terms in this expansion contain extrinsic curvatures of a surface comoving with the wall, and also contain certain functions (below denoted by C_k) that can be regarded as fields defined on that surface and coupled to the extrinsic curvatures. In the present paper, which is a sequel to [3], we show that that perturbative expansion can be generalized to the case of curved interfaces.

The Hilbert-Chapman-Enskog method and the comoving coordinates technique, which we have learned from [6,7], respectively, have already been used in theoretical investigations of planar interfaces [6], and of curved ones in superconducting films [8]. We apply these tools to curved interfaces in the three-dimensional space, in a Ginzburg-Landau type model defined by formulas (1) and (2) and Eq. (3) below. Interfaces in this model have been under investigation for a long time; see, e.g., [9]. Our main contribution consists in providing a systematic iterative scheme for generating the relevant solution in the form of a perturbative series. The role of small parameters is played by the ratios l_0/R_i, where l_0 is the width and R_i the curvature radii of the interface. In spite of the nonlinearity of the evolution equation, the perturbative contributions can be generated in a surprisingly simple manner. This is achieved by introducing the functions C_k, which saturate certain integrability conditions. As an application, we derive a formula for the local velocity of the interface with curvature corrections included, and we discuss the critical size of nucleating spherical droplets.

We consider a system described by a real, scalar, nonconserved order parameter Φ, with the free energy F of the form

$$F=\int d^3x\left(\frac{1}{2}K\frac{\partial\Phi}{\partial x^\alpha}\frac{\partial\Phi}{\partial x^\alpha}+V(\Phi)\right), \tag{1}$$

where

$$V=A\Phi^2+B\Phi^3+C\Phi^4. \tag{2}$$

Time evolution is governed by the dissipative nonlinear equation

$$\gamma\frac{\partial\Phi(\vec{x},t)}{\partial t}=K\Delta\Phi-V'(\Phi). \tag{3}$$

Here $(x^\alpha)_{\alpha=1,2,3}$ are Cartesian coordinates in the space, V' denotes the derivative $dV/d\Phi$, and K,γ,A,B,C are positive constants. The free energy of the form (1) arises in, e.g., a de Gennes–Landau description of a nematic-isotropic transition in nematic liquid crystals in a single elastic constant approximation ($L_2=0$) [10]. Then $\Phi=0$ corresponds to the isotropic liquid phase, while in the nematic phase $\Phi\neq 0$.

The concrete form (2) of the potential has the advantage that the solution of Eq. (3) describing a planar interface has a simple, explicitly known form. It can be found in, e.g., [9]. The planar interface plays an important role in the perturbative scheme: the main idea is that there exist curved interfaces that do not differ much from the planar one if considered in an appropriately chosen coordinate system (which in particular should comove with the interface). Therefore, one may hope that $\Phi(\vec{x},t)$ for such curved interfaces can be calculated perturbatively, with the planar interface giving the zeroth-order term. Toward this end, it is necessary to introduce the comoving coordinate system explicitly, and to give a prescription for the iterative computation of the perturbative corrections.

The plan of our paper is as follows. In Sec. II we describe the planar interface and the comoving coordinates. This section contains preliminary material quoted here for the convenience of the reader as well as in order to fix our notation. In Sec. III we describe the perturbative scheme for the curved interfaces. In Sec. IV we present formulas for the velocity and the free energy of the curved interface. Section V is devoted to a discussion of spherical droplets of the stable phase which nucleate during the phase transition. Several remarks are presented in Sec. VI. In Appendix A we construct solutions of linear equations obeyed for corrections to the transverse profile of the interface. Appendix B contains a brief discussion of the stability of the interface.

II. THE PRELIMINARIES

A. The homogeneous planar interface

Let us assume that the planar interface is perpendicular to the z axis ($z \equiv x^3$) and homogeneous. Then Φ depends only on z and t, and Eq. (3) is reduced to

$$\gamma \partial_t \Phi = K \partial_z^2 \Phi - V'(\Phi). \tag{4}$$

The interface type solution $\Phi(z,t)$ interpolates smoothly between minima of V when z changes from $-\infty$ to $+\infty$. V given by formula (2) has two minima,

$$\Phi_- = 0, \quad \Phi_+ = -\sqrt{K/(8Cl_0^2)},$$

where l_0 is given by formula (9) below. The corresponding phases we shall call isotropic and ordered, respectively. Let us multiply Eq. (4) by $\partial_z \Phi$ and integrate over z. The resulting identity,

$$\gamma \int_{-\infty}^{+\infty} dz \, \partial_z \Phi \partial_t \Phi = V(\Phi_-) - V(\Phi_+),$$

implies that $\partial_t \Phi \neq 0$ if the minima are nondegenerate. Further assumption that the interface moves in a uniform manner with velocity v_0, that is, that

$$\Phi(z,t) = \Phi(z - z_0 - v_0 t),$$

leads to the formula

$$\gamma v_0 \int_{-\infty}^{+\infty} (\partial_z \Phi)^2 = V(\Phi_+) - V(\Phi_-), \tag{5}$$

where

$$Z = z - z_0 - v_0 t. \tag{6}$$

Hence the interface moves towards the region of higher potential $V(\Phi_\pm)$, as expected.

The exact solution of Eq. (4) has the following form [9]:

$$\Phi_0 = -\sqrt{\frac{K}{8l_0^2 C}} \frac{1}{1+\exp(-Z/2l_0)}, \tag{7}$$

where

$$v_0 = \sqrt{\frac{9K}{32\gamma^2 C}} (B - \sqrt{9B^2 - 32AC}), \tag{8}$$

and

$$l_0^{-1} = \frac{1}{2\sqrt{2KC}} (3B + \sqrt{9B^2 - 32AC}). \tag{9}$$

The constant z_0 can be regarded as the position of the interface at $t=0$, and l_0 as its width. Φ_0 smoothly interpolates between the local minima of V: Φ_- for $z \to -\infty$ and Φ_+ for $z \to +\infty$. The corresponding values of the potential are

$$V(\Phi_-) = 0, \quad V(\Phi_+) = \frac{K\gamma v_0}{96 l_0^3 C}.$$

At $\Phi_m = -3B/4C - \Phi_+$ the potential V has a local maximum if $A > 0$. The substitutions $Z \to -Z$ and $v_0 \to -v_0$ in formulas (7) and (8) give another solution of Eq. (4), called antiinterface.

It is clear that solution (7) exists if

$$9B^2 \geqslant 32AC. \tag{10}$$

The parameter A has the following dependence on the temperature T [10]:

$$A = a(T - T_*),$$

where $a > 0$. The constants B, C, and a do not depend on temperature. Condition (10) is satisfied if the temperature T is from the interval (T_*, T_c), where T_c is determined from the equation $9B^2 = 32aC(T_c - T_*)$. It is clear that $T_c > T_*$. For temperatures in this interval one phase is stable and the other one is metastable.

The potential (2) can also lead to a static, symmetric domain wall. Namely, for the temperature T_0 such that $B^2 = 4AC$ the velocity v_0 vanishes, $V(\Phi_+) = V(\Phi_-) = 0$, and the potential can be written in the following form:

$$V = C[(\Phi - \Phi_m)^2 - \Phi_m^2]^2, \tag{11}$$

where for $T = T_0$

$$\Phi_m = -\frac{B}{4C}.$$

In this particular case there is the degenerate ground state given by $\Phi = \Phi_\pm$. The potential (11) possesses the Z_2 symmetry

$$\Phi \to 2\Phi_m - \Phi,$$

and the interface becomes a static homogeneous, symmetric domain wall with the Z_2 topological charge.

B. The comoving coordinates

Here we quote the main definitions in order to introduce our notation. A more detailed description of this change of coordinates, as well as a discussion of related mathematical questions, can be found in [3–5].

The comoving coordinates in the space denoted by $(\sigma^\alpha)=(\sigma^1,\sigma^2,\sigma^3=\xi)$, where $\alpha=1,2,3$, are defined by the following formula:

$$\vec{x}=\vec{X}(\sigma^i,t)+\xi\vec{p}(\sigma^i,t). \tag{12}$$

Here $\vec{x}=(x^\alpha)$, where x^α are the Cartesian coordinates in the space R^3. Points $\vec{X}(\sigma^i,t)$ form a smooth surface S which is parametrized by the two coordinates[1] σ^1,σ^2. In general, the interface moves in space, hence $\vec{X}$ depends on the time t.

The surface S is fastened to the interface—the shape of it mimics the shape of the interface and they move together. We shall see that for consistency of the perturbative scheme $\vec{X}(\sigma^i,t)$ has to obey certain equations from which one can determine the evolution of the surface S. The coordinate ξ parametrizes the axis perpendicular to the interface at point $\vec{X}(\sigma^i,t)$. The vector $\vec{p}(\sigma^i,t)$ is a unit normal to S at this point, that is,

$$\vec{p}^2=1, \quad \vec{p}\vec{X}_{,k}(\sigma^i,t)=0,$$

where $\vec{X}_{,k}=\partial\vec{X}/\partial\sigma^k$. The surface S is characterized in particular by an induced metric tensor on S,

$$g_{ik}=\vec{X}_{,i}\vec{X}_{,k},$$

and the extrinsic curvature coefficients

$$K_{ij}=\vec{p}\vec{X}_{,ij}.$$

The matrix (g^{ik}) is by definition the inverse of the matrix (g_{kl}), i.e., $g^{ik}g_{kl}=\delta^i_l$.

The two-by-two matrix (K_{ik}) is symmetric. Two eigenvalues k_1,k_2 of the matrix (K^i_j), where $K^i_j=g^{il}K_{lj}$, are called extrinsic curvatures of S at point $\vec{X}$. The main curvature radii are defined as $R_i=1/k_i$. Thus, by the definition

$$K^i_i=\frac{1}{R_1}+\frac{1}{R_2}, \quad \det(K^i_j)=\frac{1}{R_1R_2}.$$

In general, the curvature radii vary along S and with time.

The coordinates (σ^α) replace the Cartesian coordinates (x^α) in the vicinity of the interface. Components of the metric tensor in the space transformed to the new coordinates are denoted by $G_{\alpha\beta}$. They are given by the following formulas:

$$G_{33}=1, \quad G_{3k}=G_{k3}=0, \quad G_{ik}=N^l_i g_{lr}N^r_k,$$

where

$$N^l_i=\delta^l_i-\xi K^l_i.$$

Dependence of $G_{\alpha\beta}$ on the transverse coordinate ξ is explicit, and σ^1,σ^2 enter through the tensors g_{ik},K^l_r which characterize the geometry of surface S.

[1]The Greek indices $\alpha,\beta,\ldots$ have values 1,2,3 and they refer to the three-dimensional space, while the Latin indices $i,j,k,l,\ldots$ have values 1,2 and they refer to the inner coordinates σ^1,σ^2 on the surface S.

Components $G^{\alpha\beta}$ of the inverse metric tensor have the form

$$G^{33}=1, \quad G^{3k}=G^{k3}=0, \quad G^{ik}=(N^{-1})^i_r g^{rl}(N^{-1})^k_l,$$

where

$$(N^{-1})^i_r=\frac{1}{N}[(1-\xi K^l_l)\delta^i_r+\xi K^i_r],$$

and

$$\begin{aligned}N&=\det(N^i_k)\\&=1-\xi K^i_i+\frac{1}{2}\xi^2(K^i_iK^l_l-K^i_lK^l_i)\\&=\left(1-\frac{\xi}{R_1}\right)\left(1-\frac{\xi}{R_2}\right).\end{aligned}$$

In order to transform Eq. (3) into the comoving coordinates, we use the standard formula

$$\Delta\Phi=\frac{1}{\sqrt{G}}\frac{\partial}{\partial\sigma^\alpha}\left(\sqrt{G}G^{\alpha\beta}\frac{\partial\Phi}{\partial\sigma^\beta}\right), \tag{13}$$

where $G=\det(G_{\alpha\beta}),\sqrt{G}=\sqrt{g}N,g=\det(g_{ik})$.

The time derivative in Eq. (3) is taken under the condition that all x^α are constant. It is convenient to use a time derivative taken at constant σ^α. They are related by the formula

$$\left.\frac{\partial}{\partial t}\right|_{x^\alpha}=\left.\frac{\partial}{\partial t}\right|_{\sigma^\alpha}+\left.\frac{\partial\sigma^\beta}{\partial t}\right|_{x^\alpha}\frac{\partial}{\partial\sigma^\beta}. \tag{14}$$

Finally, let us introduce the dimensionless variables s and ϕ instead of, respectively, ξ and Φ:

$$\xi=2l_0s, \quad \Phi(\xi,\sigma^i,t)=-\sqrt{\frac{K}{8Cl_0^2}}\phi(s,\sigma^i,t). \tag{15}$$

The coordinate s gives the distance from S in the unit $2l_0$ relative to the width of the planar interface.

Using formulas (13)–(15) we can write Eq. (3) in the following form, which is convenient for construction of the expansion in width:

$$\begin{aligned}&\frac{2l_0^2\gamma}{K}\left.\frac{\partial\phi}{\partial t}\right|_{\sigma^k}-\bar{v}\frac{\partial\phi}{\partial s}-\frac{2l_0^2\gamma}{K}(N^{-1})^i_k g^{kr}\vec{X}_{,r}(\dot{\vec{X}}+2l_0s\dot{\vec{p}})\phi_{,i}\\&\quad=\frac{1}{2}\frac{\partial^2\phi}{\partial s^2}+\frac{1}{2N}\frac{\partial N}{\partial s}\frac{\partial\phi}{\partial s}+2l_0^2\frac{1}{\sqrt{g}N}\frac{\partial}{\partial\sigma^j}(G^{jk}\sqrt{g}N\phi_{,k})\\&\qquad-\alpha\phi+(1+\alpha)\phi^2-\phi^3,\end{aligned} \tag{16}$$

where

$$\bar{v}=\frac{\gamma l_0}{K}\vec{p}\dot{\vec{X}}$$

is the dimensionless transverse velocity of the surface S, the dot denotes the derivative $\partial/\partial t|_{\sigma^\alpha}$, and

$$\alpha = \frac{4Al_0^2}{K}.$$

Formula (9) and condition (10) imply that $0 \leqslant \alpha \leqslant 1$ for temperatures in the range T_*, T_c.

The homogeneous planar interface (7) can be obtained from the evolution equation written in the form (16) in the following manner. As the surface S we take a plane, hence $K^i_j = 0$. Moreover, S is assumed to move with constant velocity v_0, hence

$$\vec{p}\dot{\vec{X}}_0 = v_0 = \text{const.}$$

Finally,

$$\left.\frac{\partial \phi}{\partial t}\right|_{\sigma^\alpha} = 0$$

because we look at the interface from the comoving reference frame, and $\partial\phi/\partial\sigma^i = 0$ because of the homogeneity. Then equation (16) is reduced to

$$-\overline{v}_0 \frac{\partial \phi}{\partial s} = \frac{1}{2}\frac{\partial^2 \phi}{\partial s^2} - \alpha\phi + (1+\alpha)\phi^2 - \phi^3. \tag{17}$$

The solution previously given by formulas (6)–(9) now has the form

$$\phi = \phi_0(s), \quad \overline{v}_0 = \alpha - \frac{1}{2}, \tag{18}$$

where

$$\phi_0(s) = \frac{\exp(s - s_0)}{1 + \exp(s - s_0)}. \tag{19}$$

$\phi_0(s)$ smoothly interpolates between 0 and 1. This corresponds to interpolation between the minima Φ_-, Φ_+ of the potential V if $\vec{p}$ is directed from negative towards positive z's. If we choose the opposite direction for $\vec{p}$ we obtain the anti-interface. The constant s_0 corresponds to z_0 from formula (6).

III. EXPANSION IN WIDTH FOR CURVED INTERFACES

Let us begin with a brief description of the ideas underlying the calculations presented below. The set of solutions of the nonlinear, partial differential equation (3) is very large. We are interested here only in a rather special subset of it, consisting of solutions that represent the evolution of a smooth interface. Moreover, even within this subclass we concentrate on rather special interfaces, called by us the ''basic'' ones. Their defining feature is that one can find a comoving coordinate system in which the order parameter of the interface is essentially given by $\phi_0(s)$, formula (19), modified by small corrections that take into account the nonvanishing curvature.

By writing the evolution equation in the form (16), we have shown that l_0 can be regarded as a parameter analogous to a coupling constant—it appears in Eq. (16) only as a coefficient in several (but not all) terms. Therefore, one may hope that a systematic perturbative expansion in l_0 will turn out to be useful, as is the case with other perturbative expansions so numerous in theoretical physics. The perturbative series can be constructed in the standard manner: the solution sought ϕ and the velocity $\overline{v}$ are written in the form

$$\phi = \phi_0 + l_0\phi_1 + l_0^2\phi_2 + \ldots, \quad \overline{v} = \overline{v}_0 + l_0\overline{v}_1 + l_0^2\overline{v}_2 + \ldots, \tag{20}$$

and inserted in Eq. (16). Coefficients in front of successive powers of l_0 in this equation are equated to zero. Notice that after the rescaling $\xi = 2l_0 s$, the expansion parameter l_0 is present also in N and $(N^{-1})^i_k$. In the zeroth order we obtain the following equation:

$$-\overline{v}_0 \frac{\partial \phi_0}{\partial s} = \frac{1}{2}\frac{\partial^2 \phi_0}{\partial s^2} - \alpha\phi_0 + (1+\alpha)\phi_0^2 - \phi_0^3. \tag{21}$$

which formally coincides with Eq. (17). Therefore, we can immediately write the relevant solution

$$\overline{v}_0 = \alpha - \frac{1}{2}, \quad \phi_0(s, \sigma^i, t) = \frac{\exp[s - C_0(\sigma^i, t)]}{1 + \exp[s - C_0(\sigma^i, t)]}. \tag{22}$$

There are, however, two differences between the planar solution (18),(19) and the solution (22). First, we do not assume homogeneity of the interface; therefore, the constant s_0 from formula (19) is replaced by the function $C_0(\sigma^i, t)$ of the indicated variables. Second, the surface S is not fixed yet, while in the former case it was a plane.

It is convenient to rewrite Eq. (16) as an equation for the corrections $\delta\phi, \delta\overline{v}$, which are defined by the formulas

$$\phi = \phi_0(s, \sigma^i, t) + \delta\phi, \quad \overline{v} = \overline{v}_0 + \delta\overline{v}. \tag{23}$$

After taking into account the fact that ϕ_0 obeys Eq. (21), we obtain an equation of the form

$$\hat{L}\,\delta\phi = f, \tag{24}$$

with the linear operator $\hat{L}$ defined as

$$\hat{L} = \frac{1}{2}\frac{\partial^2}{\partial s^2} + \left(\alpha - \frac{1}{2}\right)\frac{\partial}{\partial s} - \alpha + 2(\alpha+1)\phi_0 - 3\phi_0^2,$$

and

$$\begin{aligned} f = &-\left(\frac{1}{2N}\frac{\partial N}{\partial s} + \delta\overline{v}\right)\frac{\partial \phi_0}{\partial s} + \frac{2\gamma l_0^2}{K}\left(\left.\frac{\partial \delta\phi}{\partial t}\right|_{\sigma^\alpha} - \left.\frac{\partial C_0}{\partial t}\right|_{\sigma^\alpha}\frac{\partial \phi_0}{\partial s}\right) \\ &- \frac{2l_0^2\gamma}{K}(N^{-1})^i_k g^{kr}\vec{X}_{,r}(\dot{\vec{X}} + 2l_0 s\dot{\vec{p}})\left(\delta\phi_{,i} - C_{0,i}\frac{\partial \phi_0}{\partial s}\right) \\ &- \left(\frac{1}{2N}\frac{\partial N}{\partial s} + \delta\overline{v}\right)\frac{\partial \delta\phi}{\partial s} - 2l_0^2\frac{1}{\sqrt{g}N}\frac{\partial}{\partial \sigma^j}\left[G^{jk}\sqrt{g}N\left(\delta\phi_{,k} \right.\right. \\ &\left.\left. - C_{0,k}\frac{\partial \phi_0}{\partial s}\right)\right] + (3\phi_0 - \alpha - 1)(\delta\phi)^2 + (\delta\phi)^3, \end{aligned} \tag{25}$$

where $\overline{v_0}$ and ϕ_0 are given by formulas (22). Now it is easy to see that in each order in l_0 we obtain an inhomogeneous linear differential equation of the form

$$\hat{L}\phi_n = f_n, \tag{26}$$

where $n=1,2,\ldots$. For example,

$$f_1 = (K_i^i - \overline{v_1})\frac{\partial \phi_0}{\partial s}. \tag{27}$$

In the whole perturbative scheme Eq. (21) is the only nonlinear equation for the contributions to the order parameter ϕ. We show in Appendix A that Eqs. (26) can easily be solved with the help of standard methods—one can construct the relevant Green's function for $\hat{L}$. It is remarkable that the same operator $\hat{L}$ appears in all equations (26), and that the form of it does not depend on the surface S. For these reasons, calculation of the corrections ϕ_n is reduced to the relatively simple task of finding f_n and calculating the one-dimensional integrals over s shown in the Appendix.

The perturbative Ansatz (20) and Eqs. (26) are two parts of the Hilbert-Chapman-Enskog method. The third and most crucial part consists of integrability conditions for Eqs. (26) [6]. Such conditions appear because the operator $\hat{L}^\dagger$, the Hermitian conjugate of $\hat{L}$, has a normalizable eigenstate with the eigenvalue equal to zero. Such an eigenstate is called the zero mode. Let us first find the zero mode for the operator $\hat{L}$. Inserting ϕ_0 into Eq. (21) and differentiating this equation with respect to s gives the following identity:

$$\hat{L}\psi_r = 0, \tag{28}$$

where

$$\psi_r(s,\sigma^i,t) = \frac{\partial \phi_0}{\partial s} = \frac{\exp(s-C_0)}{[1+\exp(s-C_0)]^2}. \tag{29}$$

Notice that ψ_r exponentially vanishes for $s \to \pm\infty$. Because the operator $\partial/\partial s$ is anti-Hermitian with respect to the scalar product $\langle g_1|g_2\rangle = \int_{-\infty}^{+\infty} ds\, g_1^* g_2$, the operator $\hat{L}$ is not Hermitian. Its Hermitian conjugate has the form

$$\hat{L}^\dagger = \frac{1}{2}\frac{\partial^2}{\partial s^2} - \left(\alpha - \frac{1}{2}\right)\frac{\partial}{\partial s} - \alpha + 2(\alpha+1)\phi_0 - 3\phi_0^2.$$

The operator $\hat{L}^\dagger$ has a zero mode, too, namely,

$$\hat{L}^\dagger \psi_l = 0, \tag{30}$$

where[2]

$$\psi_l = \exp[(2\alpha-1)(s-C_0)]\psi_r. \tag{31}$$

The function ψ_l vanishes exponentially for $s \to \pm\infty$ because $0<\alpha<1$ for all temperatures in the range T_*, T_c. For $\alpha = 1/2$, that is when the interface becomes the domain wall, i.e., the two zero modes coincide.

Let us multiply both sides of Eqs. (26) by $\psi_l(s)$ and take the integral $\int_{-\infty}^{+\infty} ds$. The left-hand side (lhs), of the resulting formula vanishes because of (30), hence

$$\int_{-\infty}^{+\infty} ds\, \psi_l f_n = 0 \tag{32}$$

for $n=1,2,\ldots$. It turns out that these conditions are nontrivial. In particular, they give an evolution equation for surface S.

It should be noted that the conditions (32) are in fact approximate, but the neglected terms are exponentially small. The point is that in order to obtain Eqs. (26) we use an expansion of the type

$$\frac{1}{1-2l_0 s/R_i} = \sum_{k=0}^{\infty}\left(\frac{2l_0 s}{R_i}\right)^k$$

$(i=1,2)$, which are convergent for $s<s_M$ where $s_M = \min(R_1/2l_0, R_2/2l_0)$. Therefore, when deriving conditions (32), the integration range should be restricted to $|s|<s_M$. Because of the exponential decrease of ψ_l and f_n at large $|s|$, this will give exponentially small corrections to these conditions. We assume that the ratios R_i/l_0 are so large that we may neglect those corrections.

Let us compute the basic interface up to the order l_0. For $n=1$, condition (32) gives

$$\overline{v_1} = \frac{1}{R_1} + \frac{1}{R_2} \tag{33}$$

because

$$a_0(\alpha) = \int_{-\infty}^{+\infty} ds\, \psi_l \psi_r = B(2\alpha+1, 3-2\alpha)$$

does not vanish for α in the interval (0,1). Here B denotes the Euler beta function. The function $a_0(\alpha)$ has the symmetric ''U'' shape in the interval $\alpha \in [0,1]$, with the minimum equal to 1/6 at $\alpha = 1/2$, and the upper ends reaching the value 1/3 for $\alpha = 0$ and 1. The condition (33) implies that the surface S obeys the following evolution equation:

$$\frac{\gamma}{K}\dot{\vec{X}}\vec{p} = \frac{2\alpha-1}{2l_0} + K_i^i. \tag{34}$$

It formally coincides with the well-known Allen-Cahn equation [11].

Now Eq. (26) with $n=1$ is reduced to

$$\hat{L}\phi_1 = 0. \tag{35}$$

It has the following solution, which vanishes at $s \to \pm\infty$:

$$\phi_1 = C_1(\sigma^i, t)\psi_r, \tag{36}$$

where C_1 is a smooth function of the indicated variables.

[2]The subscripts l and r stand for left and right, respectively. The point is that (30) can be written as $\psi_l \hat{L} = 0$.

For $n=2$, we obtain from formula (25), after taking into account the results (33) and (36),

$$f_2=(2sK_j^iK_i^j-\overline{v}_2)\partial_s\phi_0+(3\phi_0-\alpha-1)C_1^2(\partial_s\phi_0)^2+\frac{2\gamma}{K}(-\partial_tC_0+g^{ik}\vec{X}_{,k}\dot{\vec{X}}C_{0,i})\partial_s\phi_0+2\frac{1}{\sqrt{g}}\frac{\partial}{\partial\sigma^i}(g^{ik}\sqrt{g}C_{0,k}\partial_s\phi_0). \quad (37)$$

Straightforward integration over s as in Eq. (32) can be a little bit cumbersome. This calculation can be significantly simplified with the help of the following identity:

$$2\int ds\,\psi_l[3\phi_0-(\alpha+1)]\partial_s\phi_0\phi_n=\int ds\,\partial_s\psi_l f_n, \quad (38)$$

where in the case at hand $n=1$. Identity (38) is obtained from Eq. (26) by differentiating both sides of it with respect to s, multiplying by ψ_l, and integrating over s, just as in the derivation of the integrability conditions (32). The integrability condition gives

$$a_1(\alpha)K_i^jK_j^i-\overline{v}_2+\frac{2\gamma}{K}(-\partial_tC_0+g^{ik}\vec{X}_{,k}\dot{\vec{X}}C_{0,i})+2\Delta_2C_0+(2\alpha-1)g^{ik}C_{0,i}C_{0,k}+2C_0K_j^iK_i^j=0, \quad (39)$$

where

$$\Delta_2=\frac{1}{\sqrt{g}}\frac{\partial}{\partial\sigma^i}\left(\sqrt{g}g^{ik}\frac{\partial}{\partial\sigma^k}\right)$$

is the Laplacian on the surface S and

$$a_1(\alpha)=a_0(\alpha)^{-1}\frac{da_0(\alpha)}{d\alpha}.$$

For α from the interval [0,1] the function $a_1(\alpha)$ is almost linear. In particular, $a(0)=-3, a(1/2)=0, a(1)=3$.

The integrability condition (39) allows the freedom to choose whether we keep nonvanishing C_0 or $\overline{v}_2$. It is clear that we cannot put to zero both of them unless $K_j^iK_i^j=0$ (then S is a plane). The choice $C_0=0$ gives

$$\overline{v}_2=a_1(\alpha)K_j^iK_i^j. \quad (40)$$

This implies a correction to the Allen-Cahn equation of the form

$$\frac{\gamma l_0}{K}\delta_1(\dot{\vec{X}}\vec{p})=l_0^2\overline{v}_2,$$

where $\delta_1(\cdot)$ denotes the first-order correction to $\dot{\vec{X}}\vec{p}$. In consequence, there will be a correction to the solution $\vec{X}$ of the Allen-Cahn equation, and corrections to g_{ik} and K_{ik}. These corrections have to be taken into account when calculating f_k with $k\geqslant 3$. It is clear that this version of the perturbative scheme is rather cumbersome.

On the other hand, if we put $\overline{v}_2=0$, then the evolution of the surface S is still governed by the relatively simple Allen-Cahn equation (34). The integrability condition (39) is now saturated by the function C_0—it has the form of the evolution equation for C_0, namely,

$$\frac{\gamma}{K}[\partial_tC_0-g^{ik}\partial_{\sigma^k}\vec{X}\dot{\vec{X}}C_{0,i}]-\Delta_2C_0-\left(\alpha-\frac{1}{2}\right)g^{ik}C_{0,i}C_{0,k}-C_0K_j^iK_i^j=\frac{1}{2}\,a_1(\alpha)K_j^iK_i^j. \quad (41)$$

Let us also check the third integrability condition. Formulas (20),(25) give

$$f_3=-\overline{v}_3\psi_r+2K_i^i[3K_j^iK_i^j-(K_i^i)^2]s^2\psi_r+\frac{2\gamma}{K}\partial_t(C_1\psi_r)+\frac{2\gamma}{K}g^{ir}\vec{X}_r\dot{\vec{X}}(C_1\psi_r)_{,i}+\frac{4\gamma}{K}C_{0,i}(g^{ir}\vec{X}_r\dot{\vec{p}}K^{ir}\vec{X}_r\dot{\vec{X}})s\psi_r+2sK_j^iK_i^j\partial_s\phi_1-2\Delta_2(C_1\psi_r)+4sK_i^i\frac{1}{\sqrt{g}}(\sqrt{g}g^{jk}C_{0,k}\psi_r)_{,j}-\frac{4s}{\sqrt{g}}(\sqrt{g}g^{jk}K_l^lC_{0,k}\psi_r)_{,j}+\frac{8s}{\sqrt{g}}(\sqrt{g}K^{jk}C_{0,k}\psi_r)_{,j}+2(3\phi_0-\alpha-1)\phi_1\phi_2+\phi_1^3, \quad (42)$$

where we have put $\overline{v}_2=0$. The term with ϕ_2 contains a new function C_2 (see Appendix A); however, this function will not appear in the integrability condition because the integration over s eliminates the term proportional to C_2. This can be seen from the identity (38): on the rhs we have f_2 in which C_2 is not present. We see already from formula (42) that we can choose whether to keep nonvanishing $\overline{v}_3$ or C_1. For the same reason as in the case of $n=2$, we choose $\overline{v}_3=0$, and the integrability is saturated by C_1. The resulting evolution equation for C_1 has the following form:

$$\frac{\gamma}{K}[\partial_tC_1-g^{ik}\partial_{\sigma^k}\vec{X}\dot{\vec{X}}\partial_{\sigma^i}C_1]-\Delta_2C_1-K_j^iK_i^jC_1-\left(\alpha-\frac{1}{2}\right)g^{jk}\partial_kC_0\partial_jC_1=-\left(\frac{1}{4}a_2+a_1C_0+C_0^2\right)K_i^i[3K_j^kK_k^j-(K_l^l)^2]+2\left[g^{ij}\partial_jK_l^l-\frac{\gamma}{K}[g^{ir}(\partial_r\dot{\vec{X}}\dot{\vec{p}})+K^{ir}(\partial_r\vec{X}\dot{\vec{X}})]\right]\partial_iC_0\left(C_0+\frac{1}{2}a_1\right)-\frac{4}{\sqrt{g}}\partial_j(K^{jk}\sqrt{g}\partial_kC_0)\left(C_0+\frac{1}{2}a_1\right)-2K^{jk}\partial_kC_0\partial_jC_0\left[1+(2\alpha-1)\left(C_0+\frac{1}{2}a_1\right)\right], \quad (43)$$

where

$$a_2=\frac{a_0''}{a_0}.$$

It is easy to proceed to the second and higher orders. Using formulas from Appendix A, one can write the general solution ϕ_n of Eq. (26). It contains the function $C_n(\sigma^i,t)$, which obeys an equation analogous to (41) or (43). This equation follows from the integrability condition (32) with n replaced by $n+2$ if we put $\overline{v_n}=0$. Due to identity (38), in the derivation of that equation we do not need the explicit form of ϕ_{n+1}. In the present paper we will end our considerations at the first order.

In order to obtain a concrete basic interface solution, we have to specify initial data for equations (34),(41),(43). There is no restriction on the initial data, except the obvious requirement that perturbative corrections of a given order should be small in comparison with those of preceding orders. In particular, $l_0C_1\ll 1$, $l_0^2C_2\ll 1$, and $l_0/R_i\ll 1$. If we consider the interface solution only up to the first-order correction (36), a simplification appears: without any loss of generality we may adopt the homogeneous initial data

$$C_0(\sigma^i,t=t_0)=0, \quad C_1(\sigma^i,t=t_0)=0. \tag{44}$$

This can be justified as follows. Local deformation of S by shifting a small piece of it along the direction $\vec{p}$ results in the corresponding shift of the coordinate s. Therefore, for any given basic interface, we can choose an initial position of S such that $C_0(\sigma^i,t=t_0)=0$ in formula (22) for ϕ_0. This can be done at one instant of time, e.g., at the initial time. Values of C_0 and the position of S at later times are determined uniquely by Eqs. (34) and (41) and in general C_0 does not vanish. Notice, however, that such a shift will influence terms of the order l_0^2 and higher in formula (25) for f—due to the explicit presence of s in $N,(N^{-1})^i_k$, f is not invariant under the translations $s\to s+C_0$.

The rhs of Eq. (41) vanishes for $\alpha=1/2$, that is, in the domain wall case. In this case, the initial condition (44) implies that $C_0=0$ for all times and, in consequence, C_0 disappears from the first-order perturbative solution.

As for C_1, the reason for the homogeneous initial condition is that the interface with the first-order correction, that is,

$$\phi=\phi_0(s-C_0)+l_0C_1\psi_r(s-C_0), \tag{45}$$

can be regarded as $\phi_0(s-C_0+l_0C_1)$ to the first order in l_0, so again we can cancel C_1 at the initial time t_0 by suitably correcting the initial position of the surface S. Let us stress again that this works only at the fixed time instant.

The initial data (44) imply that at the initial time the order parameter ϕ is equal to $\phi_0(s)$. Hence, in choosing initial data for ϕ, the only freedom we still have is in the position of surface S. When the second and higher order corrections are included, one has to allow for initial data more general than (44) for C_0 and C_1. Nevertheless, the initial form of ϕ is always uniquely fixed by these data and the initial position of surface S.

To recapitulate, the perturbative solution to the first order has the form (45), where ϕ_0,ψ_r are given by formulas (22) and (29), respectively. Formula (45) gives the dependence on s explicitly. The functions C_0,C_1 are to be determined from Eqs. (41) and (43) with the initial data (44). One also has to solve the Allen-Cahn equation (34). In certain cases, these equations can be solved analytically, e.g., for the spherical interface discussed in the next section. In the general case, one will be forced to use numerical methods. In comparison with the initial evolution equation (3) the advantage is that the equations for S,C_0,C_1 involve only two spatial variables σ^1,σ^2. Such a reduction in the number of independent variables is a valuable simplification in numerical calculations.

The perturbative solution obtained above can be used in calculations of the physical characteristics of interfaces. In the following section we obtain formulas for the local velocity and surface tension of the interface. In Sec. V we discuss the evolution of a spherical interface.

IV. LOCAL VELOCITY AND SURFACE TENSION OF THE INTERFACE

Let us apply the expansion in width in order to find the local transverse velocity and surface tension of the interface. We shall use the first-order solution (45). The velocity is obtained from the condition $\phi=\text{const}$. It does not necessarily coincide with $\vec{p}\dot{\vec{X}}$ given by the Allen-Cahn equation (34). Because we neglect terms of second and higher order in l_0, we may write ϕ in the form $\phi_0(s-C_0+l_0C_1)$, from which we see that ϕ is constant on surfaces given in the comoving coordinates by the condition $s-C_0+l_0C_1=s_0$, where s_0 is a constant. It follows from formula (12) that in the laboratory Cartesian coordinate frame these surfaces are given by $\vec{x}_0(\sigma^i,t)$, where

$$\vec{x}_0(\sigma^i,t)=\vec{X}(\sigma^i,t)+2l_0(s_0+C_0-l_0C_1)\vec{p}(\sigma^i,t).$$

The transverse velocity of the interface is equal to $\dot{\vec{x}}_0\vec{p}$. In order to calculate it, we take time derivative of $\vec{x}_0$, project it on $\vec{p}$, and use equations (34), (41), and (43) with the initial data (44). The result can be written in the form

$$\frac{\gamma}{K}\vec{p}\dot{\vec{x}}_0=\frac{2\alpha-1}{2l_0}+\frac{1}{R_1}+\frac{1}{R_2}+l_0a_1\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)+l_0^2a_2\left(\frac{1}{R_1^3}+\frac{1}{R_2^3}\right). \tag{46}$$

The unit normal vector $\vec{p}$ is directed from the isotropic phase ($\Phi=\Phi_0$) to the ordered phase ($\Phi=\Phi_+$). In the next section we shall use formula (46) in the case of spherical droplets.

The surface tension is another basic characteristic of the interface. It can be determined from a formula for free energy of the interface, which is defined as follows. The surface S cuts the total volume of the sample into two regions denoted below by I and II. Let us imagine that in region I there is the homogeneous isotropic phase with a constant free energy density equal to $V(\Phi_-)=0$, and in region II the homogeneous ordered phase for which $V(\Phi_+)$

$=K\gamma v_0/(96l_0^3C)$. The normal vector $\vec{p}$ points to region II. The free energy of the interface is defined as the difference

$$F_i = F - V(\Phi_+)\mathcal{V}_{II},$$

where $\mathcal{V}_{II}$ denotes the volume of region II, and F is the total free energy of the sample given by formula (1). We shall compute F_i using the first-order solution (45). Then, without any loss of generality we may put $C_0 = C_1 = 0$ at the given time, as argued in the preceding section, while the surface S remains arbitrary. Therefore, we need only $\phi_0(s) = \exp(s)/[1+\exp(s)]$. Because the dependence on the coordinate $s = \xi/2l_0$ is explicit, we can integrate over s in formula (1) for the free energy F. The volume element and the gradient free energy are taken in the form

$$d^3x = \sqrt{G}\,d\xi d\sigma^1 d\sigma^2, \quad \frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\alpha} = G^{\alpha\beta}\frac{\partial\phi}{\partial\sigma^\alpha}\frac{\partial\phi}{\partial\sigma^\beta}.$$

Neglecting terms quadratic in l_0/R_i we obtain the following formula for the free energy of the interface:

$$F_i = \int_S \kappa dA,$$

where

$$\kappa = \frac{K^2}{96l_0^3C}\left[1 - \left(\frac{\pi^2}{3} - 2\right)(1-2\alpha)\left(\frac{l_0}{R_1} + \frac{l_0}{R_2}\right)\right] \tag{47}$$

can be regarded as the local surface tension of the interface at points $\mathbf{x}(\sigma^1,\sigma^2,t)$. R_1, R_2 are the main curvature radii of the surface S at that point, and $dA = \sqrt{g}\,d\sigma^1 d\sigma^2$ is the surface element of S. Of course, this formula for F_i can be trusted if $l_0/R_i \ll 1$.

For a spherical droplet of the ordered phase embedded in the isotropic phase, R_1, R_2 are positive (the signs follow from formulas given in Sec. II B) and of course equal to the radius of the sphere. If $\alpha < 1/2$, formula (46) implies that the droplet grows (if its radius is large enough) because $\vec{p}\dot{\vec{x}}_0 < 0$ and $\vec{p}$ is the inward normal. In this case the curvature correction diminishes the surface tension, and κ increases as the droplet grows. In the reverse situation—the isotropic phase inside and the ordered one outside—the curvature increases the surface tension and κ decreases as the droplet grows.

In the case of a growing droplet of radius R of the isotropic phase in the ordered medium κ has the same dependence on the curvature. Here $\vec{p}$ is the outward normal, $\alpha > 1/2$, $\vec{p}\dot{\vec{x}}_0 > 0$, and $R_1 = R_2 = -R$. Nevertheless, the values of surface tension in both cases are different because l_0 present in formula (47) depends on α, namely, $l_0 \sim (1+\alpha)$; see formula (A1) below. Note that the first-order curvature correction to κ vanishes in the domain wall case ($\alpha = 1/2$).

V. EVOLUTION OF THE SPHERICAL INTERFACE

Let us now apply the formalism developed in Secs. III and IV to the evolution of spherical droplets. We assume that $\alpha \neq 1/2$ in order to exclude the relatively simpler case of the domain wall. The surface S is parametrized by

$$\vec{X}_0 = \mp R(t)\vec{p}(\theta,\psi).$$

Here θ, ψ are the spherical angles, and $\vec{p}$ is the inward (outward) normal to the sphere when $\alpha < 1/2$ ($\alpha > 1/2$). Thus, $\vec{p}\dot{\vec{X}}_0 = \mp\dot{R}$, with the upper sign corresponding to $\alpha < 1/2$. The Allen-Cahn equation has the form

$$\frac{\gamma}{2K}\dot{R} = \frac{1}{R_*} - \frac{1}{R}, \tag{48}$$

where

$$R_* = \frac{4l_0}{|1-2\alpha|}.$$

Integration of Eq. (48) yields the following formula:

$$\frac{R(t)}{R_*} + \ln\left|\frac{R(t)}{R_*} - 1\right| = \frac{2K}{R_*^2\gamma}t + \frac{R(0)}{R_*} + \ln\left|\frac{R(0)}{R_*} - 1\right|. \tag{49}$$

Evolution equation (41) for C_0 now has the form

$$\frac{\gamma}{2K}\dot{C}_0 - \frac{1}{R(t)^2}C_0 = \frac{a_1}{2R(t)^2}. \tag{50}$$

It has the following solution:

$$C_0(t) = -\frac{a_1}{2}\frac{R_*}{R(t)}\frac{R(0)-R(t)}{R(0)-R_*} \tag{51}$$

which obeys the initial condition $C_0(0) = 0$.

The evolution equation for C_1 is obtained from the general equation (43). For the spherical bubble it has the form

$$\frac{\gamma}{2K}\dot{C}_1 - \frac{1}{R^2}C_1 = \mp 2\left(\frac{1}{4}a_2 + a_1C_0 + C_0^2\right)\frac{1}{R^3}, \tag{52}$$

where as usual the upper sign corresponds to $\alpha < 1/2$. We do not know the explicit solution of this equation.

There are two cases in which the spherical droplets grow: a droplet of the ordered phase when $\alpha < 1/2$, and a droplet of the disordered phase when $\alpha > 1/2$. In both cases formula (46) for the radial velocity of expansion $\dot{r}_0$ gives

$$\frac{\gamma l_0}{2K}\dot{r}_0 = \frac{|2\alpha-1|}{4} - \frac{l_0}{R} + |a_1(\alpha)|\left(\frac{l_0}{R}\right)^2 - a_2(\alpha)\left(\frac{l_0}{R}\right)^3. \tag{53}$$

The expansion velocity is identical for all surfaces of constant ϕ.

It is clear that there is a minimal R, which we denote by $R_{\min}(\alpha)$, such that $\dot{r}_0 > 0$. We have found numerically that

$$R_{\min}(\alpha) = \frac{R_*}{z(\alpha)}, \tag{54}$$

where the function $z(\alpha)$ is symmetric with respect to $\alpha=1/2$ and it has values in the interval [0.866,1.049]. For example, $z(0)=0.866, z(0.1)=0.984, z(0.20)=1.045, z(0.3)=1.040 z(0.4)=1.012, z(0.5)=1.0$. Notice that $R_{\min}(\alpha)$ diverges when $\alpha\to 1/2$. Thus, in the Ginzburg-Landau model nucleation of expanding droplets is possible only if we heat the ordered phase to a temperature above T_0, or cool the isotropic phase below T_0.

For large time t, when the droplets are very large, the velocity $\dot{r}(t)$ becomes equal to the velocity of the planar interface

$$\dot{r}_\infty(\alpha)=\frac{K}{2\gamma l_0}|1-2\alpha|,$$

as expected. Note that $R_{\min}(\alpha)$ and $\dot{r}_\infty(\alpha)$ are not independent:

$$z(\alpha)\dot{r}_\infty(\alpha)R_{\min}(\alpha)=\frac{2K}{\gamma}.$$

Parameter α is related to the temperature:

$$\alpha=\frac{2\theta}{1-2\theta+\sqrt{1-4\theta}},$$

where

$$\theta=\frac{8aC}{9B^2}(T-T_*)$$

is a reduced temperature. T_0 and T_c correspond to $\theta=2/9$ and 1/4, correspondingly. The interval $\alpha\in[0,1/2]$ corresponds to $\theta\in[0,2/9]$, and $\alpha\in[1/2,1]$ to $\theta\in[2/9,1/4]$. The temperature dependence of l_0 can be seen from formula

$$l_0=(1+\alpha)\frac{\sqrt{2KC}}{3B}, \tag{55}$$

which follows from definition (9) after some algebraic manipulations. $R_{\min}$ is proportional to $l_0/|1-2\alpha|$, which can be written in the form

$$\frac{l_0}{|1-2\alpha|}=\frac{\sqrt{2KC}}{6B}\frac{4}{|3\sqrt{1-4\theta}-1|}.$$

In the interval $\theta\in[0,2/9]$, which corresponds to $T\in[T_*,T_0]$, $l_0/|1-2\alpha|$ monotonically grows from $\sqrt{2KC}/3B$ to infinity.

Using formula (55) and the symmetry of z: $z(1/2-\delta)=z(1/2+\delta)$, we obtain the following relation:

$$\frac{R_{\min}(1/2-\delta)}{R_{\min}(1/2+\delta)}=\frac{3-2\delta}{3+2\delta}<1, \tag{56}$$

where $\delta\in(0,1/2)$. Thus, the minimal size of the droplets of the isotropic phase that appear and grow when $\alpha>1/2$ is significantly larger than the size of the droplets of the ordered phase that can appear for $\alpha<1/2$.

The velocities $\dot{r}_\infty$ depend on temperature. In particular, comparing them for temperatures below and above T_0,

$$\frac{\dot{r}_\infty(1/2-\delta)}{\dot{r}_\infty(1/2+\delta)}=\frac{3+2\delta}{3-2\delta}>1.$$

Hence, the droplets of the isotropic phase expand more slowly than the droplets with $\phi\cong 1$ inside.

Our main goal in this paper has been to develop the perturbative expansion for the curved interfaces. We plan to apply it to interfaces in liquid crystals in a subsequent work. Nevertheless, just in order to get an idea of what our formulas predict, we have estimated l_0 and $\dot{r}_\infty$ for interfaces in nematic liquid crystal MBBA. The model defined by formulas (1) and (2) and Eq. (3) can be related to the de Gennes-Landau theory in a single elastic constant approximation ($L_1=K, L_2=0$). We take data found in [12–14]: $T_*\approx 316$ K, $a\approx 0.021$ J/(cm^{3}K), $B\approx 0.07$ J/cm^3, $C\approx 0.06$ J/cm^3 (after a change in our notation), and $K\approx 6\times 10^{-12}N$, $\gamma\approx 5.2\times 10^{-2}$ kg/m s. We have identified γ with the rescaled rotational viscosity $\gamma_1 L_1/K_{11}$ at a temperature close to T_*. Then, $T_0-T_*\approx 1$ K, $T_c-T_*\approx 1.2$ K. The width l_0 and the velocity $\dot{r}_\infty$ of the planar interface are given by the following formulas:

$$l_0\approx 40(1+\alpha)\times 10^{-8}\ \text{cm}, \quad \dot{r}_\infty(\alpha)\approx 1.4\frac{|1-2\alpha|}{1+\alpha}\frac{\text{cm}}{\text{s}}.$$

Note that even for rather small droplets with a radius of several hundred Ångström, the ratio l_0/R is rather small.

VI. REMARKS

(1) We have shown how one can systematically compute curvature corrections to the transverse profile ϕ and to the local velocity $\dot{r}_0$ of the interface. Due to the presence of functions, the C_k evolution equation for the surface S has the relatively simple form (34) and there are no curvature corrections to it. The formalism for interfaces is a generalization of the one constructed for domain walls [3–5]. The main new ingredients are the C_0 function and the $(2\alpha-1)/l_0$ term in the Allen-Cahn equation (34). By including them, we have significantly enlarged the range of physical applications of the perturbative scheme. This justifies the present publication.

(2) The model we have considered is special in the sense that the exact planar interface solution ϕ_0 is known. Moreover, the solutions of the equations $\hat{L}\phi_n=f_n$ are given (almost) explicitly too, because the one-dimensional integrations in formula (60) below can be easily calculated numerically. Here the crucial point is that we know explicitly the two linearly independent solutions ψ_r and ψ_2 of the homogeneous equation $\hat{L}\psi=0$. In other models, the analogs of ϕ_0, ψ_r, ψ_2 can be found at least numerically because the pertinent equations are relatively simple differential equations with the single independent variable ξ (or s after a rescaling). In our perturbative scheme we need to perform explicitly only integrals over s, as in the integrability conditions (32) or in formula (60) for $\tilde{\phi}_n$. Such integrals can easily be calcu-

lated numerically also in the case when only numerical solutions ϕ_0, ψ_r, ψ_2 are known.

(3) In our approach, in order to describe the evolution of a curved interface we introduce the surface S, and the functions C_k which can be regarded as auxilliary fields defined on S and coupled to extrinsic curvatures of it. The corresponding evolution equations, that is, the Allen-Cahn equation (34) for S and equations (41) and (43) and the analogous equations for C_k, have one independent variable less than the original equation (3). This is a significant simplification from the viewpoint of both computer simulations and analytical approaches. Therefore, we think that our perturbative scheme is an interesting tool with which to investigate the dynamics of interfaces in Ginzburg-Landau effective models.

(4) The perturbative solution we have presented above is based on the planar homogeneous interface $\phi_0(s)$. Moreover, the dependence on the transverse coordinate s is uniquely fixed by the perturbative scheme once the initial position of the surface S and initial data for the functions C_k are fixed. This means that in our scheme we obtain a special class of interfaces distinguished by the particular form of the dependence on s. In other words, the transverse profiles of the interfaces provided by the perturbative solution are not arbitrary. Intuitively, the interfaces can be regarded as the planar interface folded to a required shape at the initial instant of time and modified by necessary curvature corrections. Therefore, it seems appropriate to regard the curved interfaces obtained in our paper as the basic ones. More general interfaces could be obtained by choosing more general initial data and solving Eq. (3). For such generic interfaces no analytic perturbative approach is available.

(5) One of the advantages of a systematic perturbative approach is that one can make reliable estimates of neglected contributions and, in consequence, check whether a given perturbative result is reliable. Formalism with that level of control can be used to make straightforward predictions of the dynamical behavior of an interface, but perhaps a more important application is to ''inverse problems,'' that is, determination of parameters of the Ginzburg-Landau effective theory. From the dynamical behavior of the interfaces one could reliably infer which values have the parameters of the model. For example, for a given liquid crystal one could experimentally determine coefficients in front of higher powers of the order parameter such as Φ^5, Φ^6 or terms of the type $\Phi\partial_\alpha\Phi\partial_\alpha\Phi$ in the formula for free energy F. For a discussion of the form of F for liquid crystals see, e.g., [16].

One could try to generalize our perturbative scheme for calculating the curvature corrections to interfaces coupled to a noise. In that case, the rhs of Eq. (3) would contain an external stochastic force which in particular would lead to fluctuations of the planar interface. In our formalism the dependence of, e.g., the surface tension κ on curvature radii R_i is explicit and it comes from purely geometric quantities such as the metric tensor or Jacobian, while numerical coefficients in front of powers of l_0/R_i are given by integrals over the s coordinate and they are determined essentially by properties of the planar interface. Therefore, one may expect that if the stochastic force is present, values of these integrals would have to be averaged over the stochastic ensemble. This is an interesting direction in which one could take the present work.

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APPENDIX A: THE EQUATIONS $\hat{L}\phi_n = F_n$

In order to determine ϕ_n we have to solve Eq. (26). Using standard methods [15], it is not difficult to obtain an appropriate solution.

Let us shift the variable s,

$$s = x + C_0,$$

in order to remove function C_0 from ϕ_0 present in the operator $\hat{L}$. Then Eq. (26) acquires the following form:

$$\tilde{L}\tilde{\phi}_n(x,\sigma^i,t) = \tilde{f}_n(x,\sigma^i,t), \tag{A1}$$

where

$$\tilde{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \left(\alpha - \frac{1}{2}\right)\frac{\partial}{\partial x} - \alpha + 2(\alpha+1)\tilde{\phi}_0(x) - 3\tilde{\phi}_0^2(x),$$

$$\tilde{f}_n(x,\sigma^i,t) = f_n(x + C_0,\sigma^i,t),$$

$$\tilde{\phi}_n(x,\sigma^i,t) = \phi_n(x + C_0,\sigma^i,t),$$

and

$$\tilde{\phi}_0(x) = \frac{\exp(x)}{1+\exp(x)}.$$

In the first step, we find two linearly independent solutions of the homogeneous equation $\tilde{L}\tilde{\psi} = 0$. The zero mode $\psi_r(x)$ is one solution of this homogeneous equation, and the other one has the form

$$\psi_2(x) = \psi_r(x)h(x),$$

where $\psi_r = d\tilde{\phi}_0/dx$ and

$$h(x) = -\frac{1}{2\alpha+1}\exp[-(2\alpha+1)x] - \frac{1}{\alpha}2\exp(-2\alpha x) + \frac{6}{1-2\alpha}\{\exp[(1-2\alpha)x] - 1\} + \frac{1}{1-\alpha}2 \times \exp[2(1-\alpha)x] + \frac{1}{3-2\alpha}\exp[(3-2\alpha)x]. \tag{A2}$$

For $\alpha \to 1/2$ the first term in the second line reduces to $6x$. With those solutions, one can construct the relevant Green's function and the solution $\tilde{\phi}_n$,

$$\tilde{\phi}_n(x,\sigma^i,t) = -2\psi_r(x)\int_0^x dy\,\psi_l(y)h(y)f_n(y,\sigma^i,t) + 2\psi_2(x)\int_{-\infty}^x dy\,\psi_l(y)f_n(y,\sigma^i,t) + C_n(\sigma^i,t)\psi_r(x), \tag{A3}$$

where

$$\psi_l(y) = \exp[(2\alpha-1)y]\psi_r(y).$$

The functions $C_n, n=1,2,\ldots,$ are utilized to saturate the integrability conditions (32).

The solution of Eq. (26) is given by the formula

$$\phi_n(s,\sigma^i,t) = \tilde{\phi}_n(s-C_0,\sigma^i,t).$$

APPENDIX B: STABILITY OF THE INTERFACE

The significance of our theoretical analysis of the interfaces depends on their stability with respect to small perturbations. It is sufficient to check the stability of the planar interface because our perturbative solution is based on it. Mathematically, the stability is related to the signs of the eigenvalues of certain operators and it is a model-dependent property. The considerations presented below apply to the model defined by formulas (1)–(3), of course. In this case the linearized evolution equation for small amplitude perturbations $\delta\phi$ of the planar interface $\phi_0(s)$, formula (19), in the comoving reference frame has the form

$$\frac{2l_0^2\gamma}{K}\partial_t\delta\phi = \hat{L}\,\delta\phi + 2l_0^2(\partial_{x^1}^2+\partial_{x^2}^2)\,\delta\phi,$$

where $\hat{L}$ has been given below formula (24), and x^1,x^2 are the two Cartesian coordinates in the plane of the interface. Because $\hat{L}$ does not depend on x^1,x^2, we may pass to Fourier modes $\tilde{\delta}\phi$ of $\delta\phi$ with respect to these coordinates. After the substitution

$$\tilde{\delta}\phi = \exp\left[\left(\frac{1}{2}-\alpha\right)s\right]\Psi$$

we obtain the following equation:

$$-\frac{2l_0^2\gamma}{K}\partial_t\Psi = \hat{N}\Psi,$$

where

$$\hat{N} = -\frac{1}{2}\partial_s^2 + \frac{1}{2}\left(\alpha+\frac{1}{2}\right)^2 + 2l_0^2k^2 - 2(\alpha+1)\phi_0 + 3\phi_0^2,$$

with $k^2=(k_1)^2+(k_2)^2$, k_1,k_2 being the wave number Fourier conjugate with x^1,x^2.

Note that the Hermitian operator $\hat{N}$ has an eigenfunction Ψ_0 given by the following formula:

$$\Psi_0 = \exp\left[\left(\alpha-\frac{1}{2}\right)s\right]\psi_r,$$

where ψ_r is the zero mode introduced in Sec. III. The corresponding eigenvalue is equal to zero. Because Ψ_0 does not vanish at any finite s, it represents the ''ground state'' of a fictitious system with $\hat{N}$ as the ''Hamiltonian.'' Hence, all other eigenvalues of $\hat{N}$ are strictly positive. Moreover, looking at the ''potential'' in the Hamiltonian $\hat{N}$, one can see that the zero eigenvalue and the next one are separated by a finite gap. The eigenmode Ψ_0 corresponds to a parallel shift of the interface as a whole in the x^3 direction. All other eigenmodes decay exponentially with a characteristic time equal to $t_c = 2l_0^2\gamma/(K\lambda)$, where λ denotes the corresponding eigenvalue. Thus, the planar interface in our model is stable. Consequently, the curved interfaces are also stable with respect to small perturbations provided that their curvature radii are large enough for our perturbative expansion.

Let us point out that from the point of view of applications in condensed matter physics, even unstable interfaces can be interesting if unstable modes grow in time so slowly that the interface manages to travel across the sample before these modes become visible. The interface has finite normal velocity $\vec{p}\dot{\vec{x}}_0$, formula (46), and in any real experiment the sample occupies a finite volume.

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