Convergence to the critical attractor of dissipative maps: Log-periodic oscillations, fractality, and nonextensivity

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For a family of logisticlike maps, we investigate the rate of convergence to the critical attractor when an ensemble of initial conditions is uniformly spread over the entire phase space. We found that the phase-space volume occupied by the ensemble $W(t)$ depicts a power-law decay with log-periodic oscillations reflecting the multifractal character of the critical attractor. We explore the parametric dependence of the power-law exponent and the amplitude of the log-periodic oscillations with the attractor's fractal dimension governed by the inflection of the map near its extremal point. Further, we investigate the temporal evolution of $W(t)$ for the circle map whose critical attractor is dense. In this case, we found $W(t)$ to exhibit a rich pattern with a slow logarithmic decay of the lower bounds. These results are discussed in the context of nonextensive Tsallis entropies.

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I. INTRODUCTION

Nonlinear low-dimensional dissipative maps can describe a great variety of systems with few degrees of freedom. The underlying nonlinearity can induce the system to exhibit a complex behavior with quite structured paths in the phase space. The sensitivity to initial conditions is a relevant aspect associated to the structure of the dynamical attractor. In general, the sensitivity is measured as the effect of any uncertainty on the system's variables. For systems exhibiting periodic or chaotic orbits, the effect of any uncertainty on initial conditions depicts an exponential temporal evolution with $\xi(t) \equiv \lim_{\Delta x(0)\to 0}\Delta x(t)/\Delta x(0) \sim e^{\lambda t}$, where λ is the Lyapunov exponent, and $\Delta x(0)$ and $\Delta x(t)$ are the uncertainties at times 0 and t. When the Lyapunov exponent $\lambda<0$, $\xi(t)$ characterizes the rate of contraction towards periodic orbits. On the other hand, for $\lambda>0$, it characterizes the rate of divergence of chaotic orbits. At bifurcation and critical points (i.e., onset to chaos) the Lyapunov exponent λ vanishes. Recently, it was shown that this feature is related to a power-law sensitivity to initial conditions on the form [1–3]

$$\xi(t) = [1+(1-q)\lambda_q t]^{1/(1-q)}, \tag{1}$$

with λ_q defining a characteristic time scale after which the power-law behavior sets up.

A quantitative way to measure the sensitivity to initial conditions is to follow, from a particular partition of the phase space, the temporal evolution of the number of cells $W(t)$ occupied by an ensemble of identical copies of the system. For periodic and chaotic orbits, $W(t) = W(0)e^{\lambda t}$. In the particular case of equiprobability, the well-known Pesin equality reads $K=\lambda$ if $\lambda \geqslant 0$ [5] with K being the Kolmogorov-Sinai entropy [4] defined as the variation per unit time of the standard Boltzmann-Gibbs entropy $S = -\sum p_i \ln p_i$. This equality provides a link between the sensitivity to initial conditions and the dynamic evolution of the relevant entropy.

At bifurcation and critical points and for an ensemble of initial conditions concentrated in a single cell, i.e., $W(0) = 1$, it has been shown that

$$W(t) = [1+(1-q)K_q t]^{1/(1-q)}, \tag{2}$$

with K_q being the generalized Kolmogorov-Sinai entropy defined as the rate of variation of the nonextensive Tsallis entropy $S_q = (1-\sum p_i^q)/(q-1)$ [6]. The Pesin equality can be generalized as $K_q = \lambda_q$ if $\lambda_q \geqslant 0$ [1]. Tsallis entropies have been successfully applied to recent studies of a series of nonextensive systems and provided a theoretical background to the understanding of some of their unusual physical properties [7,8].

The expansion towards the critical attractor of an ensemble of initial conditions concentrated around the inflection point of the map can be characterized by a proper S_q evolving at a constant rate. Scaling arguments have shown that the appropriate entropic index q is related to the multifractal structure of the critical dynamical attractor by [3]

$$\frac{1}{1-q} = \frac{1}{\alpha_{\min}} - \frac{1}{\alpha_{\max}}, \tag{3}$$

where $\alpha_{\min}$ and $\alpha_{\max}$ are the extremal singularity strengths of the multifractal spectrum of the critical attractor [9]. The

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above scaling relation has been shown to hold for the families of generalized logistic and circle maps [3,10–12].

However, the temporal evolution of critical dynamical systems can be strongly dependent on the particular initial ensemble. Although some scaling laws can be found for an ensemble of initial conditions concentrated around the map inflection point, these are usually not universal with respect to a general ensemble. In this work, we are going to numerically investigate the critical temporal evolution of the volume of the phase space occupied by an ensemble of initial conditions spread over the entire phase space. This ensemble is expected to contract towards the critical attractor. Using a family of one-dimensional generalized logistic maps having $d_f<1$, we will perform a detailed study of the parametric dependence of $W(t)$ on the fractal dimension of the critical attractor. Due to the discrete scale invariance of the critical attractor, the convergence displays log-periodic oscillations [13]. We are also going to explore the dependence of the amplitude of these oscillations with respect to the attractor's fractal dimension. Further, the behavior of $W(t)$ will be investigated for the one-dimensional critical circle map having $d_f=1$. For this map, the temporal evolution is expected to display distinct trends since the critical attractor is dense.

II. THE CONVERGENCE TO THE CRITICAL ATTRACTOR OF GENERALIZED LOGISTIC MAPS

Logisticlike maps are the simplest one-dimensional nonlinear dynamical systems that allow a close investigation of a series of critical exponents related to the onset of chaotic orbits. This family reads

$$x_{t+1}=1-a|x_t|^z,$$

$$(z>1;\quad 0<a<2;\quad t=0,1,2,\ldots;\quad x_t\in[-1,1]). \tag{4}$$

Here z is the inflection of the map in the neighborhood of the extremal point $\overline{x}=0$. These maps are well known to have topological properties not dependent of z. However, the metrical properties, such as Feigenbaum exponents [14,15] and the multifractal spectrum of the critical attractor, do depend on z. In particular, the fractal dimension of the critical attractor $d_f(z)<1$ [16] and therefore it does not fill a finite fraction of the phase space. For a set of initial conditions spread in the vicinity of the inflection point, it was found that the volume in phase space occupied by the ensemble grows following a rich pattern with the upper bounds $W_{\max}(t)$ governed by a power law $W_{\max}(t)\propto t^{1/(1-q)}$, where q is the entropic index characterizing the relevant Tsallis entropy that grows at a constant rate. It has been shown that the dynamic exponent $1/(1-q)$ is directly related to geometric scaling exponents related to the extremal sets of the dynamic attractor [3].

Due to the presence of long-range spatial and temporal correlations at criticality, one expects the critical exponent governing the temporal evolution to be sensitive to the particular initial ensemble. Indeed, the multifractal spectrum characterizing the critical dynamical attractor indicates that an infinite set of exponents are needed to fully characterize the scaling behavior. In particular, an ensemble consisting of a set of identical systems whose initial conditions are spread over the entire phase space is a common one when studying nonlinear as well as thermodynamical systems.

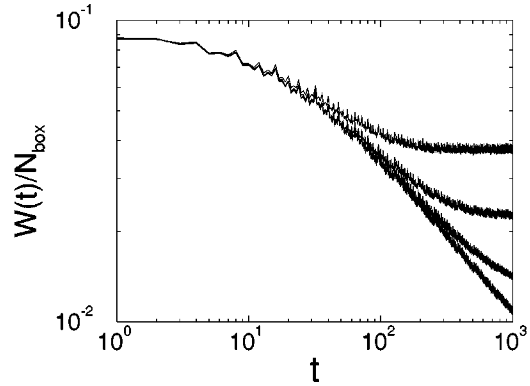

FIG. 1. The volume occupied by the ensemble $W(t)$ (number of occupied boxes) as a function of discrete time in the standard logistic map $(z=2)$ and with sampling ratio $r=0.1$. From top to bottom, $N_{\rm box}=2000$, 8000, 32 000, and 128 000.

Here, we will follow the dynamic evolution, in phase space, of an ensemble of initial conditions uniformly distributed over the phase space and explore its relation with the generalized fractal dimensions of the critical attractor. In practice, a partition of the phase space on $N_{\rm box}$ cells of equal size is performed and a set of N_c identical copies of the system is followed whose initial conditions are uniformly spread over the phase space. The ratio $r=N_c/N_{\rm box}$ is a control parameter giving the degree of sampling of the phase space.

Within the nonextensive Tsallis statistics, there is a proper entropy S_q evolving at a constant rate such that

$$K_q=\lim_{N_{\rm box}\to\infty}[S_q(t)-S_q(0)]/t \tag{5}$$

goes to a constant value as $t\to\infty$. Notice that $K_q<0$ for the process of convergence towards the critical attractor. Assuming that all cells of the partition are occupied with equal probability, the entropy $S_q(t)$ can be written as

$$S_q(t)=\frac{1-\sum_{i=1}^{W(t)}p_i^q}{q-1}=\frac{W(t)^{1-q}-1}{1-q}. \tag{6}$$

The last two equations imply that the number of occupied cells evolves in time as

$$W(t)=[W(0)^{1-q}+(1-q)K_qt]^{1/(1-q)} \tag{7}$$

with the exponent $\mu=-1/(q-1)>0$ governing the asymptotic power-law decay.

In Fig. 1, we show our results for $W(t)/N_{\rm box}$ in the standard logistic map with inflection $z=2$ and from distinct partitions of the phase space with sampling ratio $r=0.1$. We observe that, after a short transient period when $W(t)$ is nearly constant, a power-law contraction of the volume occupied by the ensemble sets up. $W(t)$ saturates at a finite fraction corresponding to the phase-space volume occupied by the critical attractor on a given finite partition. The satu-

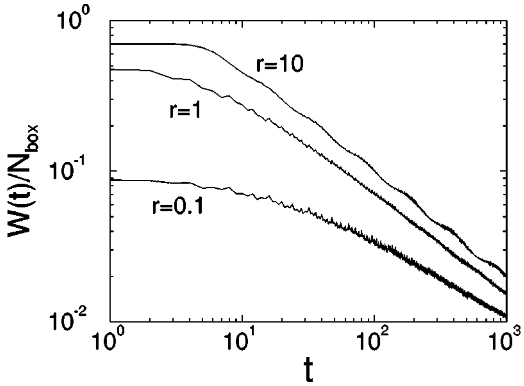

FIG. 2. The volume occupied by the ensemble $W(t)$ as a function of discrete time in the standard logistic map ($z=2$) and for a partition containing $N_{\mathrm{box}}=128\,000$ cells. Notice the emergence of log-periodic oscillations for large sampling ratios.

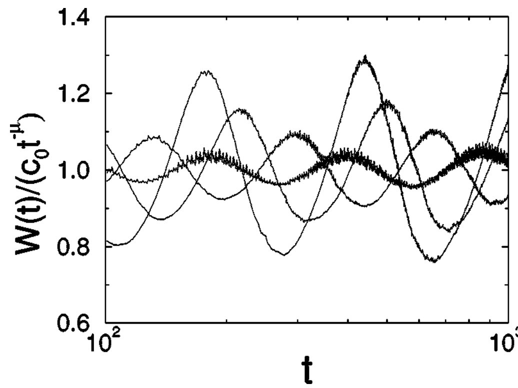

FIG. 3. The periodic function $W(t)/(c_0 t^{-\mu})$ versus discrete time within the scaling regime and for $r=10$. Data from map inflections $z=1.1,1.25,1.5,2.0$ are shown. The amplitude of the oscillations decreases monotonically as z increases, but the characteristic scaling factor between the periods of two consecutive oscillations is roughly z-independent.

ration is postponed when a finer partition is used once the fraction occupied by the critical attractor vanishes in the limit $N_{\mathrm{box}}\to\infty$.

In Fig. 2, we show $W(t)/N_{\mathrm{box}}$ for a given fine partition of the phase space and distinct sampling ratios r. We notice that the crossover regime to the power-law scaling is quite short for large values of r so that a clear power-law scaling regime sets up even at early times. This feature is consistent with Eq. (7), which states that the crossover time τ scales as $\tau \sim 1/W(0)^{q-1}$. Further, the scaling regime exhibits log-periodic oscillations once the multifractal nature of the critical attractor is closely probed by such a dense ensemble. A general form for $W(t)$ reflecting the discrete scale invariance of the attractor can be written as

$$W(t)=t^{-\mu}P\left(\frac{\ln t}{\ln \lambda}\right), \tag{8}$$

where P is a function of period unity and λ is the characteristic scaling factor between the periods of two consecutive oscillations. These log-periodic oscillations have been observed in a large number of systems exhibiting discrete scale invariance [13]. In general, the amplitude of these oscillations ranges form 10^{-4} up to 10^{-1}. Keeping only the first term in a Fourier series of $P(\ln t/\ln \lambda)$, one can write $W(t)$ in the form

$$W(t)=c_0 t^{-\mu}\left[1+2\frac{c_1}{c_0}\cos\left(2\pi\frac{\ln t}{\ln \lambda}+\phi\right)\right]. \tag{9}$$

Log-periodic modulations correcting a pure power law have been found in several systems, such as, for example, diffusion-limited aggregation [18], crack growth [19], earthquakes [20], and financial markets [21]. It has also been observed in thermodynamic systems with a fractal-like energy spectrum [22,23]. The factors controlling the log-periodic relative amplitude $2c_1/c_0$ are not well known for most of the systems where it has been observed. In the present study, we can closely investigate the factors which may control these amplitudes by measuring it as a function of the map inflection z for a fixed partition and sampling ratio (see Fig. 3). We found that these oscillations have amplitudes decaying exponentially with z as shown in Fig. 4. It is interesting to point out that the fractal dimension of the attractor is a monotonically decreasing function of z. Therefore, the above trend indicates a possible correlation between the amplitude of the log-periodic oscillations and the fractal dimension of the dynamical attractor.

We also measured the critical exponent μ as a function of the map inflection z. Our results are summarized in Table I. It is a decreasing function of z as can be seen in Fig. 5. The volume occupied by the ensemble depicts a fast contraction for $z\sim 1$ where the fractal dimension is small. On the other side, a very slow contraction is observed for large values of z, pointing towards a saturation or at most to a logarithmic decrease of $W(t)$ in the limit of dense attractors. We would like to point out here that the exponent governing the expansion of the volume occupied by an ensemble of initial conditions concentrated around the inflection point exhibits a reversed trend. Although scaling arguments have shown that this exponent can be written in terms of scaling exponents characterizing the extremal sets in the attractor, we could not devise a simple scaling relation between μ and the multifractal singularity spectrum. However, we observed that, when plotted against the fractal dimension of the attractor as shown in Fig. 6, the dynamic exponent μ is very well fitted

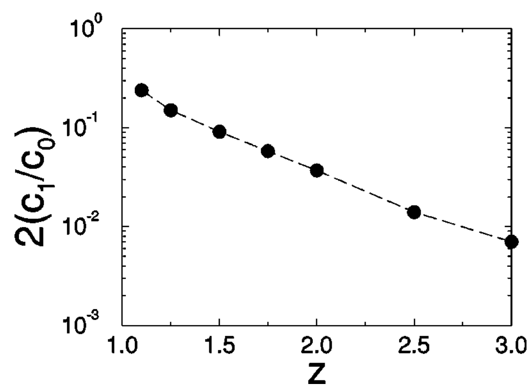

FIG. 4. The amplitude of the log-periodic oscillations $2c_1/c_0$ as a function of the map inflexion z for sampling ratio $r=10$. The monotonic decrease of the oscillations indicates a close relation between these and the fractal dimension of the underlying dynamical attractor.

TABLE I. Numerical values, within the z-generalized family of logistic maps, of (i) the dynamic exponent μ governing the contraction towards the critical attractor of the uniform ensemble; (ii) the entropic index q of the proper Tsallis entropy decreasing at a constant rate; (iii) the fractal dimension d_f of the critical attractor. These values also hold for the generalized periodic maps. The last line represents our results for the z-generalized circle maps.

z	$\mu=-1/(1-q)$	q	d_f
1.10	1.62 ± 0.02	1.62 ± 0.01	0.32 ± 0.02
1.25	1.23 ± 0.01	1.81 ± 0.01	0.40 ± 0.01
1.5	0.95 ± 0.01	2.05 ± 0.01	0.47 ± 0.01
1.75	0.80 ± 0.01	2.25 ± 0.015	0.51 ± 0.01
2.0	0.71 ± 0.01	2.41 ± 0.02	0.54 ± 0.01
2.5	0.59 ± 0.01	2.70 ± 0.02	0.58 ± 0.01
3.0	0.515 ± 0.005	2.94 ± 0.02	0.60 ± 0.01
5.0	0.395 ± 0.005	3.53 ± 0.03	0.66 ± 0.01
z-circular maps	0.0	∞	1.0

by $\mu\propto(1-d_f)^2$, which indicates d_f as the relevant geometric exponent coupled to the dynamics of the uniform ensemble. We would like to mention here that the same dynamic exponents were obtained for the generalized periodic maps, which belong to the same universality class of logisticlike maps [10].

III. THE CONVERGENCE TO THE CRITICAL ATTRACTOR OF THE CIRCLE MAP

The results from the preceding section indicate that a slow convergence to the critical attractor will be expected for dense critical attractors. However, it is not clear in what fashion this convergence will take place when the dynamical attractor fills the phase space with a multifractal probability density as occurs for the one-dimensional critical circle map,

$$\theta_{t+1}=\theta_t+\Omega-\frac{1}{2\pi}\sin(2\pi\theta_t)\ \mathrm{mod}(1), \qquad (10)$$

where $0\leqslant\theta_t<1$ is a point on a circle. The circle map describes dynamical systems possessing a natural frequency ω_1 that are driven by an external force of frequency ω_2 ($\Omega=\omega_1/\omega_2$ is the bare winding number) and belongs to the

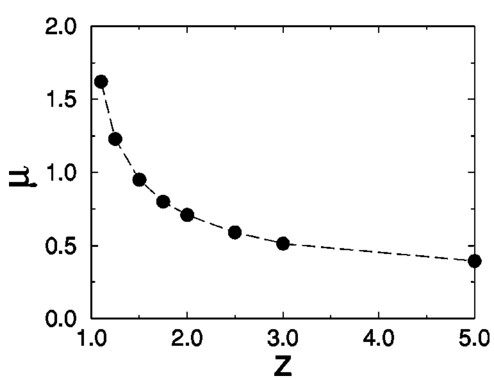

FIG. 5. The dynamic exponent μ governing the contraction of the occupied phase-space volume $[W(t)\propto t^{-\mu}]$ as a function of the map inflection z.

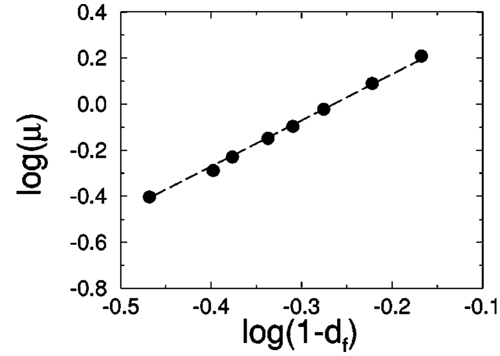

FIG. 6. $\log_{10}(\mu)$ versus $\log_{10}(d_f)$. The parametric dependence of the dynamic exponent μ with the fractal dimension d_f of the critical attractor is very well fitted to the form $\mu\propto(1-d_f)^2$. It indicates that d_f is the relevant geometric exponent coupled to the dynamics of the uniform ensemble.

same universality class of the forced Rayleigh-Bénard convection [17]. For $\Omega=0.606\,661\ldots$, the circle map has a cubic inflection ($z=3$) in the vicinity of the point $\overline{\theta}=0$. Starting from a given point on the circle, it generates a quasiperiodic orbit which fills the phase space and the dynamical attractor is a multifractal with fractal dimension $d_f=1$ [9].

In Fig. 7, we show our results for the temporal evolution of the phase-space volume occupied by an ensemble of initial conditions uniformly spread over the circle. $W(t)$ exhibits a rich pattern which resembles the one observed for the sensitivity function associated to the expansion of the phase space from initial conditions concentrated around the inflection point. However, $W(t)$ does not present any power-law regime. Instead, the lower bounds display a slow logarithmic decrease with time, saturating at a finite volume fraction. The saturation is a feature related to the finite partition used in the numerical calculation. This minimum decreases logarithmically with the number of cells in the phase space as shown in Fig. 8. We also observed the same behavior for generalized circle maps with an arbitrary inflection z [12]. The critical attractors within this family all have $d_f=1$ although they exhibit a z-dependent multifractal singularity spectra. The z-independent scenario for $W(t)$ corroborates the conjecture that d_f is the relevant geometric exponent coupled to the dynamics of the uniform ensemble.

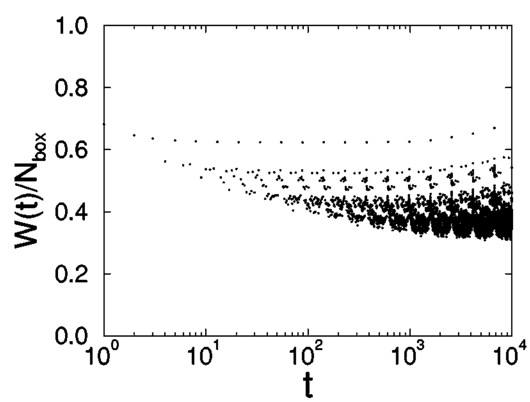

FIG. 7. The volume occupied by the ensemble $W(t)$ as a function of discrete time in the standard critical circle map. The lower bounds display a slow logarithmic decay with time saturating at a finite volume fraction due to the finite partition of the phase space.

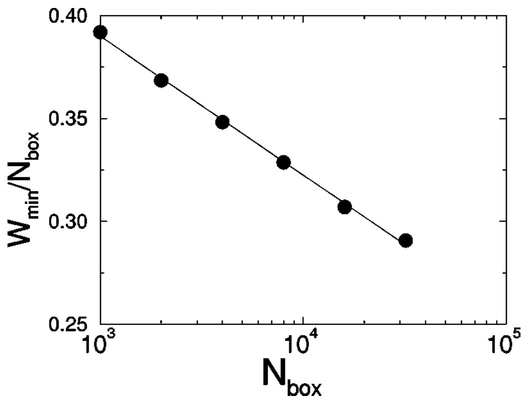

FIG. 8. The asymptotic lower bounds for the occupied volume in the phase space versus the number of cells N_{box}. The logarithmic decay agrees with the prediction that $\mu(d_f \to 1) \to 0$. The same behavior was observed for the family of generalized circle maps and corroborates the conjecture that d_f is the relevant geometric exponent coupled to the dynamics of the uniform ensemble.

IV. SUMMARY AND CONCLUSIONS

In this work, we studied the temporal evolution in phase space of an ensemble of identical copies of one-dimensional nonlinear dissipative maps. We found that the phase-space volume occupied by an initially uniform ensemble displays a power-law decay with log-periodic oscillations whenever the dynamical attractor has a fractal dimension $d_f < 1$, i.e., when the fractal attractor does not densely fill the phase space. Generally, these oscillations also emerge in open high-dimensional systems operating at a self-organized critical state. The spatiotemporal long-range correlations present on the critical state reflect the scale invariance of the dynamical attractor. Therefore, the present work corroborates the concept that the fractal nature of the dynamical attractor and the presence of a characteristic scaling factor are key ingredients for the emergence of log-periodic oscillations [13]. The amplitude of the oscillations was found to depict a monotonic parametric dependence on d_f. For dense multifractal attractors, $W(t)$ presents only a slow logarithmic contraction of its lower bounds followed by a rich pattern.

The critical exponent characterizing the contraction of the uniform ensemble was found to have no direct relation to the one governing the expansion from a set of initial conditions concentrated around the inflection point. In particular, no power law was found for the contraction in the standard and generalized circle maps, in contrast to the z-dependent power-law expansion. These results indicate that the relevant Tsallis entropy evolving at a constant rate (modulated by log-periodic oscillations) is characterized by an entropic index q that depends on the initial ensemble. It would be valuable to investigate the possible existence of classes of ensembles with a common dynamics in phase space and, therefore, characterized by the same entropic index q. The nonuniversality of q with respect to the initial ensemble is related to the multifractal character of the dynamical attractor. However, as for the ensemble concentrated at the vicinity of the inflection point, the exponent governing the dynamics of the uniform ensemble is coupled to a geometric scaling exponent, in particular to the proper fractal dimension of the attractor. Extensive numerical work would be valuable to verify the validity of the proposed relation on higher-dimensional systems. In any case, the present results come out in favor of the concept that the degree of nonextensivity of the entropy measure evolving at a constant rate is related to the fractal nature of the dynamical attractor.

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