Mesoscopic dynamics of microcracks

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(Received 31 January 2000; revised manuscript received 28 June 2000)

The mesoscopic concept is applied to the description of microcracks. The balance equations of the cracked continuum result in mesoscopic directional balances of mass, momentum, angular momentum, and energy. Averaging over the length of the cracks gives the corresponding orientational balances. A further averaging process leads to the macroscopic balance equations of the microcracked continua. Dynamic equations for the fabric tensors of different order are derived using a multipole moment expansion of the orientational crack distribution function. The simple example of Griffith cracks is treated. The role of physical assumptions in the microcrack representations and the different macroscopic internal variable representations of microcracks are discussed.

PACS number(s): 62.20.Mk, 46.50.+a, 81.40.Np

I. INTRODUCTION

To find suitable and applicable models for microstructured mechanical materials is a challenge of contemporary physics, especially of continuum mechanics and statistical physics. An important particular (and relatively simple) example in this respect is to describe the mechanical properties of microcrack systems in elastic materials. The two basic model levels are the continuum, where macroscopic variables are introduced to characterize the microcrack system, and the statistical, where the properties and interactions of single microcracks or the embedding material are considered.

On the macroscopic, continuum level *continuum damage mechanics* leads to suitable theories. In this phenomenological continuum theory thermodynamic internal variables of different tensorial order are used to calculate the influence of cracks (and other damage) on elastic properties of the material and to predict failure. It is important to remark here that the different continuum theories are far from being able to propose a single model for *all* important phenomena connected with cracking (multiaxial loading conditions, material stability, dynamics, etc.). The competing theories use different macroscopic mechanical and thermodynamic concepts. The most important aspects discussed are the nature (e.g., tensorial order, physical meaning) of the proposed macrosopic internal variables and the laws governing their time development, i.e., the corresponding macroscopic dynamical laws. We do not want to analyze the situation on the macroscopic level, but want to emphasize here that the lack of understanding at this level, that is, a macroscopic phenomenological model for the experimental observations (e.g., in the framework of irreversible thermodynamics), is a serious disadvantage in statistical physical modeling.

On the other hand, microcracking is an important problem in statistical physics and is treated with two different approaches. *Micromechanics* builds from detailed properties of single cracks and extends the results with the help of statistical methods [1]. The ''microscopic laws'' for a crack embedded in an ideal elastic continuum are well treated and known [2,3] and are difficult enough (long range, tensorial, anisotropic interactions with singularities) to mean a challenge for the basic principles of statistical physics on an equilibrium and also a nonequilibrium level. The second large group of statistical models is based on *lattice* calculations and simulations. These models introduce simple interactions between lattice elements (e.g., springs) and try to grasp some general qualitative properties of the phenomena with statistical methods. Some recent numerical and analytical investigations suggest the validity of mean field behavior in the presence of quenched disorder in isothermal systems (which we can expect in ordinary experimental situations), arguing that failure due to microcracking can be treated as a first order phase transition and the whole process as spinodal nucleation [4]. In early investigations spinodal nucleation was discussed as a thermally activated process, where the quenched disorder is irrelevant [5,6]. Some recent treatments claim that it is more realistic to consider a situation where a system is effectively at zero temperature and only the quenched disorder is relevant [7,8]. All these investigations concentrate on the avalanchelike behavior of microfracturing and calculate the scaling properties. However, the mean field behavior observed in numerical simulations of lattice models supports the view that phenomenological internal variable models can characterize the material, especially when we are far from the quasistatic regime.

In this paper we propose an idea to bridge the gap between the microscopic-statistical approach and the macroscopic-phenomenological one. We introduce a level of modeling that we call *mesoscopic*, because we go under the continuum level and use the statistical distribution function of the microcracks. However, instead of detailed microscopic modeling, general ideas are used to get the governing equations of the different distribution functions. The suggested method can be used to derive different macroscopic internal variable models that are compatible with the statistical description and to incorporate micromechanical information from single microcracks, thus connecting the statistical and phenomenological approaches.

Taking into account directional data distributions (e.g., normal vectors of planar microcracks) Kanatani [9] treated different possible statistical descriptions of directional data

and found a coordinate independent description in the form of generalized Fourier series, which is a coordinate independent form of the multipole moment expansion known from classical electrodynamics [10]. Kanatani called the corresponding moments of the directional data distribution the fabric tensors. So we can get a statistically founded classification of the macroscopic internal variables without any information on the possible dynamic properties. After that, different macroscopic thermodynamic methods, which are independent of the previous investigations, are used to get dynamic equations for the macroscopic variables.

However, in the case of another important family of microstructured continua, liquid crystals, the same moment series expansion is successfully applied to get not only the possible macroscopic thermodynamic variables but also some general information about their dynamic equations [11–18]. Here balance equations are applied to the microstructured continuum and by using them we can get some information on the mesoscopic dynamics and also derive dynamic equations for the macroscopic variables. These macroscopic variables are the same moments of the orientation of liquid crystal molecules that are used for microcracked continua; however, here they are called alignment tensors.

In this paper we apply the mesoscopic theory to get the dynamic equations of the mesoscopic variables, to introduce macroscopic variables, and to obtain the general form of their dynamic equations. These considerations give a good possibility for comparing the macroscopic consequences of the mesoscopic approach with other macroscopic theories, for example, with rational thermodynamic theories of microstructured continua, where the derivation of ''micromomentum balances'' is based on a particular application of material frame indifference [19–21]. On the other hand, we can introduce specific single crack properties (which is impossible in the case of liquid crystals) to solve the dynamic equation for the distribution function and get information far from the quasistatic range that is comparable with that from micromechanical and lattice models.

II. BASIC FIELDS AND FUNCTIONS

In liquid crystals molecules of restricted symmetry constitute the material continuum. In nematic liquid crystals the molecules are rodlike; therefore we introduce a quantity that characterizes the orientation of the molecules. There are two basic possibilities: we can give it as an additional vectorial field variable, when this (unit) vector is called the macroscopic director. In this way we can arrive at the Ericksen-Leslie-Parodi theory of nematics. The other possibility is to introduce the director as a mesoscopic variable. In this case the additional orientational information (a unit vector) plays a similar role to time and space and becomes a variable of the field quantities. In this way we arrive at the mesoscopic theory of liquid crystals, where all the field quantities are defined on the extended nematic space $S^2\times\mathbb{E}\times\mathbb{I}$, where S^2 denotes the unit sphere and $\mathbb{E}$ and $\mathbb{I}$ represent the space and the time. If the internal structure that we are modeling is more complicated then the characteristic mesoscopic variables can be more complicated, too. For example, in the case of biaxial nematics the molecules have two axes and the resulting symmetry is best described by quaternions.

What can we say about damaged materials? Here the damage can have a more complicated microscopic structure than in liquid crystals, but we can restrict ourselves to the simple and frequently investigated case of planar microcracks. Now the damage consists of small planar surface elements embedded in an elastic or elastoplastic (or any kind of) background material. In this case a crack can be represented by its surface vector. If the cracks are fixed in the material, that is, they do not move independently of the material elements, then we can apply the mesoscopic concept to describe the microstructure. Therefore a characteristic material element of the microcracked continuum is a crack together with the containing base material.

Let us observe the difference between liquid crystals and microcracks somewhat more closely. With a mesoscopic theory we intrude into the representative volume element of the continuum description and instead of a homogenization procedure (from where we would arrive at the continuum theory) we suppose that the macroscopic fields themselves depend on the microstructure and therefore we consider the statistical distribution of the orientations. In liquid crystals the shape of the molecules represents the microstructure and therefore the representative volume elements of the orientation and the other fields (especially mass) can be the same. However, in the case of microcracks the status of the microstructural information is different, because they can be considered to be embedded in an elastic (or viscoelastic, or anything but continuous) base material. In this case we are using different representative volume elements for the meso-micro transition and the macro-micro transition.

In this paper we give a mesoscopic model of a continuum that contains several randomly distributed microcracks. The microcracks are supposed to be two dimensional and flat; every microcrack is characterized by its surface ''vector'' $\mathrm{l}\in\mathbb{E}\wedge\mathbb{E}$ and spacetime position $r\in M$. Therefore the domain of all field quantities of the mesoscopic theory is interpreted on a subset of this *space*. For example, the *directional density* $\tilde{\rho}$ of the continuum is given as

$$\tilde{\rho}:\mathbb{E}\wedge\mathbb{E}\times M\rightarrow\mathbb{R}^+,\quad (\mathrm{l},r)\mapsto\tilde{\rho}(\mathrm{l},r),$$

where $\mathbb{E}$ is a three dimensional Euclidean vector space and M is the spacetime (a structured four dimensional affine space). If we are in a nonrelativistic spacetime and do not insist on a frame independent description we can introduce an inertial observer (as usual) [22]. As a final simplification we will use polar vectors instead of axial ones to represent the surfaces of the cracks, introducing the usual form and symmetry requirement for the density function:

$$\rho:\mathbb{E}_l\times\mathbb{E}\times\mathbb{I}\mapsto\mathbb{R}^+,\quad (\mathbf{l},\mathbf{x},t)\mapsto\rho(\mathbf{l},\mathbf{x},t),$$

$$\rho(\mathbf{l},\mathbf{x},t)=\rho(-\mathbf{l},\mathbf{x},t).$$

The direction, position, and time of the microcracks are denoted by $(\mathbf{l},\mathbf{x},t)\in\mathbb{E}_l\times\mathbb{E}\times\mathbb{I}$. The corresponding mesoscopic space $\mathbb{E}_l\times\mathbb{E}\times\mathbb{I}$, where $\mathbb{E}_l$ and $\mathbb{E}$ are three dimensional Euclidean spaces and $\mathbb{I}$ is a one dimensional oriented vector space, will be called the *direction space*. In the following we

suppose that the directional number density of the cracks has a finite support, that is, we consider a finite piece of material where the maximum length of the cracks is limited by the size of the sample (for example). Let us denote this maximal length by l_{max}.

A further important quantity can be introduced if we decompose the direction $\mathbf{l}$ into a length $l \in \mathbb{R}^+$ and an orientation $\mathbf{n} \in S^2$ as $\mathbf{l}=l\mathbf{n}$, where $\mathbf{n}$ is a unit vector ($\mathbf{n}^2=1$). Now the *orientational density* of the cracks is defined by the integral

$$\hat{\rho}(\mathbf{n},\mathbf{x},t)=\int_0^{l_{max}} \rho(l\mathbf{n},\mathbf{x},t) l^2 dl. \tag{1}$$

We will call the $S^2\times\mathbb{E}\times\mathbb{I}$ mesoscopic space of the orientation, position, and time $(\mathbf{n},\mathbf{x},t)$ of the microcracks the *orientation space*. For the further calculations it is very important to keep in mind the basic applicability criteria of the mesoscopic concept: the cracks are fixed in the base continuum. There is a fixed amount of base material for every microcrack. In this case, and only in this case, the density of the media will characterize the density of the number of cracks also. Taking this into account, we can write that

$$\overline{\rho}(\mathbf{x},t) := \langle \rho(\mathbf{l},\mathbf{x},t)\rangle := \tfrac{1}{2}\int_{\mathbb{R}^3} \rho(\mathbf{l},\mathbf{x},t) dV_l = \tfrac{1}{2}\int_{S^2} \hat{\rho}(\mathbf{n},\mathbf{x},t) d\mathbf{n}$$

is the macroscopic density of the microcracks at spacetime point $(\mathbf{x},t)$. Here dV_l denotes the Lebesque measure of the microcrack part of the direction space, and $d\mathbf{n}$ is the corresponding surface measure of S^2 in the orientation space. Furthermore,

$$\begin{aligned} M(t) &= \int_{\mathbb{R}^3} \langle \rho(\mathbf{l},\mathbf{x},t)\rangle dV \\ &= \int_{\mathbb{R}^3} \overline{\rho}(\mathbf{x},t) dV \\ &= \tfrac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \rho(\mathbf{l},\mathbf{x},t) dV_l\, dV \\ &= \tfrac{1}{2}\int_{\mathbb{R}^3}\int_{S^2} \hat{\rho}(\mathbf{n},\mathbf{x},t) d\mathbf{n}\, dV \end{aligned}$$

is the total mass of the sample continuum. The symmetric polar vector representation $[\rho(\mathbf{l},\mathbf{x},t)=\rho(-\mathbf{l},\mathbf{x},t)$ and $\hat{\rho}(\mathbf{n},\mathbf{x},t)=\hat{\rho}(-\mathbf{n},\mathbf{x},t)]$ necessitates the factor of $\frac{1}{2}$ in the last two integrals.

It is useful to normalize the densities by introducing the *directional probability distribution*

$$f(\mathbf{l},\mathbf{x},t) := \frac{\rho(\mathbf{l},\mathbf{x},t)}{\overline{\rho}(\mathbf{x},t)}, \tag{2}$$

orientational probability distribution

$$\hat{f}(\mathbf{n},\mathbf{x},t) := \frac{\hat{\rho}(\mathbf{n},\mathbf{x},t)}{\overline{\rho}(\mathbf{x},t)}, \tag{3}$$

and *length probability distribution*

$$f_l(\mathbf{l},\mathbf{x},t) := \frac{\rho(\mathbf{l},\mathbf{x},t)}{\hat{\rho}(\mathbf{n},\mathbf{x},t)} = \frac{f(\mathbf{l},\mathbf{x},t)}{\hat{f}(\mathbf{n},\mathbf{x},t)}. \tag{4}$$

At the end of this section let us remark that a mesoscopic theory formally resembles a mixture theory that uses the continuous directional or orientational ''index'' $\mathbf{l}$ or $\mathbf{n}$ for the ''components'' instead of a discrete one. This analogy can be a help in the interpretation of the directional and orientational ''component'' equations.

III. MESOSCOPIC KINEMATICS

The following formulas make it possible to give substantial balances in the mesoscopic continuum in addition to the local ones, so we can grasp the meaning of the corresponding mesoscopic balances more easily. Let us consider a piece of continuum material. Now we refer to the material elements with their position X at some initial instant t_0, as usual. Let us denote by x the position of the appropriate material element at the instant t. We give the position of the material element X at the time t with the map

$$\mathbf{x}: \mathbb{E}_0\times\mathbb{I} \rightarrowtail \mathbb{E}, (X,t) \mapsto \mathbf{x}(X,t).$$

Here we denoted the three dimensional Euclidean vector space of positions by $\mathbb{E}$ and the structural space of material points by $\mathbb{E}_0$. Similarly, we can give the material element at the position $\mathbf{x}$ and instant t with the map

$$\mathbf{X}: \mathbb{E}\times\mathbb{I} \rightarrowtail \mathbb{E}_0, (x,t) \mapsto \mathbf{X}(x,t).$$

The two maps have one to one correspondence and they are each other's inverse at the same instant $\mathbf{x}(\mathbf{X}(x,t),t)=x$ and $\mathbf{X}(\mathbf{x}(X,t),t)=X$.

The mesoscopic structure is characterized by the variable $l \in \mathbb{E}_l$. In the case of nematic liquid crystals this is the unit sphere S^2, for biaxial nematics it is S^3, and for planar microcracks a subset of $\mathbb{E}$. The microstructure is connected to the material element; therefore we can give its value at the instant t corresponding to the material element $X: \mathbf{l}: \mathbb{E}_0\times\mathbb{I} \rightarrowtail \mathbb{E}_l, (X,t) \mapsto \mathbf{l}(X,t)$.

Now we define the velocities

$$\mathbf{v} := \frac{\partial \mathbf{x}}{\partial t}(X,t) \quad \text{and} \quad \mathbf{v}_l := \frac{\partial \mathbf{l}}{\partial t}(X,t).$$

If a field quantity depends on the direction l and the position x we can define its material time derivative as follows:

$$\dot{f}(l,x,t) := \left(\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \mathbf{v}_l\cdot\nabla_l f\right) \circ (\mathbf{l},\mathbf{x},t)(X,t). \tag{5}$$

Until now we have considered a continuum; therefore the position x and the direction l were treated as fields. However, the situation is more difficult because we are below the continuum level. In the case of liquid crystals on the microscopic level we have single molecules. For cracks we can suppose that we are in the continuum domain as regards the mass, but the material elements contain single microcracks and therefore the direction can be discontinuous from crack to crack at this level. Our task is to get a continuum description and at the same time keep some information from below the usual macroscopic continuum level. Therefore we accomplish a second homogenization, forming a bigger material element from the micro-meso ones and introducing a center of mass X_0 for that macro material element with volume V_m:

$$X_0 = \frac{\int_{V_m} X\rho(X)dV_m}{\int_{V_m} \rho(X)dV_m}.$$

Now if we completely replaced the micro material elements with macro ones ($X \to X_0$) we would get the macroscopic director $\mathbf{l}(X_0,t)$. Instead of doing that, we would like to keep some microscopic information and therefore we make the $X \to (X_0,l)$ substitution, supposing that there is a distribution of directions inside the macro element. In this way l is no longer a field quantity as we supposed above, but stands on equal footing with X_0 and plays an independent role characterizing the macro continuum element. Therefore the previous functions defined on the micro material space become functions on the mesoscopic space (e.g., the velocities $\mathbf{v},\mathbf{v}_l$). Moreover, as the continuity of the variables is secured with the homogenization procedure we can introduce the previous derivative (5) as a material derivative on the mesoscopic space.

IV. DIRECTIONAL BALANCES

After these preparations we are ready to get the mesoscopic balance equations of the directional quantities. All of the following local balances were derived from the proper global balances using a generalized form of the Gauss-Stokes integral theorem (or Reynolds transport theorem, equivalently). The difference between the usual spacetime balances and the following generalized balances where the spacetime variables are completed with the direction is that now the ''configuration space'' of the continuum is six dimensional. Therefore the velocity space is also six dimensional; we get an additional directional velocity component. Moreover, the local balances will have an additional ''current term'' with the divergence of the directional part of the total (2×3) dimensional current densities $(\boldsymbol{\nabla}_l \cdot)$. Using the introduced mesoscopic material time derivative (5) we will give the corresponding substantial balances also.

First we can get the fundamental balance of mass

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) + \boldsymbol{\nabla}_l\cdot(\rho\mathbf{v}_l) = 0 \tag{6}$$

and

$$\dot{\rho} + \rho(\boldsymbol{\nabla}\cdot\mathbf{v} + \boldsymbol{\nabla}_l\cdot\mathbf{v}_l) = 0, \tag{7}$$

where $\mathbf{v}$ is the directional material velocity and $\mathbf{v}_l$ is the velocity of the change of crack orientation and length. Here and throughout this section all quantities are directional; their domain is the subset of the direction space.

The balance of momentum is given by

$$\frac{\partial \rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\circ\mathbf{v} - \mathbf{t}^T) + \nabla_l\cdot(\rho\mathbf{v}_l\circ\mathbf{v}) = \rho\mathbf{f} \tag{8}$$

and

$$\rho\dot{\mathbf{v}} - \boldsymbol{\nabla}\cdot\mathbf{t}^T = \rho\mathbf{f}. \tag{9}$$

Here $\mathbf{t}$ is the stress, $\mathbf{f}$ is the body force density, and $\circ$ is the notation of tensorial product in continuum physics. The superscript T denotes the transpose of the corresponding second order tensor. We assumed here that the change of momentum, even if it depends on the direction, is due only to the body and normal surface forces, that is there are no surface forces in the direction component, and that there is no conductive directional momentum current.

Similarly, for the balance of the moment of momentum we will get the form

$$\frac{\partial \rho\mathbf{s}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}\circ\mathbf{s} - \boldsymbol{\pi}^T) + \boldsymbol{\nabla}_l\cdot(\rho\mathbf{v}_l\circ\mathbf{s}) = \mathbf{t}^{as} + \rho\mathbf{g}, \tag{10}$$

where $\mathbf{s}$ is the directional spin density, $\mathbf{t}^{as}$ denotes the antisymmetric part of the stress tensor, $\boldsymbol{\pi}$ is the couple stress, and $\mathbf{g}$ is the density of the couple force. The substantial form is

$$\rho\dot{\mathbf{s}} - \boldsymbol{\nabla}\cdot\boldsymbol{\pi}^T = \mathbf{t}^{as} + \rho\mathbf{g}. \tag{11}$$

Finally we give the directional internal energy density balances. These we got by subtracting the balances of the kinetic and rotational energy from the balance of the total energy:

$$\frac{\partial \rho\epsilon}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}\epsilon + \mathbf{q}) + \boldsymbol{\nabla}_l\cdot(\rho\mathbf{v}_l\epsilon + \mathbf{q}_l) = \boldsymbol{\nabla}\circ\mathbf{v}:\mathbf{t} + \rho\Xi. \tag{12}$$

Here $\mathbf{q}$ and $\mathbf{q}_l$ are the heat current and the directional heat current, respectively (both mesoscopic). Ξ is the internal energy production related directly to the microcrack propagation. The corresponding substantial form is

$$\rho\dot{\epsilon} + \boldsymbol{\nabla}\cdot\mathbf{q} + \boldsymbol{\nabla}_l\cdot\mathbf{q}_l = \boldsymbol{\nabla}\circ\mathbf{v}:\mathbf{t} + \rho\Xi. \tag{13}$$

V. ORIENTATIONAL BALANCES

Traditionally in damage mechanics we are interested only in the orientational part of the data distributions; when the length of the cracks is supposed to be statistically independent of the orientational part of the data distribution, we use averaged, uniform size cracks in the treatment. Therefore here we give the balances of the orientational quantities also. If we want to get an orientational quantity from a directional one we should average over the microcrack length using the directional distribution function f and length distribution function f_l [see Eqs. (2) and (4)]. To do this we will integrate the directional balances over the microcrack length. It is worthy of note here that the time derivation and normal divergence commute with the integration over the length, and for an arbitrary directional function h

$$\int_0^{l_{max}} \boldsymbol{\nabla}_l h(\mathbf{l}) l^2\, dl = \boldsymbol{\nabla}_{\mathbf{n}} \int_0^{l_{max}} h(l\mathbf{n}) l^2\, dl. \tag{14}$$

The commutation properties and Eq. (14) suppose several identifications and regularity properties. For example, in calculating the formula (14) the splitting of the directional derivative was accomplished as $\boldsymbol{\nabla}_l = (\mathbf{n}\cdot\boldsymbol{\nabla}_\mathbf{l}, (\mathbf{I}-\mathbf{n}\circ\mathbf{n})\boldsymbol{\nabla}_\mathbf{n}) = (\partial/\partial l, \boldsymbol{\nabla}_\mathbf{n})$, and $(0,\mathbf{a}) = \mathbf{a}$. We will use the caret for the orientational quantities (as above) and introduce the notation

$\langle\,\rangle_l$ for the length averaging. Denoting the orientational velocities by $\hat{\mathbf{v}}:=\langle\mathbf{v}\rangle_l$ and $\hat{\mathbf{v}}_l:=\langle\mathbf{v}_l\rangle_l$, then we get the orientational mass balance

$$\frac{\partial\hat{\rho}}{\partial t}+\boldsymbol{\nabla}\cdot(\hat{\rho}\hat{\mathbf{v}})+\boldsymbol{\nabla}_{\mathbf{n}}\cdot(\hat{\rho}\hat{\mathbf{v}}_l)=0. \tag{15}$$

In substantial form this is

$$\dot{\hat{\rho}}+\hat{\rho}\boldsymbol{\nabla}\cdot\hat{\mathbf{v}}+\hat{\rho}\boldsymbol{\nabla}_{\mathbf{n}}\cdot\hat{\mathbf{v}}_l=0. \tag{16}$$

The balance of momentum in local form can be given as

$$\frac{\partial\hat{\rho}\hat{\mathbf{v}}}{\partial t}+\boldsymbol{\nabla}\cdot(\hat{\rho}\hat{\mathbf{v}}\circ\hat{\mathbf{v}}-\hat{\mathbf{t}}^T)+\boldsymbol{\nabla}_n\cdot(\hat{\rho}\hat{\mathbf{v}}_l\circ\hat{\mathbf{v}}-\hat{\mathbf{T}}^T)=\hat{\rho}\hat{\mathbf{f}}. \tag{17}$$

Here we introduced the orientational stress $\hat{\mathbf{t}}$ and orientational *microstress* $\hat{\mathbf{T}}$ as follows:

$$\hat{\mathbf{t}}=\hat{\rho}\hat{\mathbf{v}}\circ\hat{\mathbf{v}}-\int_0^{l_{max}}(\rho\mathbf{v}\circ\mathbf{v}-\mathbf{t})l^2\,dl$$

$$=\hat{\rho}(\hat{\mathbf{v}}\circ\hat{\mathbf{v}}-\langle\mathbf{v}\circ\mathbf{v}\rangle_l)+\int_0^{l_{max}}\mathbf{t}l^2\,dl,$$

$$\hat{\mathbf{T}}=\hat{\rho}\hat{\mathbf{v}}_l\circ\hat{\mathbf{v}}-\int_0^{l_{max}}\rho\mathbf{v}_l\circ\mathbf{v}l^2\,dl=\hat{\rho}(\hat{\mathbf{v}}_l\circ\hat{\mathbf{v}}-\langle\mathbf{v}_l\circ\mathbf{v}\rangle_l).$$

We can give the substantial form of the orientational momentum balance as

$$\hat{\rho}\dot{\hat{\mathbf{v}}}-(\boldsymbol{\nabla}\cdot\hat{\mathbf{t}}^T+\boldsymbol{\nabla}_n\cdot\hat{\mathbf{T}}^T)=\hat{\rho}\hat{\mathbf{f}}. \tag{18}$$

The appearance of microstress is remarkable, a conductive orientational momentum current in the orientational momentum balance.

The local balance of the moment of momentum is

$$\frac{\partial\hat{\rho}\hat{\mathbf{s}}}{\partial t}+\boldsymbol{\nabla}\cdot(\hat{\rho}\hat{\mathbf{v}}\hat{\mathbf{s}}-\hat{\boldsymbol{\pi}}^T)+\boldsymbol{\nabla}_{\mathbf{n}}\cdot(\hat{\rho}\hat{\mathbf{v}}_l\hat{\mathbf{s}}-\hat{\mathbf{\Pi}}^T)=\hat{\mathbf{t}}^{as}+\hat{\rho}\hat{\mathbf{g}}. \tag{19}$$

Here $\hat{\mathbf{s}}=\langle\mathbf{s}\rangle_l$ is the orientational spin and $\hat{\mathbf{g}}=\langle\mathbf{g}\rangle_l$ is the orientational couple force vector. However, we should be careful because, for example, $\hat{\mathbf{t}}^{as}\neq\langle\mathbf{t}^{as}\rangle_l$, but we should consider the previous definition. Moreover, the couple microstress $\hat{\boldsymbol{\pi}}$ and the new *orientational coupling microstress* $\hat{\mathbf{T}}$ are defined as

$$\hat{\boldsymbol{\pi}}=\hat{\rho}(\hat{\mathbf{v}}\circ\hat{\mathbf{s}}-\langle\mathbf{v}\circ\mathbf{s}\rangle_l)+\int_0^{l_{max}}\pi l^2\,dl,$$

$$\hat{\mathbf{\Pi}}=\hat{\rho}(\hat{\mathbf{v}}_l\circ\hat{\mathbf{s}}-\langle\mathbf{v}_l\circ\mathbf{s}\rangle_l).$$

It is easy to see that $\hat{\boldsymbol{\pi}}$ is orthogonal to $\mathbf{n}$. The corresponding substantial equation is

$$\hat{\rho}\dot{\hat{\mathbf{s}}}-(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\pi}}^T+\boldsymbol{\nabla}_{\mathbf{n}}\cdot\hat{\mathbf{\Pi}}^T)=\hat{\mathbf{t}}^{as}+\hat{\rho}\hat{\mathbf{g}}. \tag{20}$$

The orientational balance of the internal energy is very similar to the directional one, but the conductive currents and the second source term, are not simply the average of the corresponding directional quantities. The single averaged orientational term is the internal energy itself ($\hat{\epsilon}=\langle\epsilon\rangle_l$).

$$\frac{\partial\hat{\rho}\hat{\epsilon}}{\partial t}+\boldsymbol{\nabla}\cdot(\hat{\rho}\hat{\mathbf{v}}\hat{\epsilon}+\hat{\mathbf{q}})+\boldsymbol{\nabla}_{\mathbf{n}}\cdot(\hat{\rho}\hat{\mathbf{v}}_l\hat{\epsilon}+\hat{\mathbf{q}}_l)=\boldsymbol{\nabla}\circ\hat{\mathbf{v}}:\hat{\mathbf{t}}+\hat{\rho}\hat{\Xi}. \tag{21}$$

The definitions of the heat currents and the source term are as follows:

$$\hat{\mathbf{q}}=\hat{\rho}(\langle\mathbf{v}\epsilon\rangle_l-\hat{\mathbf{v}}\hat{\epsilon})+\int_0^{l_{max}}\mathbf{q}l^2dl,$$

$$\hat{\mathbf{q}}_l=\hat{\rho}(\langle\mathbf{v}_l\epsilon\rangle_l-\hat{\mathbf{v}}_l\hat{\epsilon}),$$

$$\hat{\Xi}=\hat{\rho}\langle\Xi\rangle_l+\int_0^{l_{max}}\boldsymbol{\nabla}\circ\mathbf{v}:\mathbf{t}l^2dl-\boldsymbol{\nabla}\circ\hat{\mathbf{v}}:\hat{\mathbf{t}}.$$

We can easily get the substantial form:

$$\hat{\rho}\dot{\hat{\epsilon}}+\boldsymbol{\nabla}\cdot\hat{\mathbf{q}}+\boldsymbol{\nabla}_{\mathbf{n}}\cdot\hat{\mathbf{q}}_l=\boldsymbol{\nabla}\circ\hat{\mathbf{v}}:\hat{\mathbf{t}}+\hat{\rho}\hat{\Xi}. \tag{22}$$

VI. MACROSCOPIC BALANCES

In the calculation of the macroscopic balances we can use either the directional or the orientational balances. Maybe the first way is the more convenient. We will denote the macroscopic quantities with the overbar and the averaged directional quantities that are calculated with the help of the directional distribution function (2) with angular brackets $\langle\,\rangle$. The corresponding macroscopic equations are calculated by integration of the directional balances over $\mathbf{l}$. This integration commutes with the time and space derivatives and eliminates the divergence of the directional derivative, because any direction function $\mathbf{h}$ has a compact support,

$$\int_{V_l}\boldsymbol{\nabla}_l\cdot\mathbf{h}\,dV_l=0.$$

If the macroscopic (barycentric) velocity $\overline{\mathbf{v}}=\langle\mathbf{v}\rangle$ then, with the previously introduced macroscopic density $\overline{\rho}$, the balance of mass can be written as

$$\frac{\partial\overline{\rho}}{\partial t}+\boldsymbol{\nabla}\cdot(\overline{\rho\,\mathbf{v}})=0. \tag{23}$$

The balance of momentum is

$$\frac{\partial\overline{\rho\,\mathbf{v}}}{\partial t}+\boldsymbol{\nabla}\cdot(\overline{\rho\,\mathbf{v}}\circ\overline{\mathbf{v}}-\overline{\mathbf{t}}^T)=\overline{\rho\,\mathbf{f}}, \tag{24}$$

where the macroscopic force density $\overline{\mathbf{f}}=\langle\mathbf{f}\rangle$. Again the macroscopic stress is not a simple average; it can be calculated as

$$\overline{\mathbf{t}}=\int_{V_l}\mathbf{t}\,dV_l+\rho(\overline{\mathbf{v}}\circ\overline{\mathbf{v}}-\langle\mathbf{v}\circ\mathbf{v}\rangle).$$

The balance of the moment of momentum and the balance of internal energy can be calculated similarly:

$$\frac{\partial \overline{\rho\,\mathbf{s}}}{\partial t}+\boldsymbol{\nabla}\cdot(\overline{\rho\,\mathbf{v}\circ\mathbf{s}}-\overline{\boldsymbol{\pi}})=\overline{\mathbf{t}}^{as}+\overline{\rho\,\mathbf{g}}, \tag{25}$$

where the macroscopic couple force density $\overline{\mathbf{g}}=\langle\mathbf{g}\rangle$. The macroscopic couple stress also includes a contribution due to deviations of $\mathbf{v}$ and $\mathbf{s}$ from the average; this can be calculated as

$$\overline{\boldsymbol{\pi}}=\int_{V_l}\pi\,dV_l+\rho(\overline{\mathbf{v}\circ\mathbf{s}}-\langle\mathbf{v}\circ\mathbf{s}\rangle).$$

The balance of the internal energy becomes

$$\frac{\partial \overline{\rho\epsilon}}{\partial t}+\boldsymbol{\nabla}\cdot(\overline{\rho\,\mathbf{v}\epsilon}-\overline{\mathbf{q}})=\boldsymbol{\nabla}\circ\overline{\mathbf{v}:\mathbf{t}}+\overline{\rho\Xi}, \tag{26}$$

where

$$\overline{\mathbf{q}}=\overline{\rho}(\langle\mathbf{v}\epsilon\rangle-\overline{\mathbf{v}\epsilon})+\int_{V_l}\mathbf{q}dV_l,$$

$$\overline{\rho\,\Xi}=\overline{\rho}\langle\Xi\rangle+\int_{V_l}\boldsymbol{\nabla}\circ\mathbf{v}:\mathbf{t}\,dV_l-\boldsymbol{\nabla}\circ\overline{\mathbf{v}:\mathbf{t}}.$$

Let us remark that on the directional and orientational level it was unreasonable to suppose a positive entropy production but it makes sense on a macroscopic level and we can exploit it.

VII. CRACK PROPAGATION

For the most frequently used materials in damage mechanics the balance equations given above are too general. Therefore we introduce some simplifying assumptions. (1) The base material does not have an internal spin, that is, a crack does not rotate independently from the base material. (2) There are no couple forces ($\mathbf{g}=0$) and coupling stresses ($\mathbf{II}=0$). (3) There are no external body forces ($\mathbf{f}=0$). (4) The material is in mechanical equilibrium ($\dot{\mathbf{v}}=0$). (5) The velocity does not depend on the crack size and orientation; this means it is equal to the barycentric velocity [$\mathbf{v}(\mathbf{l},\mathbf{x},t)=\overline{\mathbf{v}}(\mathbf{x},t)$].

Because of the first condition we do not need the balance of the internal energy and the third condition simplifies the spin balance to a symmetric stress. For this symmetric stress the balance of the momentum together with the fourth and fifth conditions results in an equation for the mechanical equilibrium:

$$\boldsymbol{\nabla}\cdot\mathbf{t}=0. \tag{27}$$

Very similarly, we can get in the orientational space

$$\boldsymbol{\nabla}\cdot\hat{\mathbf{t}}=0. \tag{28}$$

Moreover, the balance of mass simplifies considerably because of the last condition:

$$\frac{\partial f}{\partial t}+\overline{\mathbf{v}}\cdot\boldsymbol{\nabla}f+\boldsymbol{\nabla}_l\cdot(f\mathbf{v}_l)=0, \tag{29}$$

where $f=\rho/\overline{\rho}$ is the directional probability density as it was given in Eq. (2) and $\overline{\mathbf{v}}$ is the macroscopic velocity. The substantial form is remarkably simple:

$$\dot{f}+f\boldsymbol{\nabla}_l\cdot\mathbf{v}_l=0.$$

Integrating over the crack length we get the following orientational balance:

$$\frac{\partial \hat{f}}{\partial t}+\overline{\mathbf{v}}\cdot\boldsymbol{\nabla}\hat{f}+\boldsymbol{\nabla}_{\mathbf{n}}\cdot\hat{f}\hat{\mathbf{v}}_l=0, \tag{30}$$

where the orientational crack velocity $\hat{\mathbf{v}}_l$ was introduced. Let us observe that Eq. (29) is formally the same as Eq. (30) but the functions are different (directional/orientational).

Therefore our simplified final system of equations is Eqs. (27) and (29) in direction space, or Eqs. (28) and (30) in orientation space. Now we can consider several possibilities for a closed, soluble system. We can try to close the system on the mesoscopic or on the macroscopic level.

(a) We can consider some specific information on the crack propagation and calculate the crack growth speed $\mathbf{v}_l$. This is promising because this velocity is connected to the micromaterial element and therefore we need to investigate a single crack to calculate it. In this way, by introducing the corresponding state space and considering some constitutive assumptions on the mesoscopic stress, there is a good chance of closing the system at the mesoscopic level. The problematic point is the constitutive assumption for $\mathbf{t}$. On the mesoscopic level, without an inequality from the second law for the mesoscopic functions, the constitutive theory is more approximate.

(b) The other possibility is to calculate the macroscopic balances from the mesoscopic ones. In this case the orientational balances are more promising, because here the moment series expansion gives a familiar and understood process (see, e.g., [12]). We can try similar series expansions in the directional space too, but the most straightforward choices mix the length and orientational information and therefore the meaning of the macroscopic quantities is not evident.

A. Moment series expansion and order parameters

First we will investigate the consequences of moment series expansion of the distribution function $\hat{f}$ and Eq. (30). We can introduce the following alignment-fabric tensors:

$$\mathbf{a}^{(k)}(\mathbf{x},t):=\int_{S^2}\hat{f}(\mathbf{n},\mathbf{x},t)\,\overline{\mathbf{n}\circ\cdots\circ\mathbf{n}}\,d\mathbf{n}, \tag{31}$$

where $\overline{\cdots}$ denotes the symmetric irreducible part of a tensor [23]. It is remarkable that only the even order tensors appear in the series because the microcracks are represented by axial vectors. These damage parameters are macroscopic quantities and are called fabric tensors of the second kind in damage mechanics (see Kanatani [9] or Krajcinovic [1]). They were introduced on purely statistical grounds, without a mesoscopic foundation.

Now let us turn our attention to the series expansion of Eq. (30). We can get the following system of equations for the kth moment [24]:

$$\frac{da^{(k)}_{n_1\cdots n_k}}{dt} - k\,\overline{(\underline{\omega}\times a^{(k)})}_{n_1,\ldots,n_k} = \frac{2k+1}{k!4\pi}\sum_{l\,even}(2l-1)!!\Bigg[\left(\oint_{S^2}\delta\hat{v}^p\; n_{m_1}\cdots n_{m_l}\; n_{n_1}\cdots n_{n_k}d^2n a^{(l)}_{m_1,\ldots,m_l}\right)_{,p}$$
$$+\left(\oint_{S^2}\delta\hat{v}^p\,\overline{n_{m_1}\cdots n_{m_l}}\,\overline{n_{n_1}\cdots n_{n_k}}\,d^2n(\ln\rho)_{,p}\right.$$
$$\left.+\oint_{S^2}\overline{n_{m_1}\cdots n_{m_l}}\,\overline{n_{n_1}\cdots n_{n_k}}\,(\boldsymbol{\nabla}_n\times\delta\hat{\underline{\omega}})\cdot d\mathbf{n}\right)a^{(l)}_{m_1,\ldots,m_l}$$
$$-l\oint_{S^2}\overline{n_{m_1}\cdots n_{m_l}}\,\overline{n_{n_1}\cdots n_{n_k}}\,(\delta\hat{\underline{\omega}}\times a^{(l)})_{m_1,\ldots,m_l}\cdot d\mathbf{n}$$
$$+\oint_{S^2}\delta\hat{v}^p\,\overline{n_{n_1}\cdots n_{n_k}}\,d^2n(\ln\rho)_{,p}+\left(\oint_{S^2}\delta\hat{v}^p\,\overline{n_{n_1}\cdots n_{n_k}}\,d^2n\right)_{,p}$$
$$+\oint_{S^2}\overline{n_{n_1}\cdots n_{n_k}}\,(\boldsymbol{\nabla}_n\times\delta\hat{\underline{\omega}})\cdot d\underline{n}\Bigg],$$

where $\underline{\omega}=\frac{1}{2}\boldsymbol{\nabla}\times\mathbf{v}$, $\delta\hat{\underline{\omega}}=\hat{\underline{\omega}}-\overline{\underline{\omega}}$, $\delta\hat{\mathbf{v}}=\hat{\mathbf{v}}-\overline{\mathbf{v}}$, and we denoted the components of the velocity with the index p to avoid misunderstanding. In this way we have a whole set of possible macroscopic damage parameters together with a general form for their dynamic equation. Let us investigate more closely the dynamic equation of the second order tensor term in the expansion. It seems useful to put down the definition and the dynamic equation for that term separately as follows:

$$\mathbf{a}(x,t) := \int_{S^2}\hat{f}\,\overline{\mathbf{n}\circ\mathbf{n}}\,d\mathbf{n}, \tag{32}$$

and we get the following dynamic equation:

$$\frac{\partial\mathbf{a}}{\partial t}+\bar{\mathbf{v}}\cdot\boldsymbol{\nabla}\mathbf{a}+\int_{S^2}\overline{\mathbf{n}\circ\mathbf{n}}\,\boldsymbol{\nabla}_{\mathbf{n}}\cdot(\hat{f}\hat{\mathbf{v}}_l)=0, \tag{33}$$

or equivalently

$$\frac{\partial\mathbf{a}}{\partial t}+\bar{\mathbf{v}}\cdot\boldsymbol{\nabla}\mathbf{a}+2\langle\,\overline{\mathbf{v}_l\circ\mathbf{n}}\,\rangle=0. \tag{34}$$

Without calculating the last term we can see that **a** is a normal internal variable in the sense that a local first order differential equation describes its change.

It is worth investigating the uniaxial case separately, when the alignment tensors can be expressed in terms of the order parameters $S^{(k)}$ and a unit vector **d** in the following way:

$$\mathbf{a}^{(k)}=S^{(k)}\,\overline{\mathbf{d}\circ\cdots\circ\mathbf{d}}\qquad(k=2,4,\ldots),$$

where the value of the order parameter $S^{(k)}$ is 1 in the case of total alignment (the microcracks are parallel) and zero for randomly oriented cracks. In the case of a second order alignment tensor ($\mathbf{a}^{(2)}=S^{(2)}\,\overline{\mathbf{d}\circ\mathbf{d}}$) the following dynamic equations can be written for the order parameter:

$$\frac{dS}{dt}+e\,\mathbf{d}\cdot\langle\,\overline{\mathbf{v}_l\circ\mathbf{n}}\,\rangle\cdot\mathbf{d}=0. \tag{35}$$

Moreover, for the vector **d** we can get

$$S\frac{d\mathbf{d}}{dt}=2(\mathbf{d}\circ\mathbf{d}-\delta)\mathbf{d}:\langle\,\overline{\mathbf{v}_l\circ\mathbf{n}}\,\rangle.$$

Let us remark that there are cases when a truncation of the series leads to paradoxes. If the microcrack distribution is uniaxial then the best fitting alignment tensor can result in negative crack densities, the so called anticrack regions in the approximate data distribution [25]. This unexpected property can be removed if we use a director (vectorial) internal variable representation instead of the even order traceless tensors. The single vectorial approximation results in a macroscopic director theory.

B. Solution for the distribution function: Griffith cracks

In this section the equation of motion for the mesoscopic distribution function is specialized by considering a specific single crack model. In this case we can start from the directional level and calculate the crack size distribution function. The following additional assumptions are introduced. (1) The crack surface area can increase, but cannot decrease. (2) The crack velocity is independent of other cracks in the vicinity, i.e., cracks do not interact. (3) Crack inertia is neglected in the expression for the crack velocity, i.e., it is assumed that the crack stops enlarging instantaneously when the external load stops changing. (4) All the idealizations assumed by Griffith [26] (e.g., two dimensions, ideal elliptic cracks, etc.) are supposed here. Let us observe that these seemingly restrictive conditions are in some respects more general than the restrictions used explicitly or implicitly in models of micromechanical origin [2,1,3]. For example, we did not assume special crack orientations or definite interactions between the microcracks.

From the mesoscopic balance of mass we derived the following differential equation for the directional distribution function:

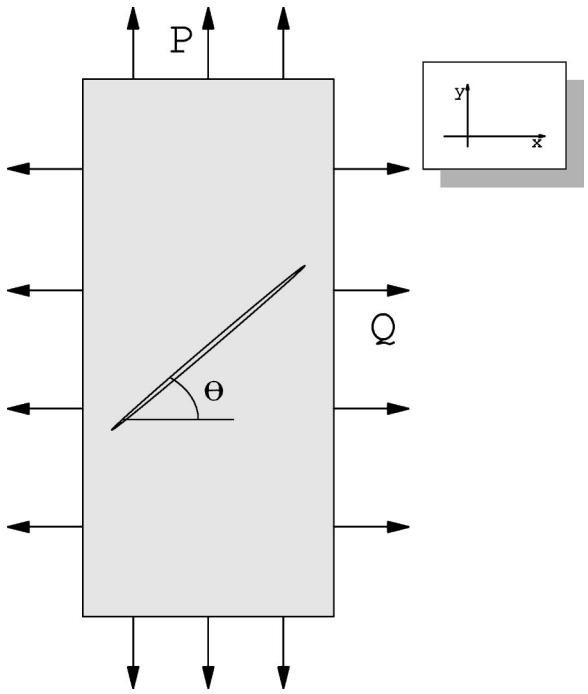

FIG. 1. The loaded sample with a crack, according to Griffith [26].

$$\frac{\partial f}{\partial t}+\overline{\mathbf{v}}\cdot\boldsymbol{\nabla} f+\boldsymbol{\nabla}_l\cdot(f\mathbf{v}_l)=0. \tag{36}$$

In the following we use spherical coordinates. In spherical coordinates the mesoscopic velocity $\mathbf{v}_l$ is decomposed into the length change velocity v_c and the orientation change velocity ω, which is zero in our model:

$$\mathbf{v}_l(\mathbf{l},\mathbf{x},t)=(v_c(\mathbf{l},\mathbf{x},t),\omega(\mathbf{l},\mathbf{x},t)). \tag{37}$$

From the model of Griffith [26] it follows that the crack length change velocity is

$$v_c=-\frac{2}{m}l^{3/2}\dot{R}, \tag{38}$$

where R is the stress at the location of the crack, and m is a material dependent constant. In the following we consider the case where slowly changing external loads P and Q are applied to the sample as shown in Fig. 1.

The R used by Griffith has been given by Inglis [27] in terms of the parameters α_0, β, and θ, where θ is the angle between the crack orientation and the y axis, and α_0 is the ratio of diameter to thickness of the crack (the ratio of the large to small axis of the ellipse describing the crack in the model of Griffith). α_0 is very large and is assumed to be constant in time according to our preliminary assumptions (because otherwise the crack length and orientation would not be the only crack variables). β is the parameter of the ellipse. According to the previous assumption ($\alpha_0\gg 1$) the stress is maximal for $\beta=\pi$, i.e., on the tip of the crack (the real maximum is very close to that, at least for Griffith cracks). Therefore we will here set $\beta=\pi$. θ is constant in time because the crack cannot change its orientation. With these assumptions we obtain from the paper by Griffith [26] the following expression for the stress change rate:

$$\begin{aligned}\dot{R}&=c_1(\dot{P}+\dot{Q})+c_2(\dot{P}-\dot{Q}),\\ c_1&=\frac{(e^{2\alpha_0}-1)\cos(2\Theta)}{\cosh(2\alpha_0)-1},\\ c_2&=\frac{\sinh(2\alpha_0)}{\cosh(2\alpha_0)-\cos(2\Theta)}.\end{aligned} \tag{39}$$

The coefficients c_1 and c_2 depend on crack orientation, but in our model not on position and time. If we average over different crack orientations, the result will depend on the order parameters introduced above.

Now the mesoscopic velocity $\mathbf{v}_l$ derived from the expressions (39) and (38) is introduced into the differential equation for the distribution f. In spherical coordinates we have

$$\begin{aligned}\boldsymbol{\nabla}_l\cdot(\mathbf{v}_l f_l)&=\frac{1}{l^2}\frac{\partial}{\partial l}(l^2 v_c f)\\ &=\frac{1}{l^2}\frac{\partial}{\partial l}\left(-\frac{2}{m}l^{7/2}[c_1(\dot{P}+\dot{Q})+c_2(\dot{P}-\dot{Q})]f\right),\end{aligned} \tag{40}$$

which results in the equation for the distribution function,

$$\frac{df}{dt}=-\boldsymbol{\nabla}_l\cdot(\mathbf{v}_l f)=\frac{1}{l^2}\frac{\partial}{\partial l}\left(\frac{2}{m}l^{7/2}[c_1(\dot{P}+\dot{Q})+c_2(\dot{P}-\dot{Q})]f\right). \tag{41}$$

Separation of the variables gives the solution of the differential equation. Moreover, we can go further, introducing the moments of the distribution function as macroscopic variables. From Eq. (41) we can derive evolution equations for particular moments also.

In the case of Griffith cracks we can introduce the orientational order parameters and the length order parameters as macroscopic quantities describing the mesoscopic distribution. However, there are several other possibilities. The question arises which macroscopic parameter is relevant for the mechanical properties of the material. Here we mention an example of a macroscopic parameter that is different from the moments. In a simple variation of the one dimensional ''loose bundle parallel bar'' model of Krajcinovic [1], the material is assumed to consist of elastic parallel bars of fixed diameter l_0. When the projection of the crack length perpendicular to the bar axis is greater than l_0, the bar is broken and does not support stresses any more. The damage parameter D is introduced as the ratio of broken bars to the whole number of bars. Translating this definition to the mesoscopic theory with Griffith cracks we can define

$$D(\mathbf{x},t)=\int_0^{l_0} f(l,\mathbf{x},t)l^2\,dl \tag{42}$$

as a new macroscopic parameter. The mesoscopic theory provides tools to deal with this damage variable also (dynamics, relation to the moment series expansion, etc.).

VIII. DISCUSSION

In this paper we investigated the applicability of the mesoscopic concept to microcracks. The physical conditions show that if we consider planar microcracks that are fixed in the surrounding medium (no diffusion, unlike dislocations) then the formalism and results developed for liquid crystals are applicable and can give some fundamental information on the possible macroscopic internal variables and also on their dynamics.

For example, according to the present investigation the moment series expansion of the orientational distribution function does not close the long discussion on the nature of the tensorial order of the internal variables in continuum damage mechanics. First of all the introduction of an orientational distribution function is only a convenient simplification of the situation and there can be cases when the length and the orientation of the cracks are statistically dependent. On the other hand the dynamics of the microcrack distribution depends on the mesoscopic space.

Sometimes a vectorial representation is simpler and fits better than a tensorial one (uniaxial case). This can be interpreted as a special case of uniaxiality in the fabric tensor description. The situation is best seen from the point of view of liquid crystal theories, where both kinds of description are present. Similar symmetry requirements as in the case of microcracks (head-tail symmetry) result in only even order terms in the alignment tensor series expansion, but the vectorial director theory of Ericksen-Leslie-Parodi-Verhás is well usable (and somewhat simpler) in many systems.

In continuum damage mechanics we can find examples of very different damage descriptors (scalars [28,29]; vectors [30–33]; second order tensors [1]; higher order tensors [34]). From a mesoscopic point of view the relation between the macroscopic theories with internal variables of different tensorial order is clear [13,14,16]. Furthermore, the mesoscopic theory gave a particular form of the possible dynamic equations on both the mesoscopic and the macroscopic level. Without calculating a particular source term we can see that it is a first order equation in the time and space derivatives. Using further specific assumptions about the dynamics of the extension of single microcracks, one can get a closed system of equations for the dynamics of the moments of the microcrack distribution and for the distribution function itself.

It is important to see that our results do not correspond to some other microstructural continuum theories [19,20,35], where a second order equation is supposed for the dynamics of the microstructure. Let us give a closer look at this proposition. According to the suggestion of Capriz [19] we include a general kinetic energy term in the energy balance and after some calculations based on the principle of material frame indifference we get for the micromomentum balance [20]

$$\rho\left[\left(\frac{\partial\dot{\kappa}}{\partial\dot{\nu}}\right)-\frac{\partial\kappa}{\partial\nu}\right]-\rho\beta+\chi=0 \tag{43}$$

where ν is a parameter of the microstructure (e.g., microcrack length); β and χ can be interpreted as ''microforces'' and ''microstresses'' and they must be given constitutively. The first term contains $\kappa(\nu,\dot{\nu})$, the ''micro kinetic energy.'' It is easy to prove that this term cannot result in a first order equation for ν.

On the other hand, we can make some remarks about the statistical approaches, too. The mesoscopic theory in some sense supports the validity of the mean field description in the case of simple crack orientation distributions when the first terms of the momentum series expansion can represent the length distribution functions. For example, this can be expected when uniaxial loading conditions are applied to an initially undamaged material, as is expected in lattice models where mean field scaling is observed [8]. However, the reason for long standing metastable states should be explained on the phenomenological level also.

ACKNOWLEDGMENTS

We thank the DAAD for sponsoring the cooperation between the Departments of Physics at the Technical Universities of Berlin and Budapest. Financial support by the VISHAY Company, 95100 Selb, Germany, is gratefully acknowledged. This research was also supported by FKFP Grant No. 0287/1997, OTKA Grant No. F02262, and by CNRT at the Energy Department of ENEA, Casaccia, Italy.

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