Unattainability of Carnot efficiency in the Brownian heat engine

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We discuss the reversibility of the Brownian heat engine. We perform an asymptotic analysis of the Kramers equation on a Büttiker-Landauer system and show quantitatively that Carnot efficiency is unattainable even in the fully overdamped limit. The unattainability is attributed to inevitable irreversible heat flow over the temperature boundary.

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How efficiently can a Brownian heat engine work? This question is important not only for the construction of a theory of molecular motors [1] but also for the foundations of nonequilibrium statistical physics. Like the Carnot cycle, the Brownian heat engine can extract work from the difference of temperature between heat baths, where the Brownian working material operates as a transducer of thermal energy into mechanical work. The features of this engine are: (1) it operates autonomously, and (2) it is driven by a *finite* difference of temperature between heat baths which both contact the working material simultaneously. Thus, this engine works because the system is out of equilibrium. Feynman *et al.* [2] devised what is called Feynman's ratchet, which can rectify thermal fluctuations to produce work using the difference of temperature between two thermal baths. Büttiker [3] and Landauer [4] proposed a simpler type of Brownian motor and pointed out that one could extract work even using a simple heat engine where a Brownian particle in a periodic potential is subject to heat baths of spatially periodic temperatures [5].

One crucial point in Brownian engines is the efficiency [2,6–17]. Feynman *et al.* claimed that their thermal ratchet can operate reversibly, resulting in Carnot efficiency. Recently, however, some authors have claimed that this is incorrect, while some have supported it. Parrondo and Español suggested that Feynman's ratchet should not work reversibly since the engine is simultaneously in contact with heat baths at different temperatures [6]. Sekimoto devised the so-called stochastic energetics and applied it to Feynman's ratchet [7]. He showed numerically that the efficiency is much less than that of Carnot. Hondou and Takagi showed that reversible operation of Feynman's ratchet is impossible by using *reductio ad absurdum* [10]. Magnasco and Stolovitzky studied how the engine generates motion by a detailed analysis of its phase space [11]. On the other hand, Sakaguchi suggested that Feynman's ratchet can operate reversibly by using a ''stochastic boundary condition'' [8]. A similar result is also found in Ref. [15] (not for Feynman's ratchet but for the Büttiker-Landauer system), which we will discuss in detail later. These studies remind us that there is difficulty concerning the energetic description of Brownian systems, because a naive application of conventional energetics formulated in a thermodynamic and/or equilibrium system to a Brownian system may lead to incorrect results.

The operation of Brownian engines is done by the engines themselves and the engines are, therefore, *out of* equilibrium. To clarify the nonequilibrium nature of Brownian heat engines and to discover how we should apply energetics to them, it is important to make a quantitative analysis of the efficiency without adopting oversimplifications for the analysis that lose the function of a heat engine. Because Feynman's ratchet is somewhat complex to make a rigorous analysis, it seems more suitable to discuss the Büttiker-Landauer [3,4] system, which is the simplest system of Brownian motors. Recently, Matsuo and Sasa analyzed the energetics of the Büttiker-Landauer system by a renormalization method [15]. They claimed that the system approaches Carnot efficiency in the overdamping limit during a quasistatic process [18]. Their analysis was based on a rigorous calculation starting from Kramers equation, and the result is clear except for one point: They assumed that the momentum degree of freedom is always in equilibrium with the heat bath because the system is overdamped [19]. This assumption is not easy for us to accept because the system is singular at the transition point [20] where the temperature of the heat bath changes suddenly. We conjectured that the essence of the mechanism of the Brownian heat engine is concentrated at these singular points and that the nature of these nonequilibrium engines will emerge by analysis of them. Thus we will discuss the energetics of the Bütikker-Landauer system paying attention to the transition points. The result will also give us insight about how we should apply ''stochastic energetics'' [7] to overdamped systems with space-dependent temperature.

Let us consider the one-dimensional Brownian system that Bütikker and Landauer discussed, where working particles operate due to the broken uniformity of the temperature of the heat baths [3,4]. While Bütikker and Landauer started their discussion from the overdamprd equation of the system, we start from the more basic standpoint of the underdamped description, from which the overdamped equa-

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tion is obtained by eliminating the momentum variable. The probability density in phase space $\rho(p,q)$ obeys the Kramers equation [21]

$$\frac{\partial\rho(p,q)}{\partial t}=-\left(\frac{\partial J_q}{\partial q}+\frac{\partial J_p}{\partial p}\right)$$
$$=-K(q)\frac{\partial\rho(p,q)}{\partial p}-\frac{p}{m}\frac{\partial\rho(p,q)}{\partial q}$$
$$+\frac{\gamma}{m}\frac{\partial}{\partial p}\left[p\rho(p,q)+mk_BT(q)\frac{\partial\rho(p,q)}{\partial p}\right], \quad (1)$$

where $K(q)=-\partial U/\partial q$; γ, m, J_q, and J_p are the friction constant, the mass of the particle, and the probability currents in space and in momentum, respectively [22]. The potential $U(q)$ satisfies $U(q)=U_r(q)+gq$, where $U_r(q+L)=U_r(q)$, g (>0) is the gradient of the global slope (the load), and L is the period. The temperature has the same spatial period as the potential, $T(q+L)=T(q)$. In this Büttiker-Landauer system, there are two heat baths with temperatures T_h (for the hot bath) and T_c (for the cold bath). Thus, there are two transition points in a spatial period, where the thermal bath affecting the particle changes. Here we restrict ourselves to the case that $T_c/T_h=O(1)$ for simplicity. The system is known to operate as a molecular engine [3,4,8,15] because the particle can *move* against the global gradient of the potential. The globally unidirectional motion is attributed to the difference between the temperatures of the baths, since the hot bath can activate the working particle more than the cold bath. Suppose that two working particle *climb* the potential, where one is in the hot bath and the other is in the cold bath. Then the working particle in the hot bath reaches the top of the potential hill more frequently than that in the cold bath, leading to global motion in the system. Thus, one can store work in proportion to the probability current. To make an energetic analysis, we consider a ''replica'' particle, of which the energy is $E=p^2/2m+U(q)$. Here, the ensemble average over the replicas corresponds to the thermodynamic limit [7].

It has been shown that this engine can have Carnot efficiency if the irreversible heat transfer at the transition points is physically negligible. Any Brownian motor is irreversible when it operates with finite probability current. Thus, operation with Carnot efficiency, if possible, must be in the ''stalled state'' [15], where the probability current in space disappears, $J_q(q)=0$ (in an overdamped description) or $\int dpJ_q(p,q)=0$ (in an underdamped description). Quasistatic operation requires this stalled state. Therefore, we evaluate whether and how heat flows irreversibly at the transition point by obtaining the stationary solution in Kramers equation in the stalled state [Eq. (1)]. For this purpose, we will restrict ourselves to the special region $q\in[-l_h,l_c]$ around the transition point, $q=0$, where l_h (l_c) is the width between the transition point and a point in the hot (cold) bath that satisfies the following inequality:

$$l_{th}\ll l_x\ll L_x \quad (x=h \text{ or } c), \quad (2)$$

where L_h (L_c) is the width of the hot (cold) bath ($L_h+L_c=L$), and l_{th} is the characteristic length scale of the transition region in which the probability density is different from that of thermal equilibrium. The length l_{th} is the product of the thermal velocity $v_{th}(\sim\sqrt{k_BT/m})$ and the velocity relaxation time $\tau(=m/\gamma)$: $l_{th}\sim v_{th}\tau$. The choice of l_x does not alter the following result as long as the inequality Eq. (2) is satisfied. Although we will discuss only one transition region, the asymptotic behavior does not differ in the other transition region. Hereafter we apply a normalization of the probability density $\rho(p,q)$ as [23]

$$\int_{-\infty}^{\infty}dp\int_{-l_h}^{l_c}dq\rho(p,q)=1. \quad (3)$$

Now, we will formulate the irreversible heat transfer from a heat bath to the working particle. The right-hand side of Kramers equation [Eq. (1)] has two parts. The first and the second terms are a Liouville operator on the probability density $\rho(p,q)$ and thus preserve the energy. The last term is what describes the energy transfer between the heat bath and the particle, for which the probability current in momentum space is written $J_p^{irr}=-(\gamma/m)[p\rho(p,q)+mk_BT(q)\partial\rho(p,q)/\partial p]$. Because the probability current disappears, $J_p^{irr}=0$, for the probability density at equilibrium $\rho(p,q)\propto\exp\{-[p^2/2m+U(q)]/k_BT\}$, the energy flow through J_p^{irr} can be sufficiently described only where $q\in[-l_h,l_c]$. The average heat transfer from the hot bath to the particle per unit time, $\langle dQ_h/dt\rangle$, is

$$\left\langle\frac{dQ_h}{dt}\right\rangle\sim\int_{-\infty}^{\infty}dp\int_{-l_h}^{0}dq\frac{\partial E}{\partial p}J_p^{irr}$$
$$=-\int_{-\infty}^{\infty}dp\int_{-l_h}^{0}dq\frac{p}{m}\frac{\gamma}{m}\left[p\rho(p,q)+mk_BT(q)\frac{\partial\rho(p,q)}{\partial p}\right]. \quad (4)$$

By integration by parts through momentum space p and the property that $\rho(p,q)$ exponentially decreases to zero as $p\to\pm\infty$, we obtain

$$\left\langle\frac{dQ_h}{dt}\right\rangle=-2\frac{\gamma}{m}\int_{-\infty}^{\infty}dp\int_{-l_h}^{0}dq\left(\frac{p^2}{2m}-\frac{k_BT(q)}{2}\right)\rho(p,q)$$
$$\equiv-2\frac{\gamma}{m}\left\langle\frac{p^2}{2m}-\frac{k_BT_h}{2}\right\rangle_h. \quad (5)$$

This is the formula for the heat transfer from the hot bath to the particle [24]. When the system is in equilibrium with the heat bath, the heat transfer $\langle dQ_h/dt\rangle$ disappears, because the theory of equipartition requires $\langle p^2/2m\rangle=k_BT/2$. This also shows that the energy exchange between the replica particle and the thermal bath is dominant only near the thermal transition point $q=0$, where the average kinetic energy $p^2/2m$ deviates from $k_BT/2$.

We discuss here how the energy flows around the transition point. As we are analyzing the stalled state, the probability density $\rho(p,q)$ is stationary. Thus, the energy density $\rho_E(q)=\int_{-\infty}^{\infty}dp[p^2/2m+U(q)]\rho(p,q)$ is stationary. Because there is no work in the stalled state, the conservation of energy requires that $d\langle Q_h+Q_c\rangle/dt=0$. This shows that the same quantity of the heat that flows from the hot bath to the particle also flows from the particle to the cold bath: $\langle dQ_h/dt\rangle = -\langle dQ_c/dt\rangle$. It should also be noted that, in the stalled state, the efficiency vanishes except when $\langle dQ_h/dt\rangle = -\langle dQ_c/dt\rangle = 0$, because work, the numerator of the efficiency, is absent here. Quasistatic operation is reversible only if Eq. (5) vanishes. Therefore the quantity $\langle dQ_h/dt\rangle$ sufficiently characterizes the operation in the stalled state and thus we will analyze it in detail. Note that the following equality is simultaneously derived [25]:

$$\left\langle \frac{dQ_h}{dt} \right\rangle = \int_{-\infty}^{\infty} dp \, \frac{p^2}{2m} \, \frac{p}{m} \rho(p,q) \bigg|_{q=0}. \tag{6}$$

This formula confirms that the irreversible heat transfer is carried microscopically as the kinetic energy of the particle at a transition point.

It is known, for example, from the kinetic theory of gases [26], that there is finite heat transfer I in the system where a Brownian particle of finite mass and friction is crossing over two regions with different temperatures, even if the two thermal baths have no direct contact. This implies that $I \equiv \langle dQ_h/dt\rangle > 0$. The authors of Ref. [15] assumed that the heat transfer should disappear in the overdamped limit, $m/\gamma \to 0$. However, their assumption is not evident *a priori*. To reveal the validity of the assumption we have to perform an appropriate energetic analysis on the Kramers equation, which includes the degree of momentum p, instead of on the overdamped Fokker-Planck equation, which does not.

Hereafter, we will consider the asymptotic behavior of the heat transfer I in the limit of the overdamped process ($\gamma \to +\infty$ and/or $m \to 0$). To find the asymptotic behavior, it is convenient to use the reference heat transfer I^* of unit mass and friction in an arbitrary set of units: $I^* \equiv I(m=1,\gamma=1)$ [27]. By Eq. (5), the reference heat transfer I^* reads:

$$\begin{aligned} I^* &= -2\left\langle \frac{p^2}{2} - \frac{k_B T_h}{2} \right\rangle_h \\ &= -2\int_{-\infty}^{\infty} dp \int_{-l_h}^{0} dq \left(\frac{p^2}{2} - \frac{k_B T_h}{2} \right) \rho^*(p,q), \end{aligned} \tag{7}$$

where $\rho^*(p,q)$ is the probability density in the reference state $m=1$ and $\gamma=1$. We call the probability density ρ and the heat transfer of arbitrary mass and friction in the units the generic probability density and the generic heat transfer. Note that the following result is not altered if we have a different reference state. The choice of the values $m=1$ and $\gamma=1$ for the reference state is only for simplicity. In the reference state, the characteristic length of the transition region l_{th}^*, where the probability density in momentum p is out of equilibrium is $l_{th}^* = v_{th}^* \tau^* = \sqrt{k_B T}$.

To evaluate the generic heat transfer [Eq. (4)] in terms of the reference heat transfer [Eq. (7)], we will find the relation between the generic probability density with arbitrary mass and friction $\rho(p,q)$ and the reference one $\rho^*(p,q)$. The potential term $K\partial\rho/\partial p$ of Kramers equation, Eq. (1), is negligible when one discusses the asymptotic behavior of the overdamping [28]. With the stationary condition $\partial/\partial t = 0$, in Eq. (1), we obtain a simple equation that describes stationary flow in phase space around the boundary $q=0$:

$$\frac{p}{m} \frac{\partial \rho(p,q)}{\partial q} = \frac{\gamma}{m} \frac{\partial}{\partial p} \left[p\rho(p,q) + m k_B T(q) \frac{\partial \rho(p,q)}{\partial p} \right]. \tag{8}$$

We find here that this equation has a scaling property in mass and friction: The generic probability density $\rho(p,q)$ is expressed using the probability density of the reference state $\rho^*(p,q)$,

$$\rho(p,q) = c\rho^*\left(\frac{p}{\sqrt{m}}, \frac{\gamma}{\sqrt{m}} q \right), \tag{9}$$

where the constant factor c should be determined by normalization [Eq. (3)] [29].

As q departs from the transition point $q=0$ farther than the characteristic length l_{th}, the probability density approaches that of equilibrium, where $\rho_h(p,q) = C_h \exp\{-p^2/2mk_B T_h\}$ (for $q \ll -l_{th}$), and $\rho_c(p,q) = C_c \exp\{-p^2/2mk_B T_c\}$ (for $q \gg l_{th}$). The coefficients C_h and C_c are then required to satisfy the condition of continuity of the probability current. Thus we have

$$C_h T_h^{3/2} = C_c T_c^{3/2}, \tag{10}$$

which is consistent with the condition derived for the overdamped limit [4]. The remaining condition that determines C_x is the normalization. Note that normalization of the probability density ρ is satisfactorily carried out even if the contribution from the transition region is neglected because the characteristic scale of the transition region l_{th} is much smaller than the width l_x: $l_{th}/l_x \ll 1$ ($x=h$ or c) [Eq. (2)]. Then C_h and C_c are determined as

$$\begin{aligned} C_h &= \frac{1}{\sqrt{2\pi m k_B T_h}} \, \frac{T_c}{T_c l_h + T_h l_c}, \\ C_c &= \frac{1}{\sqrt{2\pi m k_B T_c}} \, \frac{T_h}{T_c l_h + T_h l_c}. \end{aligned} \tag{11}$$

With these solutions and Eq. (9), we obtain the relation between the two normalized probability densities ρ and ρ^* [29]:

$$\rho(p,q) = \frac{1}{\sqrt{m}} \rho^*\left(\frac{p}{\sqrt{m}}, \frac{\gamma}{\sqrt{m}} q \right). \tag{12}$$

Note that this equation is valid even within the transition region.

We can now express the heat transfer I in terms of the reference heat transfer I^*. We rewrite I as

$$I = -2\frac{\gamma}{m} \int_{-\infty}^{\infty} dp \int_{-l_h}^{0} dq \left(\frac{p^2}{2m} - \frac{k_B T_h}{2} \right) \rho(p,q). \tag{13}$$

By a change of variables such that $p' = p/\sqrt{m}$, $q' = (\gamma/\sqrt{m})q$ [29], we obtain

$$I=-2\int_{-\infty}^{\infty}dp\int_{-\gamma l_h/\sqrt{m}}^{0}dq\left(\frac{p^2}{2}-\frac{k_BT_h}{2}\right)\rho\left(\sqrt{m}p,\frac{\sqrt{m}q}{\gamma}\right). \tag{14}$$

This yields, using Eq. (12),

$$I=-2\int_{-\infty}^{\infty}dp\int_{-\gamma l_h/\sqrt{m}}^{0}dq\left(\frac{p^2}{2}-\frac{k_BT_h}{2}\right)\frac{1}{\sqrt{m}}\rho^*(p,q). \tag{15}$$

This integrand is dominant only near the transition point $q=0$, with characteristic length l_{th}^*. As we are analyzing the asymptotic behavior such that $m\to 0$ and/or $\gamma\to\infty$, the inequality $(l_{th}^*\ll)l_h\ll\gamma l_h/\sqrt{m}$ is satisfied. Because the contribution from the interval $q\in[-\gamma l_h/\sqrt{m},-l_h]$ to the integral is negligible in Eq. (15) compared with that from $q\in[-l_h,0]$, the interval of this integral may adequately be replaced by $q\in[-l_h,0]$. Using Eq. (7), we obtain one of the main results of our paper [30]:

$$I\sim-\frac{2}{\sqrt{m}}\int_{-\infty}^{\infty}dp\int_{-l_h}^{0}dq\left(\frac{p^2}{2}-\frac{k_BT_h}{2}\right)\rho^*(p,q)=\frac{1}{\sqrt{m}}I^*. \tag{16}$$

Because the characteristic length of the transition region vanishes in the overdamped limit, the scaling property is exact asymptotically.

From this result, we learn that the irreversible heat transfer at the transition point does not decrease when one takes the overdamped limit, which is in contrast to the claim in Ref. [15]. One way to take this limit is to increase the friction constant γ; then the heat transfer does not decrease, because the heat transfer I does not depend on γ. The other way is to decrease the mass m: then the heat transfer does not decrease either, it increases with the power of $1/\sqrt{m}$. The result justifies the intuitive estimation by Derényi and Astumian [19]. The heat flow is the result of broken symmetry of the probability density in momentum at the transition point, because the heat transfer disappears if the probability density is symmetric in phase space, as shown by Eq. (6). Since an overdamped equation has no degree of freedom to describe the irreversible flow caused by the discontinuity of the temperature, the previous literature found Carnot efficiency [15].

Up to now, we have discussed how heat transfer between the two heat baths behaves in the overdamping process. We found that the irreversible heat transfer does not decrease in the process. One finds, however, that the possible work out of the system may also vary according to the overdamped limit, because the probability current may vary due to change of the parameters γ and m. Thus, it is not yet obvious whether nonvanishing heat transfer I itself reveals that the system cannot attain Carnot efficiency in any condition including the nonstalled state. Thus, in addition to the irreversible heat transfer discussed above, we will estimate the work and work-induced heat transfer in an overdamped process.

We will return to the original Kramers equation [Eq. (1)] for a Büttiker-Landauer system. We have analyzed this equation retaining both degrees of freedom p and q. However, we do not have to consider a momentum degree of freedom when we discuss work out of the Brownian system, because the work is a function only of the displacement of position. Thus, we start the evaluation of the work using the overdamped Fokker-Planck equation for the probability density $P(q)$ of the system:

$$\frac{\partial P(q)}{\partial t}=-\frac{\partial}{\partial q}J(q)=\frac{1}{\gamma}\frac{\partial}{\partial q}\left[\frac{\partial U(q)}{\partial q}+\frac{\partial(k_BT(q))}{\partial q}\right]P(q), \tag{17}$$

where a periodic boundary condition is applied: $P(0)=P(L)$ and $(dP/dq|_{q=0})=(dP/dq)|_{q=L}$. Explicit mass dependence on the displacement of the system disappears in the overdamped limit. In stationary state, $\partial P(q)/\partial t=0$, the probability current $J(q)$ is independent of q. The probability current J reads

$$J=-\frac{1}{\gamma}\left[\frac{\partial U(q)}{\partial q}+\frac{\partial[k_BT(q)]}{\partial q}\right]P(q). \tag{18}$$

The equation for $P(q)$ reads

$$\frac{\partial}{\partial q}\left[\frac{\partial U(q)}{\partial q}+\frac{\partial[k_BT(q)]}{\partial q}\right]P(q)=0. \tag{19}$$

This equation shows that change of the friction constant γ does not alter the probability density $P(q)$. Thus, with Eq. (18), the probability current J scales as $J\propto\gamma^{-1}$. For a fixed load potential, the work per unit time dW/dt is proportional to the probability current. Thus the sole operation $\gamma\to\infty$ does not lead the system to Carnot efficiency, because the induced work (proportional to J) decreases while the irreversible heat [Eq. (16)] does not decrease.

To find the mass dependence of the work, we consider a working particle obeying Stokes' law with radius r_B, where the mass and the friction are specified by one parameter r_B: $\gamma\propto r_B$ and $m\propto r_B^3$. Thus we have $dW/dt\propto J\propto\gamma^{-1}\propto r_B^{-1}$. The irreversible heat transfer dQ_{irr}/dt that is independent of work is just the heat transfer I [Eq. (16)]. Thus we have $dQ_{irr}/dt\propto m^{-1/2}\propto r_B^{-3/2}$. The work-induced heat transfer Q_W that is proportional to the work W is proportional to the probability current J [15]. Thus we obtain $dQ_W/dt\propto J\propto r_B^{-1}$. The three components determine the efficiency. We have

$$\eta=\frac{dW/dt}{dQ_W/dt+dQ_{irr}/dt}=\frac{c_1r_B^{-1}}{c_2r_B^{-1}+c_3r_B^{-3/2}}=\frac{c_1}{c_2+c_3/\sqrt{r_B}}, \tag{20}$$

where c_1, c_2, and c_3 are constants. The result shows that the efficiency decreases monotonically to zero when one takes the overdamped limit $r_B\to 0$. This result is not altered even if one includes another transition point in the same period, because the asymptotic behavior of the two is the same.

In this paper, we have analyzed the energetics of a Brownian motor of Büttiker-Landauer type. We showed quantitatively that irreversible heat transfer does not disappear even if one takes the overdamped limit ($\gamma\to+\infty$ and/or $m\to 0$). This result is in contrast to the claims in Refs. [8,15]. The mass dependence of the irreversible heat is consistent with the intuitive estimation in Ref. [19]. We further analyzed the effect of nonvanishing irreversible heat transfer on the efficiency and showed that, even in the fully overdamped limit, Carnot efficiency is unattainable for a particle obeying Stokes' law. This shows that the maximum efficiency of the Brownian motor is not attained in the stalled state. The result revealed that the Brownian heat engine is qualitatively different from heat engines for which the most efficient operation is quasistatic: A quasistatic process is the worst condition for the Brownian heat engine to work, while it is the best for the Carnot cycle.

The location of the irreversible heat transfer is the transition region characterized by the thermal length l_{th}. It is certain that the characteristic length l_{th} can disappear in the fully overdamped limit. However, the irreversible effect in the transition region cannot be eliminated. From the result we also learn how to apply energetics to overdamped systems with a space-dependent temperature: We should apply energetics *before* taking the overdamped limit. Otherwise, we might fail in proper evaluation of the irreversible heat transfer within the transition region [8,15], because energetic interaction between the heat bath and the particle is carried out by the momentum exchange between them. When the particle has smaller kinetic energy than that expected by the equipartition theorem, it receives kinetic energy from the heat bath on average. Thus, if we lose the degree of momentum as in the overdamped equation, we cannot describe this existing physical process properly.

The present system cannot have maximum efficiency in a quasistatic condition. This means that the maximum efficiency is achieved with finite probability current, which is therefore accompanied by irreversible dissipation. Thus, the next challenging question is, ''Is there any principle that determines the optimal efficiency in Brownian heat engines?''

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[1] See for a review, F. Jülicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997).

[2] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures in Physics* (Addison-Wesley, Reading, MA, 1966).

[3] M. Büttiker, Z. Phys. B: Condens. Matter **68**, 161 (1987).

[4] R. Landauer, J. Stat. Phys. **53**, 233 (1998).

[5] Energetics was not discussed quantitatively at that time, because little was known about the energetic evaluation of Brownian systems.

[6] J.M.R. Parrondo and P. Español, Am. J. Phys. **64**, 1125 (1996).

[7] K. Sekimoto, J. Phys. Soc. Jpn. **66**, 1234 (1997).

[8] H. Sakaguchi, J. Phys. Soc. Jpn. **67**, 709 (1998).

[9] H. Kamegawa, T. Hondou, and F. Takagi, Phys. Rev. Lett. **80**, 5251 (1998).

[10] T. Hondou and F. Takagi, J. Phys. Soc. Jpn. **67**, 2974 (1998).

[11] M.O. Magnasco and G. Stolovitzky, J. Stat. Phys. **93**, 615 (1998).

[12] T. Shibata and S. Sasa, J. Phys. Soc. Jpn. **67**, 2666 (1998).

[13] J.M.R. Parrondo, Europhys. Lett. **43**, 248 (1998).

[14] K. Sekimoto, J. Phys. Soc. Jpn. **68**, 1448 (1999).

[15] M. Matsuo and S. Sasa, Physica A **276**, 188 (2000).

[16] A. Parmeggiani *et al.*, Phys. Rev. E **60**, 2127 (1999).

[17] F. Takagi and T. Hondou, Phys. Rev. E **60**, 4954 (1999).

[18] A similar claim and assumption are found in Ref. [8].

[19] I. Derényi and R.D. Astumian, Phys. Rev. E **59**, 6219 (1999).

[20] The term ''transition'' was introduced in Ref. [4] in the sense that the probability density in the overdamped limit changes suddenly.

[21] H.A. Kramers, Physica (Amsterdam) **7**, 284 (1940).

[22] In this paper, we discuss the case in which the friction coefficient γ is independent of the temperature T for simplicity. The friction coefficient may depend on the temperature as discussed in Ref. [4]. However, this does not alter the central result, because the asymptotic property of the overdamped process is the same for a temperature-dependent friction coefficient.

[23] The choice of the interval $q \in [-l_h, l_c]$ changes the factor of the probability density $\rho(p,q)$. However, the choice does not alter the following asymptotic estimation in an overdamped process.

[24] This formula is consistent with that derived in a different way in Ref. [6].

[25] The equation can be obtained even just *at* the transition point, because $\rho(p,q)$ is continuous.

[26] See, for example, G. M. Barrow, *Physical Chemistry*, 4th ed. (McGraw-Hill, New York, 1979).

[27] We require the reference state to hold, Eq. (2): $l_{th}^{*} \ll l_h \ll L_h$.

[28] If one takes the overdamped limit, the characteristic length of the transition region l_{th} decreases to zero. Thus, the effect of a potential force K in the transition region is negligible in comparison to the interaction between the heat bath and the system.

[29] Here, the parameters m and γ are the ratios between those in the generic state and those in the reference state. Thus they are dimensionless.

[30] The same result can also be obtained using Eq. (6) instead of Eq. (5).