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Entropy and Wigner functions

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The properties of an alternative definition of quantum entropy, based on Wigner functions, are discussed. Such a definition emerges naturally from the Wigner representation of quantum mechanics, and can easily quantify the amount of entanglement of a quantum state. It is shown that smoothing of the Wigner function induces an increase in entropy. This fact is used to derive some simple rules to construct positive-definite probability distributions which are also admissible Wigner functions.

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I. INTRODUCTION

Entropy is the central concept of thermodynamics and statistical mechanics. It was introduced by Clausius in the mid-19th century as a phenomenological variable that quantifies the intrinsic irreversibility of thermodynamic processes. It was Boltzmann who recognized the link between entropy and the lack of information about a system, defined as the number Γ of microstates which have the same macroscopic properties. The celebrated formula

$$S_B = k_B \ln \Gamma, \tag{1}$$

where k_B is the Boltzmann constant, establishes such a link in a mathematically rigorous manner (in the rest of this paper we shall use units for which $k_B = 1$: with this prescription, entropy becomes a dimensionless quantity). Boltzmann, of course, derived this formula in the context of classical statistical mechanics. In classical physics, microstates are defined as points in a continuous $2D$-dimensional phase space (D is the number of degrees of freedom of the system under consideration), and cannot be "counted" in any meaningful sense. Therefore, Boltzmann took as the number Γ of microstates the available volume in phase space Ω divided by the volume of a unit cell (unspecified at the time when Boltzmann published his work, but which will turn out to be Planck's constant, raised to the appropriate power, h^D): $\Gamma = \Omega/h^D$. In quantum mechanics, a microstate is described by a wave function, which contains all the information about the state of the system. In contrast to the classical case, now there is no ambiguity, since quantum states are discrete in principle. Hence, although the macrostate has a huge number of possible microstates consistent with it, this number, Γ, is nevertheless definite and finite.

The most general quantum system is described by a density matrix, i.e., a positive-definite, Hermitian operator, with unit trace. In terms of the density matrix ρ, the entropy can be expressed in the following way, due to Von Neumann [1]:

$$S_{\mathrm{VN}} = -\mathrm{Tr}\,\rho \ln \rho. \tag{2}$$

This is the standard definition of entropy, which generalizes Boltzmann's expression to quantum mechanics. Although unambiguously defined, however, S_{VN} can be extremely difficult to compute in practice, since one would need to diagonalize ρ in order to compute the trace of its logarithm. Von Neumann's entropy (VN) has a number of good properties, which will be detailed in the following sections. Here we note that, if $\alpha_i \geqslant 0$ are the eigenvalues of the density matrix ($\Sigma_i \alpha_i = 1$), the VN entropy becomes $S_{\mathrm{VN}} = -\Sigma_i \alpha_i \ln \alpha_i$. Therefore, $S_{\mathrm{VN}} \geqslant 0$, and the equality holds only if we have complete information, i.e., if only one of the eigenvalues is different from zero: in this case, the system is in the pure state corresponding to this eigenvalue. Another crucial property of S_{VN} is that it is conserved as ρ evolves according to the quantum Liouville equation

$$i\hbar \frac{\partial \rho}{\partial t} = H\rho - \rho H, \tag{3}$$

where H is the Hamiltonian. Indeed, the trace of any functional F of the density matrix $\mathrm{Tr}F(\rho)$ is also conserved. This fact can be used to define other entropylike quantities. Not all of these quantities are equivalent, however, and we will show in the following section that only one of them is particularly adapted to the Wigner representation of quantum mechanics.

The classical limit (CL) of the Von Neumann entropy, Eq. (2), is obtained by replacing the density matrix with the phase-space probability distribution $f(x,p)$ (for simplicity, we will consider systems with only one degree of freedom, $D = 1$), and the trace with the integral in phase space. One obtains the following expression, due to Gibbs:

$$S_{\mathrm{CL}} = -\int f \ln(fh)\, dx\, dp, \tag{4}$$

and the probability distribution is positive and normalized to unity. Note that the classical entropy is defined up to an additive constant, which means that the constant h in the argument of the logarithm in Eq. (4) can be chosen arbitrarily, although it seems reasonable to use Planck's constant $h = 2\pi\hbar$. Indeed, if f is constant inside a certain phase-space volume Ω and zero elsewhere (i.e., at thermodynamic equilibrium), then $S_{\mathrm{CL}} = \ln(\Omega/h)$, in agreement with Boltzmann's

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original definition, Eq. (1). We also stress that $S_{\rm CL}$ can take negative values, in contrast with $S_{\rm VN}$, which is always nonnegative. From the previous discussion, it is easy to conclude that $S_{\rm CL}$ will be negative when $\Omega < h$. This means that we are trying to localize a particle on a phase-space region smaller than Planck's constant, and therefore violate the uncertainty principle. For probability distributions that satisfy the uncertainty principle, the classical entropy is positive. Similarly to the quantum-mechanical case, the classical entropy is conserved for a Hamiltonian process, i.e., when the probability distribution evolves according to the classical Liouville equation. Again, the phase-space integral of any functional $F(f)$ is also conserved (indeed, f itself is conserved, since it is just transported along the classical trajectories).

In this paper, we discuss the properties of an alternative definition of quantum entropy, based on Wigner functions. Although this entropy has already been known for some time (generally expressed in terms of the density matrix), we feel that its properties are not fully appreciated. In particular, it will be shown that such a definition of entropy emerges naturally from the Wigner representation of quantum mechanics. It has therefore a privileged status compared to the many other definitions proposed in the literature, and deserves to be studied in some depth.

The Wigner representation [2] is a useful tool to express quantum mechanics in a phase-space formalism (for reviews, see [3,4]). Although it was derived by Wigner for technical purposes, this approach has recently attracted much interest, since it is well-suited to analyze the transition from classical to quantum dynamics. The Wigner representation can deal with both pure and mixed quantum states, and is completely equivalent to the more usual picture based on the density matrix. In this representation, a quantum state is described by a Wigner function (i.e. a function of the phase-space variables—see the next section), and the Wigner equation provides an evolution equation for the state which is equivalent to the quantum Liouville equation (3). It will be shown that, if one tries to define an entropy functional in the framework of Wigner's representation, only one ''reasonable'' choice is possible, and this is discussed in the next section. Subsequently, we will discuss the properties of such an entropy (Sec. III), and present some examples of its applications in Secs. IV and V.

II. QUANTUM ENTROPY

The quantum distribution function $W(x,p)$ is defined in terms of the density matrix $\rho(x,y)$ for a quantum mixed state,

$$W(x,p)=\frac{1}{2\pi\hbar}\int \rho\left(x-\frac{\lambda}{2},x+\frac{\lambda}{2}\right)\exp\left(\frac{ip\lambda}{\hbar}\right)d\lambda, \quad (5)$$

or in terms of the wave function $\psi(x)$ for a pure state,

$$W(x,p)=\frac{1}{2\pi\hbar}\int \psi\left(x-\frac{\lambda}{2}\right)\psi^*\left(x+\frac{\lambda}{2}\right)\exp\left(\frac{ip\lambda}{\hbar}\right)d\lambda. \quad (6)$$

The function $W(x,p)$ possesses many of the properties of a phase-space probability distribution: it is real, normalized to unity, and, when integrated over x or p, gives the correct marginal distribution, e.g., $\int W\,dp=\rho(x,x)=$ spatial density. Furthermore, it can be used to compute averages of any dynamical variable $A(x,p)$: $\langle A\rangle=\int WA\;dx\,dp$. Note, however, that, since some terms in $A(x,p)$ may not commute, it is necessary to establish a nonambiguous correspondence between classical variables and quantum operators (Weyl's rule) [4]. Despite these good properties, the Wigner function cannot be interpreted as a probability distribution, since it can assume negative values. The only pure state whose Wigner function is positive definite is given by the minimum uncertainty packet (i.e., a Gaussian wave function).

The evolution of $W(x,p,t)$ is governed by the Wigner equation, which replaces the classical Liouville equation:

$$\frac{\partial W}{\partial t}+\frac{p}{m}\frac{\partial W}{\partial x}=\frac{i}{2\pi\hbar^2}\int\left[\Phi\left(x-\frac{z}{2}\right)-\Phi\left(x+\frac{z}{2}\right)\right] \times\exp\left(-\frac{i}{\hbar}(p-p')z\right)W(x,p',t)dz\,dp', \quad (7)$$

where $\Phi(x)$ is the potential. The Wigner equation is equivalent to the quantum Liouville equation (3), and can describe the evolution of both pure states and mixtures. However, in the present work, we shall favor the Wigner formalism over the density matrix one, since it is easier to represent in the classical phase space, and it allows a more straightforward treatment of the semiclassical limit.

We would like to define an entropy functional in terms of Wigner functions. The classical choice, Eq. (4), obviously cannot work, since W can assume negative values. It is easy to show the existence of two simple functionals of W that are invariant under Eq. (7): the first is the total probability $\int W\;dx\,dp=1$; the second invariant is $\int W^2\;dx\,dp$, which has no obvious physical meaning. We stress that this is a property of Eq. (7), and does not depend on whether W represents a pure state, a mixture, or even a state which violates the uncertainty principle. However, the fact that the latter expression is indeed invariant suggests that we introduce the following definition of entropy:

$$S_2=1-(2\pi\hbar)^D\int W^2\;dx\,dp, \quad (8)$$

where D is the number of degrees of freedom: except where otherwise stated, we will always work with systems for which $D=1$.

The S_2 entropy can be expressed in terms of the density matrix ρ

$$S_2=1-\mathrm{Tr}\,\rho^2, \quad (9)$$

a result which follows from the fact that W is related to the Fourier transform of ρ. Equation (9) has been used in the literature as an entropylike quantity [5], and is sometimes referred to as the linear entropy. Its relevance to Wigner functions has been noticed by some authors [6], but its full implications have not, to our knowledge, been appreciated and developed. We first notice that this is the only expression of entropy having the same functional form when expressed in terms of either W or ρ (for example, $\int W^4$ is *not* simply related to Tr ρ^4). Second, and most importantly, the very structure of Wigner's equation selects the functional S_2 as a special candidate for a definition of entropy. It is therefore important to study its properties and implications.

When W is an admissible Wigner function (i.e., when it represents either a pure or a mixed quantum state), the previous entropy satisfies the relation $0 \leqslant S_2 \leqslant 1$, and $S_2 = 0$ holds for a pure state, which is a reasonable result, since pure states contain the maximum information available. Indeed, it is possible to define *quantum information* as the complement of S_2 to unity, $I = 1 - S_2$. Note that S_2 can become negative only for states that violate the uncertainty principle, as will be explained in Sec. III. We point out that $S_2 = 0$ is a necessary, but definitely not sufficient, condition for the corresponding Wigner function to represent a pure state [3]. This can be shown by finding a counterexample. Let us define the Wigner function as $W = \Sigma_{i=1}^{3} \alpha_i W_i$, where the W_i are orthogonal pure states, and $\alpha_1 = \alpha_2 = \frac{2}{3}$, $\alpha_3 = -\frac{1}{3}$. Even though the coefficients α_i sum up to unity, W does not represent an admissible Wigner function, since one of the coefficients (which represent probabilities) is negative. However, it is simple to prove that $S_2[W] = 0$. Incidentally, this example has shown the existence of phase-space functions which represent neither pure states nor mixtures. This point will be discussed in more detail in the next section.

This entropy is related to a formula proposed by Tsallis [7], which has stimulated much work in the past decade (see, for example, [8] and references therein). If $\{\alpha_i\}$ is a set of probabilities adding up to unity, Tsallis entropy is defined by

$$S_q = \frac{1 - \sum_i \alpha_i^q}{q-1}, \tag{10}$$

where q is a real, not necessarily positive, number, and the standard entropy is recovered for $q \to 1$. Tsallis entropy is a possible, and indeed useful, way to generalize the Boltzmann–Von Neumann expression, and has been employed by several authors to study the thermodynamics of strongly correlated systems, such as self-gravitating gases and inviscid fluids [8].

Equation (8) is the continuous counterpart of the discrete Tsallis entropy with $q = 2$. The continuous formula can be recovered by the following heuristic argument. Let us cover the phase space with cells of size $\Delta x \Delta p$. The discrete probabilities are then $\alpha_i = W(x_i, p_i) \Delta x \Delta p$, and the discrete entropy becomes

$$S_2 = 1 - \Delta x \Delta p \sum_i W^2(x_i, p_i) \Delta x \Delta p. \tag{11}$$

The sum in Eq. (11) gives the integral $\int W^2 \, dx \, dp$. However, we cannot let the factor $\Delta x \Delta p$ in front of the sum go to zero, since this would violate the uncertainty relation. Indeed, we obtain the correct continuous formula [Eq. (8) with $D = 1$] by taking for $\Delta x \Delta p$ the smallest value allowed by quantum mechanics, i.e., Planck's constant $h = 2\pi\hbar$.

Another way to go from the continuous to the discrete formula is to consider a Wigner function that is the sum of N orthogonal pure states $W(x,p) = \Sigma_{i=1}^{N} \alpha_i W_i(x,p)$. Of course, W represents a quantum mixture. We recall the following useful relation, valid for orthogonal pure states:

$$\int W_i W_j \, dx \, dp = \delta_{ij} / 2\pi\hbar, \tag{12}$$

where δ_{ij} is the Kronecker delta. By developing W in terms of the W_i in Eq. (8), and making use of Eq. (12), we obtain Tsallis discrete entropy $S_2 = 1 - \Sigma_{i=1}^{N} \alpha_i^2$. We stress again that the above properties are valid for the quadratic entropy S_2, but do not hold for other functionals involving higher powers of W.

It is interesting to show that a local entropy σ and an entropy flux J_S can also be defined:

$$\sigma(x,t) = \int W \, dp - 2\pi\hbar \int W^2 \, dp,$$

$$J_S(x,t) = \int \frac{p}{m} W \, dp - 2\pi\hbar \int \frac{p}{m} W^2 \, dp. \tag{13}$$

Of course, one has $S_2 = \int \sigma \, dx$. By multiplying Eq. (7) by W and integrating over momentum space, one can prove that the local entropy obeys a continuity equation:

$$\frac{\partial \sigma}{\partial t} + \frac{\partial J_S}{\partial x} = 0, \tag{14}$$

which shows that entropy can be transfered from one spatial location to another, but is globally conserved. The physical meaning of σ is easier to grasp if we express it in terms of the density matrix in the position representation. With the help of Eq. (5) one finds (we drop the time dependence)

$$\sigma(x) = \rho(x,x) - \int |\rho(x - \lambda/2, x + \lambda/2)|^2 d\lambda. \tag{15}$$

Equation (15) shows that entropy is closely related to the off-diagonal terms of the density matrix. For a pure state, $\rho(x,y) = \psi(x)\psi^\star(y)$ (ψ is the wave function), and the local entropy can be expressed in terms of the spatial density $n(x) = |\psi(x)|^2 = \rho(x,x)$,

$$\sigma(x) = n(x) - \int n\left(x - \frac{\lambda}{2}\right) n\left(x + \frac{\lambda}{2}\right) d\lambda \equiv n(x) - \iota(x), \tag{16}$$

where we have defined the *local quantum information* $\iota(x)$ so that $I = \int \iota \, dx$. It appears that $\iota(x)$ is a density autocorrelation function, which shows that, in quantum mechanics, information and spatial correlations are intimately close concepts.

III. PROPERTIES OF QUANTUM ENTROPY

The expression given in Eq. (8) has proven to be a fruitful tool to quantify some key properties of quantum systems, such as nonlocal correlations. In order to be an appropriate definition of entropy, it should nevertheless satisfy some standard properties [9], among which concavity and additivity are particularly fundamental. Some of these properties were previously studied by Tsallis [7] for the discrete case.

1. Concavity. This means that, if $W=\Sigma_{i=1}^{N}\alpha_i W_i$ (where the W_i are not necessarily pure orthogonal states), then the following inequality holds:

$$S_2[W] \geqslant \sum_{i=1}^{N} \alpha_i S_2[W_i]. \tag{17}$$

The proof is obtained by direct calculation for $N=2$, and is then easily extended to higher N by recursive arguments.

Note that we can also prove an upper bound for S_2,

$$S_2[W] \leqslant \sum_{i=1}^{N} \alpha_i^2 S_2[W_i] + 1 - \sum_{i=1}^{N} \alpha_i^2, \tag{18}$$

which holds for W_i representing both pure states or mixtures. The term $1-\Sigma_i \alpha_i^2$ represents the so-called mixing entropy. The proof of Eq. (18) relies on the following inequality [3]:

$$\int W_i W_j \, dx\, dp \geqslant 0, \tag{19}$$

which is valid for all admissible Wigner functions, pure or mixed states (see Sec. IV for a definition of admissibility). When the W_i represent pure states, then $S_2[W_i]=0$, and Eq. (18) becomes

$$S_2[W] \leqslant 1 - \sum_{i=1}^{N} \alpha_i^2. \tag{20}$$

The equality sign holds when the W_i are also orthogonal, as was shown in Sec. II.

2. Additivity. Let us consider two independent subsystems A and B. The Wigner function W describing the total system $A \cup B$ is simply given by the product of the Wigner functions W_A and W_B for the two subsystems,

$$W(x_A, p_A, x_B, p_B) = W_A(x_A, p_A) W_B(x_B, p_B). \tag{21}$$

It is easy to show that both the classical entropy, Eq. (4), and the Von Neumann entropy, Eq. (2), are additive [9], i.e., $S[W]=S[W_A]+S[W_B]$. This is a key property, since it enables one to identify the statistical entropy with the thermodynamical entropy, which is also additive.

By contrast, our definition of entropy is not additive in the usual sense. Let us first notice that, whereas the number of degrees of freedom of each subsystem is $D=1$, the total system has $D=2$. Therefore, the information is defined as $I[W_{A,B}]=h\int W_{A,B}^2$ for each subsystem and $I[W]=h^2\int W^2$ for the total system. With this in mind, it is easy to establish the following expression for the quantum information:

$$I[W] = I[W_A] I[W_B], \tag{22}$$

which shows that, since $I<1$, the information contained in the total system is smaller than the information of each subsystem, except for pure states, for which $I=1$. In terms of the entropy $S_2=1-I$, Eq. (22) becomes

$$S_2[W] = S_2[W_A] + S_2[W_B] - S_2[W_A] S_2[W_B]. \tag{23}$$

The total entropy is therefore smaller than the sum of the partial entropies, but larger than each of them. Note that when the subsystems are "almost pure" quantum states, then $S_2[W_{A,B}] \ll 1$, and the nonadditive correction to Eq. (23) becomes of higher order. In this case, approximate additivity is recovered.

It is also interesting to note that Eq. (23) is formally identical to the expression for the probability of the union of two subsets A and B, which reads

$$\text{prob}(A\cup B) = \text{prob}(A) + \text{prob}(B) - \text{prob}(A\cap B), \tag{24}$$

and $\text{prob}(A\cap B)=\text{prob}(A)\ \text{prob}(B)$ for statistically independent systems. The analogy of S_2 as probability is also consistent with the normalization $0 \leqslant S_2 \leqslant 1$.

3. Subadditivity. If the subsystems A and B are not independent, the Wigner function cannot be factored as in Eq. (21). The Wigner function of each subsystem is then defined by integrating over the other system's variables, for instance

$$W_A(x_A, p_A) = \int W(x_A, p_A, x_B, p_B)\, dx_B\, dp_B, \tag{25}$$

and similarly for W_B. For the Boltzmann–Von Neumann entropy, one can prove that $S[W] \leqslant S[W_A]+S[W_B]$, and the equality sign holds when the two subsystems are independent [9]. This means that the total system $A\cup B$ contains *more* information than the sum of its parts, which is natural, since the two subsystems are correlated. However, no such relation can be proven for S_2: this entropy is therefore not subadditive. Note that this fact is consistent with the analogy of S_2 as probability given by Eq. (24). Indeed, when the subsets A and B are not independent, the probability of their intersection $\text{prob}(A\cap B)$ can be either smaller or larger than the product $\text{prob}(A)\ \text{prob}(B)$, corresponding to either negative or positive correlation.

4. Microcanonical ensemble. We want to extremize the entropy S_2 with the constraint $\int W\, dx\, dp = 1$. Using Lagrange multipliers, it is easy to show that the entropy is maximum when $W=\text{const}=\Omega^{-1}$ within a phase-space region of volume (area) equal to Ω, and $W=0$ elsewhere. In this case the entropy is

$$S_2 = 1 - \frac{h}{\Omega}, \quad (h = 2\pi\hbar). \tag{26}$$

This is the analog of Boltzmann's formula, Eq. (1), when the appropriate additive constant is used, i.e., $S_B=\ln(\Omega/h)$. For both expressions, $S=0$ when $\Omega=h$ (minimum uncertainty), and the entropy becomes negative when $\Omega<h$, i.e., when the uncertainty relation is violated. In the limit $\Omega\to\infty$, S_2 is bounded, and tends to unity (least information). With this notation, information $I=1-S_2$ is just the inverse of the number of available microstates Ω/h.

5. Canonical ensemble. We now extremize S_2 with the constraints $\int W\, dx\, dp=1$ and $\int WE\, dx\, dp = U$, where $E(x,p)=p^2/2m+\Phi(x)$, and U is the average energy. Again using Lagrange multipliers, we find the following equilibrium distribution:

$$W_{eq}(x,p)=Z^{-1}[1-\beta E(x,p)], \quad \beta E<1,$$
$$W_{eq}(x,p)=0, \quad \beta E\geqslant 1, \tag{27}$$

where β is the Lagrange multiplier corresponding to the energy constraint, and can be interpreted in the usual fashion as the inverse temperature $\beta=1/T$; Z is a normalization constant. For energies such that $\beta E\ll 1$, Eq. (27) becomes identical with the standard exponential Boltzmann factor $\exp(-\beta E)$. Since W_{eq} is a linear function of the energy, we have been forced to introduce a cutoff, otherwise W_{eq} would diverge for large values of E. Physically, this means that states with energy $E>T$ are forbidden at equilibrium. Note the difference with standard thermodynamics, where such states are highly improbable (because Boltzmann's factor decreases exponentially), but not forbidden in principle.

An interesting fact is that Eq. (27) is a stationary solution of the Wigner equation (7)—indeed, we are aware of no other stationary solution that is also a function of the energy $E(x,p)$ alone. This is easy to prove when the right-hand side of Eq. (7) is written as

$$\sum_{n=0}^{\infty} c_n \frac{\partial^{2n+1}\Phi}{\partial x^{2n+1}} \frac{\partial^{2n+1}W}{\partial p^{2n+1}},$$

where the c_n are constants. The $n=0$ term yields the classical part of Wigner's equation, whereas all other terms do not provide any contribution, since W_{eq} is quadratic in p. Moreover, since W_{eq} is a function of the energy alone, it is a stationary solution of the classical Liouville equation, so that we have finally $\partial W_{eq}/\partial t=0$. The fact that maximizing the entropy S_2 naturally yields a Wigner function which is both stationary and a function of the energy alone is in itself remarkable. At the present stage, it is premature to make any statement about the role of W_{eq}, but the subject certainly deserves further attention. For example, it would be interesting to know if, and under what constraints, W_{eq} can act as an attractor in a relaxation process.

IV. SMOOTHED WIGNER FUNCTIONS

The Wigner function cannot be interpreted as a genuine probability distribution because it almost always takes negative values. The only pure state whose Wigner function is positive is given by the minimum uncertainty Gaussian wave packet:

$$\psi(x)=(2\pi)^{-1/4}\sigma^{-1/2}\exp(-x^2/4\sigma^2), \tag{28}$$

whose Wigner function is also Gaussian,

$$G(x,p)=\frac{1}{\pi\hbar}\exp\left(-\frac{x^2}{2\sigma^2}-\frac{2p^2\sigma^2}{\hbar^2}\right). \tag{29}$$

A possible way to obtain a positive distribution is to smooth a pure Wigner function $W(x,p)$ using a kernel $K(x,p)$ which is itself a Wigner function corresponding to a pure state [10]. The smoothing operation is represented mathematically by a convolution in phase space. The smoothed Wigner function $\overline{W}(x,p)$,

$$\overline{W}(x,p)=\int W(x',p')K(x-x',p-p')dx'dp'\equiv W\circ K, \tag{30}$$

is then positive and normalized to unity, so that it can be interpreted as a probability distribution.

In the past, the most common choice of the smoothing kernel has been the minimum uncertainty Gaussian $G(x,p)$, as given in Eq. (29) [10]. The resulting smoothed Wigner function is sometimes referred to as the Husimi function. This choice is, however, quite arbitrary, and no argument has ever been proposed, to our knowledge, in order to justify its privileged status. We shall now prove that smoothing with a Gaussian kernel *does* have some special properties, and should therefore be regarded as the correct way to obtain positive smoothed Wigner functions. In particular, it will be shown that, when the smoothing is performed with a Gaussian kernel, *the result is still an admissible Wigner function*.

First of all, we need a precise definition of an admissible Wigner function. Of course, not all functions of the phase-space variables are admissible: for example, those functions which violate the uncertainty principle are clearly not admissible. Functions that can be constructed by summing orthogonal pure states, such as $W=\Sigma_i\alpha_i W_i$, are not admissible if some of the α_i are negative: this was the example analyzed in Sec. IV. Our definition of an admissible Wigner function is rather standard [4], and is based on the density-matrix formalism. According to standard quantum theory, a density matrix ρ must satisfy three properties in order to describe a quantum mixed state: (i) it must have unit trace Tr $\rho=1$; (ii) it must be Hermitian $\rho(x,y)=\rho^*(y,x)$; and (iii) its eigenvalues must be non-negative. While the first two properties are easy to verify, the third is much harder to test, since one would need to diagonalize ρ in order to compute its eigenvalues. Property (iii) can also be expressed in the following way:

$$\int \psi(x)\rho(x,y)\psi^*(y)dx\,dy\geqslant 0, \quad \forall \quad \psi, \tag{31}$$

where the inequality must hold for *all* wave functions ψ. This makes it even more apparent that property (iii) cannot be used as an operational test.

Now, the previous properties can be transposed to Wigner functions by making use of the definition, Eq. (5). In particular, we would like to know whether the *smoothed* Wigner function $\overline{W}$ is in general admissible or not. Properties (i) and (ii) simply require that $\overline{W}$ be real and normalized to unity. Property (iii) can be written in the following form [4]:

$$\int \overline{W}(x,p)F(x,p)dx\,dp\geqslant 0, \quad \forall \quad F(x,p)=\text{pure state}. \tag{32}$$

The equivalence between Eqs. (31) and (32) can be verified by noting that $\overline{W}$ and F are the Wigner transform of, respectively, ρ and ψ, as defined in Eqs. (5) and (6). It is clear that, in order to check the admissibilty of $\overline{W}(x,p)$, one should perform an infinite number of integrals involving test Wigner functions $F(x,p)$ that represent pure states. However, Eq.

(32) can be used to prove that smoothing with a Gaussian kernel yields a smoothed Wigner function which is itself admissible.

In order to do so, let us plug Eq. (30) into the left-hand side of Eq. (32). We obtain (W is the original Wigner function, K is the smoothing kernel, and F is the test function: all three represent pure states)

$$\begin{aligned}\int W(x-x',p-p')F(x,p)K(x',p')dx'dp'dx\,dp \\ =\int K(x',p')dx'dp' \\ \times\int W_1(x-x',p-p')F(x,p)dx\,dp \\ =\int K(x',p')[W_1\circ F](x',p')dx'dp',\end{aligned} \tag{33}$$

where $W_1(x,p)=W(-x,-p)$ is the Wigner function corresponding to the wave function $\psi(-x)$ [whereas W corresponds to $\psi(x)$]. The term $W_1\circ F$ is certainly a positive function, since it is the convolution product of two Wigner functions. It follows that a sufficient condition for Eq. (32) to be satisfied is that $K(x,p)$ be positive. But the only pure state Wigner function which is also positive is the Gaussian $G(x,p)$ [Eq. (29)]. This proves that, when the smoothing kernel is Gaussian, the inequality given in Eq. (32) is verified, and the smoothed Wigner function $\overline{W}(x,p)$ is therefore admissible. In this case, the density matrix $\overline{\rho}$ corresponding to $\overline{W}$ can be written as

$$\begin{aligned}\overline{\rho}(x,y)=\frac{1}{\sqrt{2\pi}\sigma}\int W(q,p)\exp\left(-\frac{(x-q)^2}{4\sigma^2}+\frac{ipx}{\hbar}\right) \\ \times\exp\left(-\frac{(y-q)^2}{4\sigma^2}-\frac{ipy}{\hbar}\right)dq\,dp.\end{aligned} \tag{34}$$

The previous result can be easily checked by computing the Wigner function $\overline{W}$ associated to $\overline{\rho}$ via Eq. (5), and realizing that it can be written as $\overline{W}=W\circ G$. Equation (34) expresses the density matrix as a continuous sum of localized states in phase space (''coherent states'' [11]). Note that the coefficients in this sum [i.e., $W(x,p)$ itself] are not necessarily positive numbers. The reason for this is that the set of coherent states is ''overcomplete,'' meaning that the representation of an arbitrary quantum state in terms of coherent states is not unique. However, thanks to the previous theorem, we know that a diagonal representation of $\overline{\rho}$ with non-negative coefficients does exist, although we are not generally able to construct it explicitly.

So far we have proven that smoothing with a Gaussian kernel yields a function $\overline{W}$ which is itself an admissible Wigner function. Nothing definite can be said when the smoothing is performed using a different kernel. However, we are able to produce a counterexample, i.e., a pure state Wigner function which, after smoothing with a non-Gaussian kernel, does not satisfy Eq. (32), and is therefore not admissible. Let us consider the wave function

$$\psi(x)=2(2/\pi)^{1/4}x\exp(-x^2), \tag{35}$$

and call $W(x,p)$ its Wigner transform. Now we smooth W using as a kernel W itself:

$$\overline{W}=W\circ W. \tag{36}$$

In order to be an admissible Wigner function, $\overline{W}$ must satisfy Eq. (32) for every test function F. Let us use as a test function once again W itself, and compute the integral in Eq. (32). We obtain (details are in the Appendix)

$$\int \overline{W}(x,p)W(x,p)dx\ dp=-\frac{1}{27\pi\hbar}<0. \tag{37}$$

This result shows that not all ways of smoothing Wigner functions are equivalent: only by smoothing with a Gaussian kernel are we certain to obtain a function that is positive and also represents an admissible quantum state (i.e., a state defined by a density matrix with real non-negative eigenvalues).

Furthermore, Eq. (33) suggests another way to construct a phase-space distribution which is both positive and admissible [satisfying Eq. (32)]. Let us take for $W(x,p)$ an *arbitrary* positive function of phase-space variables, and smooth it with a Gaussian kernel $G(x,p)$: $\overline{W}=W\circ G$. We want to prove that $\overline{W}$ is admissible. Equation (32) yields (using the fact that G is even)

$$\begin{aligned}\int W(x-x',p-p')F(x,p)G(x',p')dx'dp'dx\,dp \\ =\int W(x',p')dx'dp'\int G(x'-x,p'-p) \\ \times F(x,p)dx\,dp \\ =\int W(x',p')[G\circ F](x',p')dx'dp'>0.\end{aligned} \tag{38}$$

The result follows from the fact that the convolution product is positive, since both F and G are pure state Wigner functions, and $W>0$ because we chose it to be so. This proves that $\overline{W}(x,p)$ is an admissible Wigner function, and is also positive, since it is the convolution product of two positive functions. The density matrix corresponding to $\overline{W}$ is again $\overline{\rho}$, as given by Eq. (34). Physically, the smoothed function $\overline{W}=W\circ G$ can be interpreted as the admissible quantum state that best approximates the classical state W for a given value of $\hbar$.

To conclude this section, we restate the two main results that have been obtained here. We have shown two possible ways to construct a phase-space distribution that is both positive and an admissible quantum state. This can be performed (a) by smoothing a pure state Wigner function with a Gaussian kernel, or (b) by smoothing an arbitrary (but positive) function of phase-space variables, again with a Gaussian kernel. Therefore, the Gaussian function $G(x,p)$ given in Eq. (29) has a privileged status as a smoothing kernel. Note, however, that G is not unique, since it depends on the parameter σ.

Although such results were derived for a pure state Wigner function, they can easily be generalized to mixtures. It follows that, when smoothing several times with a Gaussian kernel, we still remain within the class of admissible Wigner functions. This class is therefore closed with respect to this particular operation.

V. ENTROPY AND SMOOTHED WIGNER FUNCTIONS

The smoothing operation has the effect of erasing some of the correlations in the phase space. We expect therefore that smoothing should increase the entropy. This is not difficult to prove. In order to do this, we need to define the double Fourier transform of a Wigner function $W(x,p)$,

$$W(k,\lambda)=\int\int W(x,p)\exp(-ikx-i\lambda p)dx\,dp. \quad (39)$$

By means of Eqs. (6) and (39), one obtains for a pure state

$$W(k,\lambda)=\int \psi\left(x-\frac{\lambda\hbar}{2}\right)\psi^*\left(x+\frac{\lambda\hbar}{2}\right)\exp(-ikx)dx. \quad (40)$$

We can then easily prove the following lemma:

$$|W(k,\lambda)|^2\leqslant\int\left|\psi\left(x-\frac{\lambda\hbar}{2}\right)\right|^2dx\int\left|\psi^*\left(x+\frac{\lambda\hbar}{2}\right)\right|^2dx=1, \quad (41)$$

where use has been made of Schwartz's inequality.

Now, let us take an arbitrary Wigner function $W(x,p)$ and smooth it with a kernel $K(x,p)$ which is a pure state: $\overline{W}(x,p)=W(x,p)\circ K(x,p)$. In Fourier space we have $\overline{W}(k,\lambda)=W(k,\lambda)K(k,\lambda)$. The quantum information $I[\overline{W}]=2\pi\hbar\int\overline{W}^2\,dx\,dp$ relative to $\overline{W}$ satisfies the inequalities

$$\begin{aligned}I[\overline{W}]&=\frac{\hbar}{2\pi}\int|\overline{W}(k,\lambda)|^2\,dk\,d\lambda\\&=\frac{\hbar}{2\pi}\int|W(k,\lambda)|^2|K(k,\lambda)|^2dk\,d\lambda\\&\leqslant\max|K(k,\lambda)|^2\frac{\hbar}{2\pi}\int|W(k,\lambda)|^2\,dk\,d\lambda\\&=\max|K(k,\lambda)|^2 2\pi\hbar\int W^2\,dx\,dp\\&\leqslant 2\pi\hbar\int W^2\,dx\,dp=I[W],\end{aligned} \quad (42)$$

where we have used the previous lemma [Eq. (41)] for K, as well as Parseval's identity in the form

$$\int W^2(x,p)dx\,dp=\frac{1}{4\pi^2}\int|W(k,\lambda)|^2\,dk\,d\lambda. \quad (43)$$

Equation (42) implies that

$$S_2[\overline{W}]\geqslant S_2[W], \quad (44)$$

i.e., the smoothing operation has increased the entropy. Note that, in order to obtain this result, the smoothing kernel need not be a Gaussian.

Now we turn to the case where the smoothing kernel is indeed Gaussian. In this case, a relatively simple expression for $I[\overline{W}]$ can be obtained. The double Fourier transform of the Gaussian defined in Eq. (29) is

$$G(k,\lambda)=\exp\left(-\frac{k^2\sigma^2}{2}-\frac{\lambda^2\hbar^2}{8\sigma^2}\right). \quad (45)$$

The Fourier transform of the Wigner function W to be smoothed is given by Eq. (40). Let us compute the information:

$$\begin{aligned}I[\overline{W}]&=2\pi\hbar\int\overline{W}^2\,dx\,dp\\&=\frac{\hbar}{2\pi}\int|W(k,\lambda)|^2|G(k,\lambda)|^2\,dk\,d\lambda.\end{aligned} \quad (46)$$

Expressing W and G by means of Eqs. (40) and (45) one obtains, after some algebra,

$$\begin{aligned}I[\overline{W}]=&\frac{1}{2\sigma\sqrt{\pi}}\int\psi\left(x-\frac{\lambda}{2}\right)\psi^*\left(x+\frac{\lambda}{2}\right)\psi\left(x'+\frac{\lambda}{2}\right)\\&\times\psi^*\left(x'-\frac{\lambda}{2}\right)\exp\left(-\frac{\lambda^2+(x-x')^2}{4\sigma^2}\right)dx\,dx'\,d\lambda.\end{aligned} \quad (47)$$

We now change the integration variables, using the following unitary transformation:

$$\begin{aligned}x&=\frac{1}{2}w+\frac{1}{2}y+\frac{1}{2}z,\\x'&=\frac{1}{2}w-\frac{1}{2}y-\frac{1}{2}z,\\\lambda&=-y+z.\end{aligned} \quad (48)$$

After some algebra, the following result is obtained:

$$I[\overline{W}]=\frac{1}{\sigma\sqrt{\pi}}\int dw\left|\int\psi(w+y)\psi(w-y)\exp\left(-\frac{y^2}{2\sigma^2}\right)dy\right|^2, \quad (49)$$

which expresses the quantum information in terms of the wave function ψ corresponding to the unsmoothed Wigner function W. Equation (49) may be usefully employed to monitor the time evolution of the entropy in a numerical simulation: $\psi(x,t)$ would then evolve according to the time-dependent Schrödinger equation.

Finally, we show that a more stringent bound than the one expressed by Eq. (44) can be obtained when the smoothing kernel is Gaussian. By using again Schwartz's inequality, we have from Eq. (49)

$$I[\overline{W}] \leqslant \frac{1}{\sigma\sqrt{\pi}} \int dw \int dy \times \exp\left(-\frac{y^2}{\sigma^2}\right) \int dy' |\psi(w+y')\psi(w-y')|^2 = \int dw \int dy' |\psi(w+y')\psi(w-y')|^2 = \tfrac{1}{2}. \quad (50)$$

In terms of the entropy, this becomes

$$S_2[\overline{W}] \geqslant \tfrac{1}{2}, \quad (51)$$

a result that is valid when smoothing a pure Wigner function with a Gaussian kernel. Note that we still have some freedom in the choice of the kernel, since the width σ of the Gaussian in Eq. (29) is still unspecified. It would be interesting, for example, to know which value of σ minimizes the entropy $S_2[\overline{W}]$, within the bounds given by Eq. (51). We have not been able to obtain a general result, but some indication can be obtained from the following example. Let us suppose that the function W to be smoothed is also a Gaussian, as in Eq. (29), but with spatial variance μ instead of σ. The smoothed Wigner function is then

$$\overline{W}(x,p) = W \circ G = \frac{1}{2\pi\Sigma_x\Sigma_p} \exp\left(-\frac{x^2}{2\Sigma_x^2} - \frac{p^2}{2\Sigma_p^2}\right), \quad (52)$$

with

$$\Sigma_x^2 = \sigma^2 + \mu^2, \quad \Sigma_p^2 = \frac{\hbar^2}{4}\left(\frac{1}{\sigma^2} + \frac{1}{\mu^2}\right).$$

The information corresponding to $\overline{W}$ is

$$I[\overline{W}] = 2\pi\hbar \int \overline{W}^2\, dx\, dp = \frac{\hbar}{2\Sigma_x\Sigma_p}. \quad (53)$$

After some algebra, one obtains the following expression:

$$I[\overline{W}] = I(z) = \frac{z}{1+z^2}, \quad (54)$$

where $z = \sigma/\mu$. The function $I(z)$ attains its maximum for $z = 1$, i.e., when $\sigma = \mu$, and the kernel has the same variance as the Wigner function to be smoothed. In this case, $S_2[\overline{W}] = \frac{1}{2}$, which represents the lower bound of Eq. (51). We could conjecture, although we do not have a formal proof, that this is the general result: the minimum entropy increase due to smoothing with a Gaussian kernel is attained when the width of the kernel is close to the width of the function to be smoothed.

Another interesting example is provided by the harmonic oscillator, whose Hamiltonian is

$$H(x,p) = \frac{p^2}{2m} + m\omega^2 \frac{x^2}{2}. \quad (55)$$

The eigenstates can be expressed in terms of Hermite polynomials $H_n(\xi)$ ($H_0 = 1,\ H_1 = 2\xi,\ H_2 = 4\xi^2 - 2, \ldots$)

$$\psi_n(x) = (2^n n!)^{-1/2} \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n(x\sqrt{m\omega/\hbar}). \quad (56)$$

The corresponding Wigner functions are [3]

$$W_n(x,p) = \frac{(-1)^n}{\pi\hbar} \exp\left(-\frac{2H}{\hbar\omega}\right) L_n\left(\frac{4H}{\hbar\omega}\right), \quad (57)$$

where $H(x,p)$ is the Hamiltonian, and the $L_n(\xi)$ are Legendre polynomials ($L_0 = 1$, $L_1 = 1 - \xi$, $L_2 = 1 - 2\xi + \xi^2/2, \ldots$). We now smooth such Wigner functions with a Gaussian kernel, and find [3]

$$\overline{W}_n(x,p) = (2\pi\hbar n!)^{-1} (H/\hbar\omega)^n \exp(-H/\hbar\omega). \quad (58)$$

Note that this relatively simple result for $\overline{W}_n$ is obtained only in the case when the square variance of the smoothing kernel [see Eq. (29)] is $\sigma^2 = \hbar/2m\omega$; in all other cases, the smoothed Wigner function is not a function of the energy only. We are now in a position to compute the information $I[\overline{W}_n] \equiv \overline{I}_n = 2\pi\hbar \int \overline{W}_n^2\, dx\, dp$. Let us first change to polar coordinates (r,θ) in the phase space,

$$\frac{p^2}{m} + m\omega^2 x^2 = \hbar\omega r^2, \quad dx\, dp = \hbar r\, dr\, d\theta. \quad (59)$$

One obtains, after integration over θ,

$$\overline{I}_n = (n!)^{-2} \int_0^\infty (r^2/2)^{2n} \exp(-r^2) r\, dr, \quad (60)$$

and finally, changing variable again $z = r^2/2$,

$$\overline{I}_n = (n!)^{-2} \int_0^\infty z^{2n} \exp(-2z) dz = \frac{(2n)!}{2^{2n+1}(n!)^2}. \quad (61)$$

We first note that $\overline{I}_0 = \frac{1}{2}$, in agreement with previous results, since the ground state of the harmonic oscillator is a Gaussian, and we are smoothing with another Gaussian of identical width. It can also be shown that $\overline{I}_n$ is a decreasing function of n. The asymptotic expansion (for $n \gg 1$) is obtained by taking the logarithm of Eq. (61) and making use of Stirling's formula,

$$\ln X! \sim X \ln X - X + \tfrac{1}{2} \ln X \ (X \gg 1),$$

which yields

$$\overline{I}_n \sim n^{-1/2}. \quad (62)$$

In terms of the entropy, we have in summary

$$S_2[\overline{W}_0] = \tfrac{1}{2},$$
$$S_2[\overline{W}_{n+1}] > S_2[\overline{W}_n], \quad (63)$$
$$\lim_{n\to\infty} S_2[\overline{W}_n] = 1.$$

The latter results mean that the entropy increase is larger when smoothing a semiclassical state. Asymptotically, the entropy of the smoothed Wigner function approaches unity. On the other hand, when smoothing a ''fully quantum'' state (i.e., a state with small quantum numbers), the entropy increase is moderate. Although these results were obtained for the special case of the harmonic oscillator, we are confident that they remain qualitatively correct for other (classically integrable) Hamiltonians.

VI. DISCUSSION

In this paper we have presented several results related to a new definition of quantum entropy, denoted S_2. Although it has already been used in the past in the framework of the density-matrix formalism, such entropy becomes particularly interesting when applied to Wigner functions. It is then possible to show that S_2 possesses a number of interesting properties—most importantly, for example, it is an invariant for the Wigner equation, which governs the evolution of Wigner functions. S_2 is related to the Tsallis entropy, although the latter is usually defined for a discrete set of probabilities, rather than for a continuous distribution. An advantage of this entropy, compared to the quantum Von Neumann entropy, is that the Wigner function is all that one needs to compute S_2. No knowledge of the density matrix is required, nor does it need to be diagonalized, as is the case for the Von Neumann entropy.

The standard properties of entropy (concavity, additivity, subadditivity) have been examined. This has revealed some interesting facts, which would require further investigations. For instance, it has been proven that S_2 (unlike ordinary entropy) behaves like a probability with respect to additivity properties, which is also consistent with the normalization $0 \leqslant S_2 \leqslant 1$. Second, the analysis of the canonical ensemble has enabled us to derive a Wigner function W_{eq} that maximizes the entropy under certain constraints. W_{eq} turns out to be both a function of the energy alone and a stationary solution of the Wigner equation. The relevance of W_{eq} is still unclear, but one could reasonably conjecture that it plays a role in some relaxation processes. Numerical experiments could clarify this point.

An ''unpleasant'' property of S_2 is that, keeping the Wigner function fixed, and letting Planck's constant go to zero, one obtains $S_2 = 1$. Thus it would seem that all classical states have unit entropy. The point is that this is not the correct procedure to obtain a classical state: indeed, if the original Wigner function is negative somewhere, we would obtain a classical state with a nonpositive probability distribution, which is of course meaningless. The correct procedure is instead to smooth the Wigner function W with an appropriate kernel, which must also be a Wigner function in order to ensure positivity. A crucial point, however, is that the smoothed Wigner function $\overline{W}$ should be itself an *admissible* quantum state, i.e., one that can be described by a density matrix with non-negative eigenvalues. We have been able to prove that, when smoothing with a minimum uncertainty Gaussian packet, the result is always admissible, although this is *not* necessarily the case when smoothing with another Wigner function. This is, to our knowledge, the first rigorous argument showing that Gaussian smoothing possesses some privileged status.

It has also been proven that smoothing increases the entropy: in particular, when smoothing a pure state with a Gaussian kernel, one has $S_2[\overline{W}] \geqslant \frac{1}{2}$. It would be interesting to know how to minimize $S_2[\overline{W}]$. This could be done by varying the width σ of the Gaussian kernel, which is still a free parameter. Although we are not able to derive a rigorous result, we have conjectured (and shown explicitly on a particular example) that $S_2[\overline{W}]$ is minimum when the width of the Gaussian kernel is close to the width of the Wigner function to be smoothed. This would not be unreasonable from the information point of view: it would mean that we can minimize the entropy increase if we have some prior knowledge of the function to be smoothed.

As a further example, we have computed the entropy of the (smoothed) stationary states of the harmonic oscillator. It was shown that S_2 increases with quantum number, therefore semiclassical states yield a larger entropy increase. Again, we have conjectured that this behavior is universal (at least for confining and classically integrable Hamiltonians), and not specific to the harmonic oscillator. We are rather confident that our conjecture is correct since the larger entropy increase for semiclassical states is mainly due to the fact that their Wigner function displays short-wavelength oscillations in the phase space, which are easily erased by the smoothing procedure.

It would be interesting to know how the previous results generalize to classically nonintegrable Hamiltonians. For the harmonic oscillator, it was found that the information of the smoothed stationary states behaves as $\overline{I_n} \sim n^{-1/2}$. Although the exponent $-\frac{1}{2}$ might be specific to the harmonic oscillator, a polynomial law may be universal for the class of integrable Hamiltonians. On the other hand, one could conjecture that, for nonintegrable Hamiltonians, the decrease is faster, perhaps exponential.

From the physical point of view, this result means that semiclassical states are highly unstable under generic perturbations (among which smoothing is a relevant example). This is reminiscent of the so-called ''predictability sieve,'' a concept introduced by Zurek and co-workers [5] in the more general framework of decoherence [12,13]. Zurek *et al.* [5] construct a model for the interaction of a quantum system with an environment at thermodynamic equilibrium, and compute the rate at which initially pure states deteriorate into mixtures by coupling with the environment. This process is known as decoherence. Subsequently, they look for the set of states which are least prone to deterioration, and find that such states are those which yield the minimum entropy increase. By estimating the entropy production, they obtain that the minimum-entropy increase is attained for the ground state of the harmonic oscillator, i.e., a minimum uncertainty Gaussian wave packet. This coincides with our results of Sec. V.

The main difference from our approach is that Zurek and co-workers [5] analyze a *dynamical* situation, while in our case the entropy-producing effect is the smoothing, which is a static process. Since both cases appear to give the same result, it is reasonable to conjecture that smoothing may represent a (simplified) model for the interaction of a quantum system with an open environment. The price to pay for our approach is that we do not have a first-principle based derivation of such an interaction. The advantage is that the model is simple enough to obtain a number of rigorous results.

These considerations may shed some new light on the semiclassical limit. We distinguish two kinds of pure quantum states: fully quantum (FQ) states W_{FQ} (with low quantum numbers), and semiclassical (SC) states W_{SC} (with large quantum numbers). For both $S_2 = 0$, i.e., they contain the same amount of information. However, after the smoothing, one obtains $S_2[\overline{W}_{FQ}]\simeq\frac{1}{2}$ and $S_2[\overline{W}_{SC}]\rightarrow 1$, i.e., the smoothed FQ state contains *more* information than the smoothed SC state. In other words, although both original states contain the same information, this is of different ''quality:'' robust for the FQ state and highly prone to deterioration for the SC state. It is not surprising, therefore, that coupling to an environment has the effect of erasing such information less easily in the former case than in the latter. These results could open new avenues for further research, particularly with computer experiments [14], to investigate the dynamical behavior of the entropy defined in this paper.

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APPENDIX

We want to prove the result of Eq. (37). Let us use the identity

$$\int f(x,p)g(x,p)dx\,dp=\frac{1}{4\pi^2}\int f(k,\lambda)g^*(k,\lambda)dk\,d\lambda,$$

with $f=\overline{W}$ and $g=W$. Since $\overline{W}(x,p)=W\circ W$, the double Fourier transform of $\overline{W}$ is $W^2(k,\lambda)$. In addition, it will turn out that $W(k,\lambda)$ is real for the case under consideration here. Therefore, by making use of the previous identity, the left-hand side of Eq. (37) becomes

$$\int \overline{W}(x,p)W(x,p)dx\,dp=\frac{1}{4\pi^2}\int W^3(k,\lambda)dk\,d\lambda.$$

The double Fourier transform $W(k,\lambda)$ is given by Eq. (40). For our example, the wave function is the one of Eq. (35), and we obtain

$$W(k,\lambda)=4\sqrt{2/\pi}\,\exp(-\lambda^2\hbar^2/2)\int\left(x^2-\frac{\lambda^2\hbar^2}{4}\right)\times\exp(-2x^2)\exp(-ikx)dx.$$

Now, by using the integrals

$$\int \exp(-2x^2)\exp(-ikx)dx=\sqrt{\frac{\pi}{2}}\exp(-k^2/8),$$

$$\int x^2\exp(-2x^2)\exp(-ikx)dx=\sqrt{\frac{\pi}{8}}\left(1-\frac{k^2}{4}\right)\exp(-k^2/8),$$

we obtain, after some straightforward algebra,

$$W(k,\lambda)=\left(1-\frac{k^2}{4}-\lambda^2\hbar^2\right)\exp\left(-\frac{\lambda^2\hbar^2}{2}-\frac{k^2}{8}\right).$$

We are now ready to compute the integral $\int W^3\,dk\,d\lambda$. Let us change integration variables $(k,\lambda)\rightarrow(r,\varphi)$

$$r^2=\frac{k^2}{4}+\lambda^2\hbar^2,\quad r\,dr\,d\varphi=\frac{\hbar}{2}d\lambda\,dk.$$

After integration over φ, one obtains

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W^3\,dk\,d\lambda=\frac{4\pi}{\hbar}\int_0^{\infty}(1-r^2)^3\exp\left(-\frac{3}{2}r^2\right)r\,dr.$$

Changing the integration variable to $y=r^2$ and using integrals of the type

$$\int_0^{\infty}y^n\exp(-ay)dy=\frac{n!}{a^{n+1}},$$

we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W^3\,dk\,d\lambda=\frac{2\pi}{\hbar}\int_0^{\infty}(1-y)^3\exp\left(-\frac{3}{2}y\right)dy=-\frac{4\pi}{27\hbar},$$

which, once divided by $4\pi^2$, yields the result of Eq. (37).

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