Calculation of ground-state entropies of highly frustrated systems on fractal lattices

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(Received 18 May 2000)

The extensive ground-state entropy of frustrated systems on fractal lattices is investigated. Two methods of calculation are proposed, namely, recursive and factorization approaches. In the recursive approach the calculation is based on exact recursion relations for the total number of ground states. The latter procedure, which is in principle an approximation, is proposed as an alternative for dealing with complicated systems (for cases where the recursive approach may become impracticable), such as randomly frustrated models; it consists of factorizing the total number of ground states in terms of the number of ground states at each hierarchy level. Some examples of antiferromagnetic Ising models on different fractal lattices are considered, for which both procedures are applied. It is shown that the factorization approach may lead, in some cases, to the exact ground-state entropy, whereas in other cases it yields very accurate (although slightly lower) estimates.

PACS number(s): 05.50.+q, 64.60.Ak, 65.50.+m

I. INTRODUCTION

Frustrated magnetic models [1] represent one of the most interesting classes of systems in statistical mechanics. Roughly speaking, the frustration [2] is a result of the competition between interactions, in such a way that the minimization of the energy occurs with some interactions unsatisfied. Since one can usually vary the set of broken interactions, frustration leads to a multiplicity of states with the same energy. At low temperatures the frustration effects play a crucial role in such a way that the physics of frustrated systems may present many new features quite different from those of nonfrustrated models. In particular, the possible presence of a nonzero ground-state (GS) entropy per particle (usually composed of residual entropy), makes the frustrated systems exceptions to the third law of thermodynamics.

Many real systems display frustration, such as ice [1], spin glasses [3–5], and diluted antiferromagnets [6,7]. For spin glasses, frustration is combined with disorder in such a way that the existence of many low-temperature states leads to a very slow dynamics, associated with the phenomenon of aging, with the system remaining out of equilibrium even on macroscopic time scales [8].

Theoretical investigations on models characterized by frustration has attracted the attention of many workers [1–5]. A large diversity of uniform (with no randomness), fully frustrated models has been considered [1]; for such systems, the total number of GS's, N_{GS}, usually increases exponentially with the number of sites N,

$$N_{\mathrm{GS}} \sim \exp(hN), \tag{1.1}$$

where h is some positive finite number [for Ising systems, $0 \leqslant h \leqslant \log 2$, since the maximum number of states is $2^N = \exp(N \log 2)$]. In the thermodynamic limit, one obtains that $h = s_0$, where s_0 denotes the residual entropy (herein we work in units of $k_B = 1$). The prototype of fully frustrated models is the antiferromagnetic Ising model on a triangular lattice, for which the residual entropy has been calculated exactly [9], $s_0 \cong 0.323\,066$. For randomly frustrated systems, like spin glasses, one has to deal with the average number of GS's, $[N_{\mathrm{GS}}]_J$ where $[\,]_J$ represents an average over the disorder; due to the averaging process, h is not related to the GS entropy, but rather, to the GS complexity [10]. A behavior similar to that of Eq. (1.1) holds for $[N_{\mathrm{GS}}]_J$ in the infinite-range-interaction Ising spin glass [11], with $h \approx 0.20$ [12]. The average number of GS's has been calculated for short-range Ising spin glasses on diamond hierarchical lattices, with different probability distributions for the couplings [13]: one finds a zero GS complexity per particle in the case of continuous probability distributions, whereas for a bimodal ($\pm J$) distribution an exponential increase in $[N_{\mathrm{GS}}]_J$ has been verified, on lattices of fractal dimensions $d_l \leqslant d \leqslant 5$ (where $d_l \approx 2.58$ represents the respective spin-glass lower-critical dimension), with h varying roughly from 0.16 (for $d = d_l$) to 0.27 (for $d = 5$). However, an outstanding question concerns the average numbers of GS's in nearest-neighbor-interaction spin glasses defined on Bravais lattices [5].

The study of magnetic models on fractal lattices [15], besides serving in practice to model natural materials such as porous rocks, aerogels, sponges, etc., has provided useful results that contribute to our comprehension of the corresponding systems on Bravais lattices. In particular, hierarchical lattices (HL's)—generated through recursive procedures—are much easier to handle [under the real-space renormalization group (RG)], in such a way that exact results may be obtained for short-range systems [16,17]. For pure systems defined on Bravais lattices, one may obtain RG equations through a spin-decimation process; in the corre-

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sponding HL, such a procedure is exact for discrete classical spin variables if, within a few RG steps one obtains nonproliferated RG equations connecting two successive hierarchy levels. Some particular HL's have been very successful in mimicking Bravais lattices [17], e.g., providing exact critical temperatures and exponents of magnetic models on the square lattice.

In the present work we present two methods for calculating the GS degeneracy of frustrated systems defined on HL's. In the first method, herein called the recursive approach (RA), one calculates the GS degeneracy through exact recursion relations based on the recursive properties of the particular lattice. In the second method, the factorization approach (FA), the total number of GS's at hierarchy level n is expressed as a product of a properly defined partial number of GS's at hierarchy levels $n, n-1, \ldots, 1$ [13]. As will be seen below, both approaches lead to the exact GS degeneracy only for very simple systems. The RA, which in principle always yields the exact GS degeneracy, may become difficult to operationally implement in some cases; for complicated systems, such as spin glasses, the FA appears as a simple and good approximation for the estimation of the GS degeneracy [13]. The present paper is organized as follows. In Sec. II we discuss the RG transformation at zero temperature. In Sec. III we present the methods for calculating the GS degeneracies. In Sec. IV we apply both methods to antiferromagnetic Ising models defined on some HL's. Finally, in Sec. V we present our conclusions.

II. THE RENORMALIZATION AT ZERO TEMPERATURE

The most common real-space RG recipe [18] consists of summing over some spin variables of the system, a procedure known as a partial trace. As a result of such an operation, one obtains an effective (or renormalized) interaction between the remaining spins, with a relation between the renormalized and original interactions. In a HL, a partial trace over the spins of the kth hierarchy leads to an effective interaction among the spins of the $(k-1)$th hierarchy level, and to a corresponding recursive relation between the interactions in successive hierarchy levels. In the case of discrete classical spin variables, the RG procedure may lead to an exact recursion relation for a HL, in the case of pure systems, if no proliferation of RG equations occurs [16,17]; one says that the space of parameters is closed under the RG process.

Herein we shall restrict ourselves to Ising spin systems defined in terms of the Hamiltonian,

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j \quad (S_i = \pm 1), \tag{2.1}$$

where the sum $\Sigma_{\langle ij \rangle}$ is restricted to nearest-neighbor pairs of spins on a given HL. For pure systems [where the coupling constants $\{J_{ij}\} = J$ $(\forall \text{pairs} \langle ij \rangle)$], one may obtain exact recursion relations $t' = f(t)$ for the thermal transmittivities [17] $t = \tanh[J/(k_B T)]$ of two successive hierarchy levels, t' [hierarchy level $(k-1)$] and t (hierarchy level k). Such recursion relations lead to plots t' versus t, like the ones exhibited in Fig. 1. In fact, Fig. 1(a) refers to a HL generated in such a way that at each step a single bond is replaced by the planar $b=3$ Wheatstone-bridge-like cell of Fig. 2(a), whereas for Fig. 1(b) a bond is replaced by the nonplanar Wheatstone-bridge cell shown in Fig. 2(c).

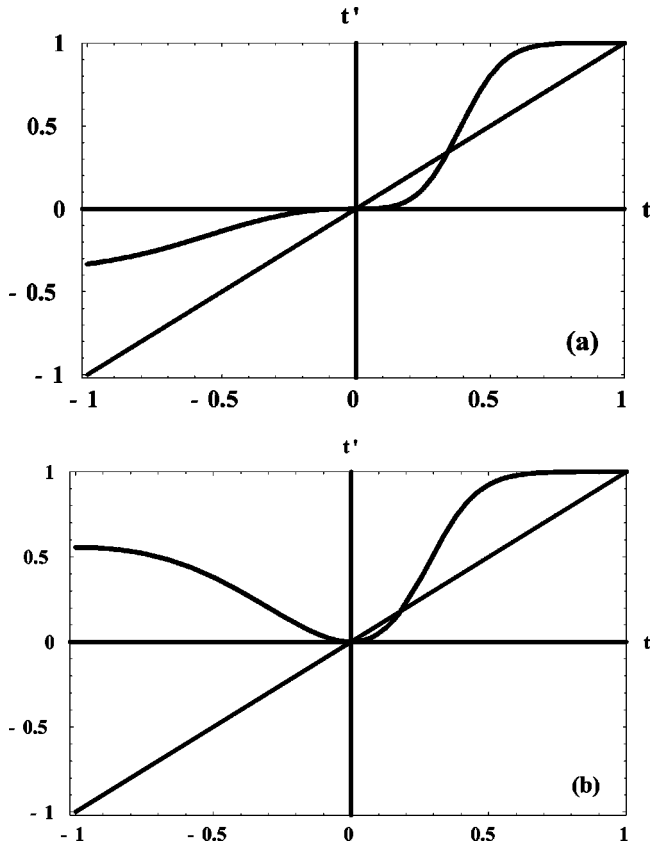

FIG. 1. Plots of the renormalized transmissivity t' vs the original transmissivity t $(t = \tanh[J/(k_B T)])$ for the Ising model on HL's composed of unit cells: (a) The $b=3$ Wheatstone-bridge-like cell [see Fig. 2(a)]. (b) The $b=2$ Wheatstone-bridge nonplanar cell [see Fig. 2(c)]. The zero-temperature point $(t=-1)$ is given by $a = \lim_{T \to 0} t'(T) = -1/3$ in case (a), and $a = 5/9$ in case (b). For the cells of Figs. 2(b) and 2(d), one obtains plots similar to the one shown in (b), with $a=1$ and $a = \frac{1}{2}(14 - 8\sqrt{3}) \approx 0.0718$, respectively.

Let us now consider, in such systems, the zero-temperature limit, $T \to 0$. If the interactions at the last hierarchy level (say the nth hierarchy) are ferromagnetic $(J > 0)$, there are no frustrations, and the system is dominated by the zero-temperature fixed point $t' = t = 1$, with positive finite effective interactions between the spins at the lowest hierarchy levels (hierarchies $n-1, \ldots, 1, 0$). However, if the interactions at the nth hierarchy are antiferromagnetic $(J < 0)$, there are frustrations in the system, in such a way that the examples exhibited in Fig. 1 present no typical zero-temperature point, i.e., for $t = -1$ one obtains $\lim_{T \to 0} t'(T) = a$, where a is a constant $(a \neq \pm 1)$. Such a behavior has also been observed in other systems, like the antiferromagnetic three-state Potts model on diamond HL's [19]. One obtain that the effective interactions between the spins at the $(n-1)$th hierarchy level,

$$J' = \lim_{T \to 0} k_B T \tanh^{-1}[t'(T)], \tag{2.2}$$

are *driven to zero*, at the first renormalization step, if $a \neq$

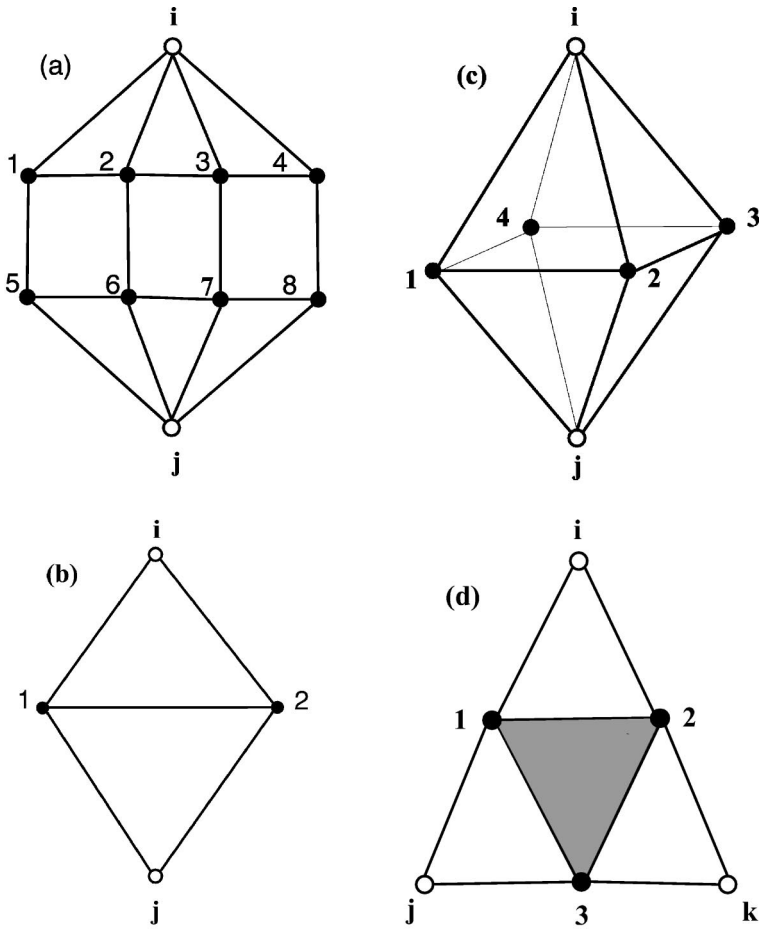

FIG. 2. The basic unit cells that define the HL's considered. (a) The $b=3$ planar Wheatstone-bridge-like cell. (b) The $b=2$ planar Wheatstone-bridge cell. (c) The $b=2$ non-planar Wheatstone-bridge cell. (d) The $b=2$ Sierpinski gasket cell. The spins at the terminal sites (empty circles, denoted by letters) belong to previous hierarchy levels and are connected to other spins of the lattice; the spins at the internal sites (black circles, denoted by numbers) are decimated throughout the renormalization process. In all cases the basic unit cells correspond to the HL at its hierarchy level $k=1$.

± 1. Therefore, further RG steps (hierarchy levels $n-1,n-2,\ldots,0$) become trivial, with no interactions between the spins.

Such a curious zero-temperature behavior implies that the spins belonging to hierarchy levels $n-1,n-2,\ldots,0$ are completely uncorrelated (as in a high-temperature phase), contributing the maximum number of states to the GS degeneracy. This is a crucial point for the calculation of the GS degeneracy of some frustrated HL's, as will be seen below.

III. METHODS OF CALCULATION

Let us consider a HL defined in such a way that at hierarchy level 0 one has $N^{(0)}$ sites connected by $N_b^{(0)}$ bonds. The lattice is generated through a given recursive rule: in Fig. 2 we represent typical cells corresponding to the hierarchy level 1 of some HL's; such cells will be considered as the basic unit cells for the lattice at a given kth hierarchy level. For the cells shown in Figs. 2(a)–2(c), one has $N^{(0)}=2$ (sites i and j) and $N_b^{(0)}=1$; the HL is generated in such a way that, at each step, a single bond is replaced by a unit cell. In Fig. 2(d) one has the unit cell of a Sierpinski gasket [15] with a scaling factor $b=2$, for which $N^{(0)}=3$ (sites i, j, and k) and $N_b^{(0)}=3$; the Sierpinski gasket is generated by removing, at each step, one-fourth of the area [the gray triangular part of Fig. 2(d)] from each triangle of the lattice. A given HL, at its kth hierarchy level, will be composed of $N_c^{(k)}$ unit cells, each of them with P terminal spins (due to the recursive rule for the generation of the lattice, one has that $P=N^{(0)}$).

Let us now fix the terminal spins of each unit cell; for an Ising system, there are 2^P ways of doing this for a single cell. For each fixed configuration of terminal spins in a unit cell, one may have a certain number of GS's associated with the internal spins of the cell. We shall denote by $\{g_\alpha\}$ the possible set of GS degeneracies associated with a unit cell of the HL; the label α, which refers to configurations of terminal spins of the cell presenting different values of degeneracies, may vary, in principle, from 1 to 2^{P-1} (since time-reversed states contribute to the same GS degeneracies). In what follows, we introduce two different methods for the calculation of GS degeneracies of frustrated HL's.

A. Recursive approach

The RA is based on the recursive properties of the particular HL; the central idea is to express GS degeneracies at a given hierarchy level in terms of those of the previous hierarchy. By fixing the spins of the zeroth hierarchy level, one has a set of $\{G_\alpha^{(k)}\}$ possible degeneracies at an arbitrary hierarchy level k, in such a way that for each value of α one has a recursion relation

$$G_\alpha^{(k)}=\Psi_\alpha(G_1^{(k-1)},G_2^{(k-1)},\ldots). \tag{3.1}$$

Since one may compute easily the set of degeneracies at hierarchy level 1 $\{G_\alpha^{(1)}\}\equiv\{g_\alpha\}$, the recursion relations in Eq. (3.1) may be followed up to any desired hierarchy level. The total number of GS's of the HL at its nth hierarchy level is expressed as

$$N_{\mathrm{GS}}^{(n)}=\sum_{\alpha} a_{\alpha} G_{\alpha}^{(n)}, \tag{3.2}$$

where the coefficients a_{α} count how many different configurations of the spins at the zeroth level contribute to the same $G_{\alpha}^{(n)}$.

If one succeeds in obtaining the recursion relations in Eq. (3.1) exactly, the RA yields the exact number of GS's of the HL at its nth hierarchy level. However, this turns out to be a difficult task in some cases, like for disordered systems. Next we introduce a much simpler method, as an alternative for working with systems for which the recursion relations in Eq. (3.1) are difficult to obtain.

B. Factorization approach

An alternative form of this method was already employed for the calculation of GS degeneracies of Ising spin glasses on diamond HL's [13]; herein we shall define the FA for the simpler case of pure systems. Let us consider a given HL at its nth hierarchy level; one may partially count the number of GS's of the HL by fixing the terminal spins of each unit cell. We shall denote the number representing this partial counting by $\Gamma^{(n)}$. In a HL all unit cells present terminal spins belonging to lowest-level hierarchies; under the RG procedure, each terminal spin will become an internal one at its respective hierarchy level. Therefore, one may write

$$N_{\mathrm{GS}}^{(n)}=\Gamma^{(n)}\Gamma^{(n-1)}\Gamma^{(n-2)}\cdots\Gamma^{(1)}A, \tag{3.3}$$

where the factor A corresponds to the number of states associated with hierarchy level 0. It is important to mention that the total number of GS's, as written in Eq. (3.2), may be factorized in the form of Eq. (3.3) only for very particular types of the recursion relations (3.1); in most of the cases, Eq. (3.3) represents an approximation for $N_{\mathrm{GS}}^{(n)}$. Typical examples for which the FA may be considered as a useful approximation are randomly frustrated systems, like Ising spin glasses on hierarchical lattices [13].

One may see easily that, for an arbitrary hierarchy level k, the partial counting may be written as

$$\Gamma^{(k)}=\prod_{\alpha} (g_{\alpha})^{N_{c,\alpha}^{(k)}}, \tag{3.4}$$

where $N_{c,\alpha}^{(k)}$ denotes the number of unit cells with degeneracy g_{α} in the HL at its kth hierarchy level (obviously, the total number of cells at the kth hierarchy level, $N_c^{(k)}$, may be obtained from $\Sigma_{\alpha}N_{c,\alpha}^{(k)}=N_c^{(k)}$). In simple systems, $N_{c,\alpha}^{(k)}$ may be calculated exactly, whereas in more complicated problems one may replace $N_{c,\alpha}^{(k)}$ with the average value [13]

$$\phi_{\alpha}^{(k)}=N_c^{(k)}F_{\alpha}^{(k)}, \quad \sum_{\alpha}\phi_{\alpha}^{(k)}=N_c^{(k)}, \tag{3.5}$$

where $F_{\alpha}^{(k)}$ represents the probability of finding a unit cell of type α at hierarchy level k [14]. Such a procedure leads to an average estimate

$$[\Gamma^{(k)}]_{av}=\prod_{\alpha} (g_{\alpha})^{\phi_{\alpha}^{(k)}}. \tag{3.6}$$

Next we apply our methods to four examples of frustrated HL's; we show that in two of them, the FA yields the exact residual entropy in the thermodynamic limit, whereas for the other two, the FA provides a lower estimate for the number of GS's.

IV. APPLICATIONS

Let us consider the antiferromagnetic Ising model defined through the Hamiltonian in Eq. (2.1) [with the coupling constants $\{J_{ij}\}=J<0$ ($\forall$pairs$\langle ij\rangle$)], on HL's composed of the unit cells shown in Fig. 2. For the cells in Figs. 2(a)–2(c) (which belong to the family of Wheatstone-bridge cells [17]) the corresponding HL's are generated in such a way that at each step, a single bond is replaced by a unit cell. The scaling factor is $b=3$ for the cell of Fig. 2(a), whereas $b=2$ for those in Figs. 2(b) and 2(c); the fractal dimensions of the respective HL's are $d=(\log 18/\log 3)\cong 2.631$ [cell in Fig. 2(a)], $d=(\log 5/\log 2)\cong 2.322$ [cell in Fig. 2(b)], and $d=(\log 12/\log 2)\cong 3.585$ [cell in Fig. 2(c)]. In Fig. 2(d) one has the unit cell of a $b=2$ Sierpinski gasket [15]; since each triangle at hierarchy level k is replaced by three new triangles with a half of the side at level $k+1$, i.e., each triangle will generate three new sites [sites 1, 2, and 3 in Fig. 2(d)], the fractal dimension of such a HL is $d=(\log 3/\log 2)\cong 1.585$. In what follows, we shall apply both methods described above to such systems.

A. $b=3$ planar Wheatstone-bridge-like HL

Let us now consider the antiferromagnetic Ising model on the HL composed of the unit cells shown in Fig. 2(a). At an arbitrary hierarchy level k, one has the total number of bonds and unit cells given, respectively, by

$$N_b^{(k)}=18^k, \quad N_c^{(k)}=18^{k-1}, \tag{4.1}$$

whereas the number of sites belonging to level k, $\tilde{N}^{(k)}$, and the total number of sites of the HL, $N^{(k)}$:

$$\tilde{N}^{(k)}=8N_c^{(k)}, \quad N^{(k)}=2+\frac{8}{17}(18^k-1). \tag{4.2}$$

It is important to remember that the RG transformation for the present HL leads to the plot t' versus t shown in Fig. 1(a), where in the zero-temperature limit ($t=-1$) one obtains $a=\lim_{T\to 0}t'(T)=-1/3$. According to Eq. (2.2), the interactions become zero after the first RG step; therefore, for a HL at the nth hierarchy level, the interactions are to be considered as antiferromagnetic at $k=n$ and zero for $k=n-1,n-2,\ldots,0$.

Let us now apply the RA for the present system; we shall first consider the case $n=1$. Since the terminal spins (S_i and S_j) belong to the zeroth hierarchy level, the effective interaction between them is zero. Therefore, the GS degeneracies of the cell in Fig. 2(a) may be easily calculated, by considering all possible states for the terminal spins, as shown below.

(i) Terminal spins parallel ($S_i=S_j$)—there are two GS configurations ($g_1=2$), given by four broken bonds each: either one breaks the set of bonds $\{J_{i1},J_{i3},J_{j6},J_{j8}\}$ or $\{J_{i2},J_{i4},J_{j5},J_{j7}\}$.

(ii) Terminal spins antiparallel ($S_i \neq S_j$)—there are also two GS configurations ($g_2 = g_1 = 2$), given by four broken bonds each: either one breaks the set of bonds $\{J_{i1}, J_{i3}, J_{j5}, J_{j7}\}$ or $\{J_{i2}, J_{i4}, J_{j6}, J_{j8}\}$.

Since $g_2 = g_1$, for the total number of GS's at hierarchy level 1 [see Eq. (3.2)] one may write

$$N_{\mathrm{GS}}^{(1)} = a_1 g_1, \tag{4.3}$$

where $a_1 = 4$, i.e, $N_{\mathrm{GS}}^{(1)} = 8$.

The case $n = 2$ may also be worked out easily, by considering antiferromagnetic interactions for $k = 2$ and zero interactions for $k = 1$ and 0. Since each bond of hierarchy level 1 will generate a unit cell with degeneracy g_1 at hierarchy level 2, one obtains

$$N_{\mathrm{GS}}^{(2)} = a_1 G_1^{(2)}, \quad G_1^{(2)} = 256 g_1^{18}, \tag{4.4}$$

where the factors $2^8 = 256$ and $a_1 = 4$ come from the sums over the spins of hierarchy levels 1 and 0, respectively. The procedure may be carried for the nth hierarchy level:

$$N_{\mathrm{GS}}^{(n)} = a_1 G_1^{(n)}, \quad G_1^{(n)} = 256 (G_1^{(n-1)})^{18}. \tag{4.5}$$

Using the fact that $G_1^{(1)} = g_1 = 2$, one may obtain the GS entropy per spin in the thermodynamic limit,

$$s_0 = \lim_{n \to \infty} \frac{1}{N^{(n)}} \log N_{\mathrm{GS}}^{(n)} = \frac{25}{144} \log 2 = 0.120\,338 \ldots, \tag{4.6}$$

which represents about 17.36% of the maximum possible entropy per spin.

Let us now apply the FA for this system. In Eq. (3.3), one has that

$$\Gamma^{(n)} = (g_1)^{N_c^{(n)}}, \quad \Gamma^{(k)} = 2^{\tilde{N}^{(k)}} \quad (k = 1,2, \ldots, n-1), \qquad A = 4. \tag{4.7}$$

Substituting such results into Eq. (3.3), and using the fact that

$$\sum_{k=1}^{n-1} \tilde{N}^{(k)} = \frac{8}{17} (18^{n-1} - 1), \tag{4.8}$$

one may easily obtain the same residual entropy of Eq. (4.6).

In this simple example, the FA yields the exact residual entropy as well. This occurs because the degeneracies of the unit cell of Fig. 2(a) are the same in both situations of parallel and antiparallel terminal spins. In this case, Eq. (4.5) may be written as

$$N_{\mathrm{GS}}^{(n)} = a_1 \mathrm{Tr}_{\{k=1,2,\ldots,n-1\}} (g_1)^{N_c^{(n)}} = a_1 (g_1)^{N_c^{(n)}} \mathrm{Tr}_{\{k=1,2,\ldots,n-1\}} 1, \tag{4.9}$$

where $\mathrm{Tr}_{\{k=1,2,\ldots,n-1\}}$ denotes a trace over all possible states of the spins belonging to hierarchies $1,2,\ldots,n-1$. Comparing with Eq. (4.7), one recovers the product of Eq. (3.3), characteristic of the FA. It should be mentioned that it is straightfoward to show that the FA also leads to the exact residual entropy for the cases where the unit cells present different degeneracies, bu the total number of ground states may be expressed as a single product of powers of such degeneracies.

It should be mentioned that it is straightfoward to show that the FA also leads to the exact residual entropy for the cases where the unit cells present different degeneracies, but the total number of ground states may be expressed as a single product of powers of such degeneracies.

B. $b = 2$ planar Wheatstone-bridge HL

In this example, we consider the antiferromagnetic Ising model on the HL defined through the unit cell exhibited in Fig. 2(b). The RG transformation for the present HL leads to a plot t' versus t similar to the one shown in Fig. 1(b), but in the zero-temperature limit ($t = -1$) one gets $a = \lim_{T \to 0} t'(T) = 1$. This leads to ferromagnetic interactions just after the first RG step; for a HL at the nth hierarchy level, the interactions are to be considered as antiferromagnectic at $k = n$ and ferromagnetic for $k = n-1, n-2, \ldots, 0$.

In order to minimize the energy of a given cell, its terminal spins (which belong to a hierarchy level $k < n$) should be parallel to one another [$S_i = S_j$ in the cell of Fig. 2(b)]. Clearly, one sees that there is a single ground-state configuration, characterized by the breaking of its central bond (J_{12}). Since this is valid for all cells of the HL, both methods (RA and FA) yield the trivial result for the total number of ground states, $N_{\mathrm{GS}}^{(n)} = 2$. Although the HL composed of the cells of Fig. 2(b) appears to be, at first glance, a fully frustrated system, it indeed corresponds to a simple model with no residual entropy; this is a consequence of the fact that each pair of triangular plaquettes in a unit cell share a single common bond.

In the next two examples, one will get more complicated recursion relations for the total number of GS's in such a way that the FA will lead to an approximate estimate for $N_{\mathrm{GS}}^{(n)}$; in fact, the FA result will be a lower estimate, as compared to the exact value obtained through RA.

C. $b = 2$ nonplanar Wheatstone-bridge HL

We shall now investigate the antiferromagnetic Ising model on the HL defined through the unit cell exhibited in Fig. 2(c). The number of bonds, cells, sites belonging to level k, and total number of sites, at an arbitrary hierarchy level k, are given respectively, by

$$N_b^{(k)} = 12^k, \quad N_c^{(k)} = 12^{k-1}, \tag{4.10}$$

$$\tilde{N}^{(k)} = 4 N_c^{(k)}, \quad N^{(k)} = 2 + \frac{4}{11} (12^k - 1). \tag{4.11}$$

The RG transformation yields the plot t' versus t shown in Fig. 1(b), with the zero-temperature limit given by $t = -1$ and $a = \lim_{T \to 0} t'(T) = 5/9$. As discussed above, this leads to independent spins right after the first RG step; therefore, for a HL at the nth hierarchy level, the interactions will be considered as antiferromagnetic at $k = n$ and zero for all other hierarchy levels.

Let us now implement the RA, starting, as usual, with the case $n=1$. The GS configurations and corresponding degeneracies associated with the cell of Fig. 2(c) are described below.

(i) Terminal spins parallel ($S_i=S_j$)—there are three GS configurations ($g_1=3$), given by four broken bonds each: $\{J_{12},J_{23},J_{34},J_{41}\}$ or $\{J_{i2},J_{i4},J_{j2},J_{j4}\}$ or $\{J_{i1},J_{i3},J_{j1},J_{j3}\}$.

(ii) Terminal spins antiparallel ($S_i\neq S_j$)—there are two GS configurations ($g_2=2$), given by four broken bonds each: $\{J_{i2},J_{i4},J_{j1},J_{j3}\}$ or $\{J_{i1},J_{i3},J_{j2},J_{j4}\}$.

For the total number of GS's at hierarchy level 1, one may write

$$N_{\mathrm{GS}}^{(1)}=a_1g_1+a_2g_2, \tag{4.12}$$

where $a_1=a_2=2$, i.e, $N_{\mathrm{GS}}^{(1)}=10$. The case $n=2$ may also be worked out easily; each of the 12 bonds of the cell in $k=1$ will generate a new cell presenting either one of the degeneracies g_1 or g_2, depending on its terminal spins. One obtains

$$N_{\mathrm{GS}}^{(2)}=a_1G_1^{(2)}+a_2G_2^{(2)}, \tag{4.13a}$$

$$G_1^{(2)}=g_1^{12}+4g_1^8g_2^4+4g_1^6g_2^6+7g_1^4g_2^8, \tag{4.13b}$$

$$G_2^{(2)}=2g_1^8g_2^4+12g_1^6g_2^6+2g_1^4g_2^8. \tag{4.13c}$$

For a HL at its nth hierarchy level,

$$N_{\mathrm{GS}}^{(n)}=a_1G_1^{(n)}+a_2G_2^{(n)}, \tag{4.14a}$$

$$G_1^{(n)}=(G_1^{(n-1)})^{12}+4(G_1^{(n-1)})^8(G_2^{(n-1)})^4+4(G_1^{(n-1)})^6(G_2^{(n-1)})^6+7(G_1^{(n-1)})^4(G_2^{(n-1)})^8, \tag{4.14b}$$

$$G_2^{(n)}=2(G_1^{(n-1)})^8(G_2^{(n-1)})^4+12(G_1^{(n-1)})^6(G_2^{(n-1)})^6+2(G_1^{(n-1)})^4(G_2^{(n-1)})^8, \tag{4.14c}$$

where $G_1^{(1)}=g_1=3$ and $G_2^{(1)}=g_2=2$. The recursion relations in Eqs. (4.14) may be iterated numerically; by doing so, one may obtain the residual entropy s_0 for a HL at a given hierarchy level, as shown in Table I. One observes a rapid convergence to the thermodynamic limit, in such a way that for $n=7$ we obtain $s_0=0.269\,816\,3(3)$, which represents about 38.93% of the maximum possible entropy per spin.

We shall now treat the problem within the FA; one obtains that $A=4$, whereas

$$\Gamma^{(n)}=(g_1)^{N_{c,1}^{(n)}}(g_2)^{N_{c,2}^{(n)}},\quad \Gamma^{(k)}=2^{\tilde{N}^{(k)}}\quad (k=1,2,\ldots,n-1), \tag{4.15a}$$

with

$$\sum_{k=1}^{n-1}\tilde{N}^{(k)}=\frac{4}{11}(12^{n-1}-1). \tag{4.15b}$$

Since one has zero couplings for hierarchies $k<n$, $F_1^{(n)}=F_2^{(n)}=\frac{1}{2}$, and so we replace $N_{c,1}^{(n)}$ and $N_{c,2}^{(n)}$ by $\phi_1^{(n)}=\phi_2^{(n)}=\frac{1}{2}N_c^{(n)}$ [see Eq. (3.5)]. Using such results one obtains that

$$\log N_{\mathrm{GS}}^{(n)}\cong\left[2+\frac{4}{11}(12^{n-1}-1)\right]\log 2+\frac{1}{2}12^{n-1}\log 3+\frac{1}{2}12^{n-1}\log 2, \tag{4.16}$$

and so, in the thermodynamic limit, the residual entropy becomes

$$s_0\cong\frac{19}{96}\log 2+\frac{11}{96}\log 3=0.263\,068\ldots, \tag{4.17}$$

which yields a relative discrepancy of about 2.5% with respect to the value computed iteratively from the RA.

The FA, which is much simpler to implement than the RA, leads in this case to a slightly lower estimate for the residual entropy, as may be seen in Table I and Fig. 3(a). Below, we give a proof that the FA should yield a lower estimate than the RA, for any HL satisfying the following requirements.

(i) The unit cells exhibit two different GS degeneracies (g_1 and g_2).

(ii) The interactions are antiferromagnetic in last hierarchy and zero for the lower hierarchy levels.

(iii) The recursion relations are written as sums of products of different powers of g_1 and g_2 [like the ones in Eqs. (4.14)].

In fact, Eqs. (4.14) lead to the following form for the number of GS's:

$$N_{\mathrm{GS}}^{(n)}=\sum_{\mu}\ (g_1)^{N_{c,1}^{(n)}\{\mu\}}(g_2)^{N_{c,2}^{(n)}\{\mu\}}, \tag{4.18}$$

where Σ_μ denotes a sum over all frustrated configurations up to hierarchy $n-1$, whereas $N_{c,\alpha}^{(n)}\{\mu\}$ denotes the number of cells, of configuration μ, with degeneracy g_α ($N_{c,1}^{(n)}\{\mu\}+N_{c,2}^{(n)}\{\mu\}=N_c^{(n)}$). One may also write

$$N_{\mathrm{GS}}^{(n)}=(g_2)^{N_c^{(n)}}\sum_{\mu}\left(\frac{g_1}{g_2}\right)^{N_{c,1}^{(n)}\{\mu\}}. \tag{4.19}$$

Since $(g_1/g_2)>1$, the right-hand side of the equation above is a convex function, and using the general property $1/N\Sigma_i f(x_i)\geqslant f(\overline{x})$ one obtains that

$$N_{\mathrm{GS}}^{(n)}\geqslant(g_1)^{\phi_1^{(n)}}(g_2)^{\phi_2^{(n)}}N_{\mathrm{GS}}^{(n-1)}=[\Gamma^{(n)}]_{av}N_{\mathrm{GS}}^{(n-1)}, \tag{4.20}$$

where $[\Gamma^{(n)}]_{av}$ is defined in Eq. (3.6). Since the spins in the hierarchy levels $k=n-1,n-2,\ldots,0$ are subject to zero interactions, the number of GS's at the $(n-1)$th hierarchy level, $N_{\mathrm{GS}}^{(n-1)}$, factorizes in the FA form of Eq. (3.3). Therefore, one obtains

$$N_{\mathrm{GS}}^{(n)}\geqslant[\Gamma^{(n)}]_{av}\Gamma^{(n-1)}\Gamma^{(n-2)}\cdots\Gamma^{(1)}A, \tag{4.21}$$

where the left- and right-hand sides of the equation above represent the total number of GS's of the HL calculated throught the RA and FA methods, respectively.

TABLE I. Residual entropy s_0 of the antiferromagnetic Ising model on HL's composed of unit cells in Figs. 2(c) and 2(d), calculated by two different methods, RA (exact) and FA (approximate), for typical lattice sizes (n denotes the hierarchy level). One observes that the first two columns [Fig. 2(c), HL] exhibit a much quicker convergence to the thermodynamic-limit value of s_0 than the other two [Fig. 2(d), HL].

n	Fig. 2(c) HL (RA)	Fig. 2(c) HL (FA)	Fig. 2(d) HL (RA)	Fig. 2(d) HL (FA)
2	0.282369	0.276101	0.513020	0.511365
3	0.270853	0.264185	0.500154	0.498381
4	0.269888	0.263161	0.495447	0.493631
5	0.269820	0.263076	0.493826	0.491995
6	0.269817	0.263069	0.493280	0.491444
7	0.269816	0.263068	0.493098	0.491260
11			0.493007	0.491169
14			0.493006	0.491168
15			0.493006	0.491167

D. $b=2$ Sierpinski gasket

Let us now consider the fully frustrated $b=2$ Sierpinski gasket, as defined above [see cell in Fig. 2(d)]. It is important to remember that the residual entropy of this system has already been calculated exactly, through other methods [20,21]; in such works, the basic unit of the HL was considered as a single triangle (i.e., the system at its zeroth hierachical level). Herein we shall keep the same convention used in the previous applications, i.e., our unit cell will be the one shown in Fig. 2(d), i.e., the HL at hierarchy level $k=1$. Therefore, the HL will be composed of unit cells with three terminal spins, as well as three internal spins each; the number of cells, sites belonging to level k, and total number of sites, at an arbitrary hierarchy level k, are given, respectively, by

$$N_c^{(k)}=3^{k-1}, \quad \tilde{N}^{(k)}=3N_c^{(k)},$$

$$N^{(k)}=3+\frac{3}{2}(3^k-1)=\frac{3}{2}(3^k+1). \tag{4.22}$$

For this system, the RG yields a plot t' versus t similar to the one shown in Fig. 1(b), with the zero-temperature limit $a=\lim_{T\to 0}t'(T)=\frac{1}{2}(14-8\sqrt{3})\cong 0.072$; therefore, the antiferromagnetic interactions of the last hierarchy level will lead to zero interactions for all lower hierarchy levels.

Let us now consider this system under the RA, starting with $n=1$. As described below, there are only two distinct GS degeneracies for the cell of Fig. 2(d), depending on whether the terminal spins at sites i, j, and k are all parallel, or one of them is antiparallel to the remaining ones.

(i) Terminal spins parallel ($S_i=S_j=S_k$)—there are four GS configurations ($g_1=4$), given by three broken bonds each: $\{J_{12},J_{23},J_{31}\}$ or $\{J_{i1},J_{j1},J_{23}\}$ or $\{J_{i2},J_{k2},J_{13}\}$ or $\{J_{j3},J_{k3},J_{12}\}$. There are two possible configurations for terminal spins parallel, and so $a_1=2$.

(ii) One terminal spin different from the other two ($S_i=S_j\neq S_k$)—there are three GS configurations ($g_2=3$), given by three broken bonds each: $\{J_{i2},J_{k3},J_{13}\}$ or $\{J_{i2},J_{j3},J_{23}\}$ or $\{J_{j3},J_{k2},J_{12}\}$. By permutation of the terminal spins, one may obtain a total of six possible configurations with such a GS degeneracy, and so $a_2=6$.

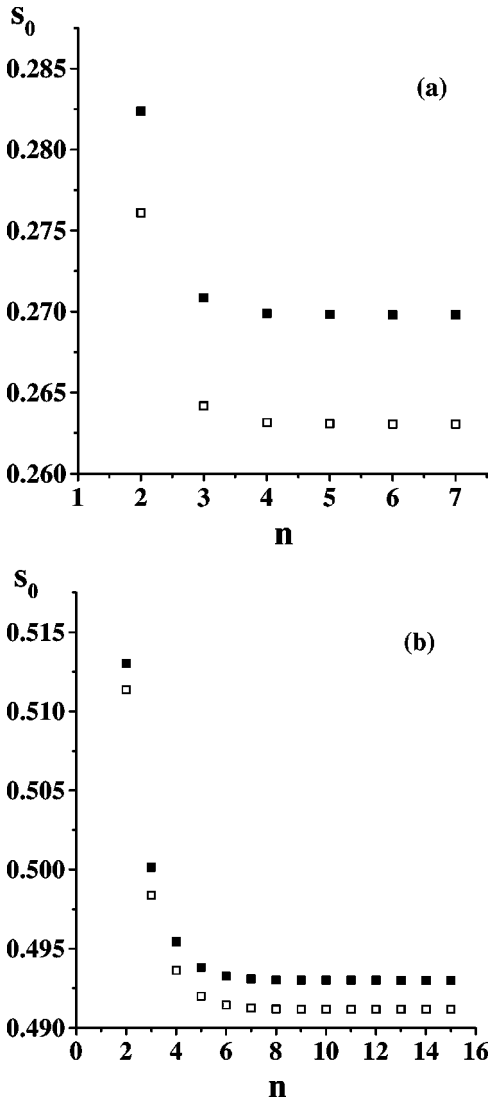

FIG. 3. The GS entropy per spin (residual entropy) for different hierarchy levels n, as calculated by the RA (black squares) and FA (empty squares) for HL's composed of unit cells: (a) The $b=2$ non-planar Wheatstone-bridge cell. (b) The $b=2$ Sierpinski gasket cell.

One obtains an equation similar to Eq. (4.12), leading to $N_{\rm GS}^{(1)}=24$. The Sierpinski gasket at $n=2$ is composed by three unit cells, which may present either one of degeneracies g_1 or g_2 each. One obtains an equation similar to Eq. (4.13a), with

$$G_1^{(2)}=g_1^3+3g_1g_2^2+4g_2^3, \tag{4.23a}$$

$$G_2^{(2)}=g_1^2g_2+4g_1g_2^2+3g_2^3. \tag{4.23b}$$

The above results may be generalized for the Sierpinski gasket at its nth hierarchy level,

$$N_{\rm GS}^{(n)}=a_1G_1^{(n)}+a_2G_2^{(n)}, \tag{4.24a}$$

$$G_1^{(n)}=(G_1^{(n-1)})^3+3G_1^{(n-1)}(G_2^{(n-1)})^2+4(G_2^{(n-1)})^3, \tag{4.24b}$$

$$G_2^{(n)}=(G_1^{(n-1)})^2G_2^{(n-1)}+4G_1^{(n-1)}(G_2^{(n-1)})^2+3(G_2^{(n-1)})^3, \tag{4.24c}$$

where $G_1^{(1)}=g_1=4$ and $G_2^{(1)}=g_2=3$. The residual entropy may be obtained by iterating numerically the above recursion relations. In Table I we present the values of s_0 for different hierarchy levels. For $n=15$ we obtain $s_0=0.493\,006\,12(2)$, which is in agreement, up to seven digits, with the value $s_0=0.493\,006\,107\ldots$ [21].

Let us now consider the present system under the FA; one obtains that $A=8$, whereas

$$\Gamma^{(n)}=(g_1)^{N_{c,1}^{(n)}}(g_2)^{N_{c,2}^{(n)}}, \quad \Gamma^{(k)}=2^{\tilde{N}^{(k)}} \quad (k=1,2,\ldots,n-1) \tag{4.25a}$$

and

$$\sum_{k=1}^{n-1}\tilde{N}^{(k)}=\frac{3}{2}(3^{n-1}-1). \tag{4.25b}$$

As above, we replace each $N_{c,\alpha}^{(n)}$ with the corresponding average value $\phi_\alpha^{(n)}$; since one has zero couplings for hierarchies $k<n$, Eq. (3.5) gives $\phi_1^{(n)}=\frac{1}{4}N_c^{(n)}$ and $\phi_2^{(n)}=\frac{3}{4}N_c^{(n)}$. Therefore,

$$\log N_{\mathrm{GS}}^{(n)}\cong\left[\frac{2}{3}3^n+\frac{3}{2}\right]\log 2+\frac{3}{4}3^{n-1}\log 3, \tag{4.26}$$

and so, in the thermodynamic limit,

$$s_0\cong\frac{4}{9}\log 2+\frac{1}{6}\log 3=0.491\,167\ldots, \tag{4.27}$$

representing a relative discrepancy of about 0.4% with respect to the value computed iteratively from the RA.

The proof carried in Eqs. (4.18)–(4.21) also holds for the present case, in such a way that the total number of GS's calculated through the FA is always a lower estimate with respect to that of the RA [see Table I and Fig. 3(b)].

V. CONCLUSION

We have calculated the ground-state entropy of frustrated Ising systems on hierarchical lattices. Two methods were introduced: the recursive and factorization approaches. The first one is based on exact recursion relations for the total number of ground states, whereas in the latter one writes the total number of ground states as a product of the number of ground states at each hierarchy level by fixing the spins of the lower-level hierarchies. Whenever one succeeds in finding the exact recursion relations, the recursion approach yields the exact ground-state entropy. The factorization approach yields, in principle, an approximate number of ground states; however, it may lead to the exact ground-state entropy, in the thermodynamic limit, for some simple systems, presenting one of the following properties: (i) systems for which the unit cell presents the same ground-state degeneracy for different configurations of its terminal spins; and (ii) systems where the total number of ground states may be expressed as a single product of powers of the degeneracies of the unit cell. For more complicated models, such as disordered systems, the recursive approach becomes a difficult task; in such cases, the factorization method is very useful. We have shown that the factorization method, applied for pure systems characterized by a unit cell presenting two distinct ground-state degeneracies, yields an estimate for the ground-state entropy per spin, which is always equal to [in case (ii) mentioned above] or less than the exact value.

We have applied the above-mentioned methods to the antiferromagnetic Ising model defined on four hierarchical lattices containing triangular plaquettes. For all cases, we have succeeded in calculating the exact residual entropy; in two of them (one of which is trivial), the factorization approach provided the exact answer, whereas for the other two, such a method yielded a lower, although very accurate, estimate (the maximum relative discrepancy found was 2.5%). One of the systems considered is fully frustrated, namely, the Sierpinski gasket, with a scaling factor $b=2$; the calculated residual entropy was $s_0\cong 0.493$, which should be compared with the well known result for the antiferromagnetic Ising model on the triangular lattice [9], $s_0\cong 0.323$. The HL generated by the planar Wheatstone bridge with scaling factor $b=2$ is shown as an example of an apparently fully frustrated system, but that in fact is a trivial case, with no effective frustration. For the Sierpinski gasket, our calculation is in full agreement with previous works [20,21]. As pointed out before, the Sierpinksi gasket may be seen as a triangular lattice with holes; such holes, which cancel some of the interactions of the corresponding triangular lattice, contribute to an increase in the ground-state degeneracy.

The application of the present methods for the study of other frustrated systems seems to be promising and may reveal interesting results on open problems.

ACKNOWLEDGMENTS

We thank S. Coutinho for fruitful discussions. The partial financial support of FAPERJ (Fundação de Apoio à Pesquisa do Estado do Rio de Janeiro), CNPq and Pronex/MCT (Brazilian agencies) are acknowledged. F.D.N. acknowledges CBPF (Centro Brasileiro de Pesquisas Físicas) for the warm hospitality during a visit in which this work was developed.

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