Dynamics of quasicollapse in nonlinear Schrödinger systems with nonlocal interactions

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We study the effect of nonlocality on some dynamical properties of a self-focusing nonlocal nonlinear Schrödinger system. Using a combination of moment techniques, time dependent variational methods, and numerical simulations, we present evidence in support of the hypothesis that nonlocal attractively interacting condensates cannot collapse under very general forms of the interaction. Instead there appear oscillations of the wave packet with a localized component whose size is of the order of the range of interactions. We discuss the implications of the results to collapse phenomena in Bose-Einstein condensates.

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I. INTRODUCTION

When a system of weakly interacting bosons with two body interactions is cooled down below the Bose-Einstein transition temperature, it may be well described by the so called Gross-Pitaevskii equation for the order parameter of the superfluid Ψ [1]

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V(\mathbf{r})\Psi + U\Psi\int K(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2 d\mathbf{r}', \qquad (1.1)$$

where m is the mass of the bosons, $U=4\pi\hbar^2 a/m$ characterizes the two-body interaction, $V(\mathbf{r})$ is a real function describing an external action on the system, and $K(\mathbf{r}-\mathbf{r}')$ is the function which possesses information on the mutual interaction between the bosons. This model has been long known in the framework of the theory of ultracold systems [2]. Equation (1.1) is a nonlocal, nonlinear wave equation whose analysis is not trivial. Only recently have there been studies considering particular cases in the contexts of nonlinear optics [3–6], Quantum mechanics [7], electromagnetic wave self-action [8], and other fields [9]. The applicability of these types of models has grown enormously after the experimental realization of Bose-Einstein condensation with ultracold atomic gases [10,11] for which they provide an accurate description if the temperature is low enough.

Presently only few rigorous results and some qualitative estimates are available for particular cases. To our knowledge, the only related field where nonlocal equations of nonlinear Schrödinger (NLS) type have been studied in detail and with mathematical rigor is scattering theory [12], but that field is completely disconnected with our purposes here. This difficulty is the reason why all the theoretical efforts related to a description of current experiments on Bose-Einstein condensation have concentrated on the case where the interactions between the constitutive bosons are very short ranged, which is a good approximation for neutral atoms in normal situations. In this limit, to be described in detail below, the interaction kernel is taken as a Dirac δ function, and Eq. (1.1) becomes a local cubic NLS equation for which many more things are known. There is, however, at least one situation where this approximation is not valid, which is the case of collapse.

The problem of collapses in nonlinear wave equations was studied extensively in the literature [13–16]. In the collapse phenomenon the amplitude of a physical quantity becomes infinite at a particular time, and usually the mathematical model is no longer representative of the physics of the problem. This is the case in many particular instances of collapses in plasma physics [17], nonlinear optics [18], and many other examples.

In the NLS equation the cubic nonlinearity arising in the case of local interactions is characterized by a parameter, the scattering length a, whose sign determines the type of interactions. When $a<0$ the interaction among the particles in the condensate is attractive, and self-interaction leads to collapse in dimensions higher than or equal to 2 [19–21], a result expected from the usual case in NLS models without external potentials $(V=0)$. In fact, experiments which tried to generate Bose-Einstein condensates, with atoms having negative scattering lengths found a critical number of par-

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ticles [11] above which the condensate is unstable and destroyed by the collapse phenomenon [11,19–21]. Despite the difficulties of generating large negative scattering length condensates by evaporative cooling [11,22] one might generate them using other procedures [23,24]. In this case the question is what would be the dynamics of a large condensate with a negative scattering length. Since collapse implies the generation of large amplitude localized density peaks, it is clear that the local interaction approximation breaks down, and one has to consider the problem in more detail, including nonlocal interactions. There are other cases where nonlocal interactions appear, such as Bose-Einstein condensates with magnetic dipole forces [25].

This is the motivation for our analysis, which will concern the existence of collapse, and the dynamics of quasicollapse events, as well as other dynamical features of the problem. In fact this problem is not completely new; three works concentrated on it previously. The first one [5] considered only the one-dimensional case, and is not relevant for our purposes. The second one [26] was devoted to an analysis of the case without an external trapping force $V(\mathbf{r})=0$. It was proven in Ref. [26] for several physically relevant particular potentials that solutions realizing the absolute minimum of the respective nonlocal Hamiltonian are stable. To the best of our knowledge, Ref. [26] is the first place where the idea that nonlocal interactions could suppress collapse may be found. Finally, in the context of Bose-Einstein condensation there was a work in which nonlocality effects were considered [27]. The authors used the gradient expansion to obtain a local interaction energy, and then performed a time independent variational analysis with a Gaussian ansatz to conclude that nonlocal interactions could prevent collapse. However, the results obtained so far either were based on qualitative arguments (the gradient expansion is not applicable in the region where the narrowing of the wave packet is stopped), or obtained for particular models and concentrated on static predictions. All of them leave open a question about the dynamics of the nonlocal interacting condensate ''near the collapse.''

In our paper we complement qualitative arguments, justifying that a nonlocal potential of a *rather general* type should prevent collapse, by a time dependent variational analysis, numerical simulations, and the analysis of the strongly nonlocal limit. Our interest will be twofold: first, from the point of view of nonlinear science, we are interested in understanding the effect of nonlocal terms on the collapse problem; second, although the results are to appear in the NLS equation in any field, we will interpret them from the viewpoint of Bose-Einstein condensation phenomena, and thus the external trapping potential will be included in our model.

Our detailed plan is as follows. In Sec. II we present model equations with a nonlocal term. A formal analysis of the strongly nonlocal limit analysis and wave packet width evolution using the moment method is done in Sec. III. In Sec. IV we present some analytical approximations to obtain a qualitative description of the collapse dynamics in all the regimes. In Sec. V we present the results of numerical simulations of the full model, and compare the results with analytical predictions of previous sections. Finally, in Sec. VI we summarize our conclusions.

II. MATHEMATICAL MODEL

Let us specify the problem: introduce the normalization for Ψ as $N=\int|\Psi|^2\, d^3\mathbf{r}$, and define the parabolic trapping potential by $V(\mathbf{r})=\frac{1}{2}m\nu^2(\lambda_x^2x^2+\lambda_y^2y^2+\lambda_z^2z^2)$ (see, e.g., in Ref. [20]). As discussed above, we keep the nonlocal interaction as it appears in the Hartree-Fock theory.

Let us write the equations using the new variables $\mathbf{r}_0=(x_0,y_0,z_0)=(x,y,z)/a_0$, $\tau=\nu t$, $\psi(\mathbf{r}_0)=\Psi(\mathbf{r})\sqrt{a_0^3/N}$, and $a_0=\sqrt{\hbar/m\nu}$. With these definitions Eq. (1.1) simplifies to

$$i\frac{\partial\psi}{\partial\tau}=-\frac{1}{2}\nabla_0^2\psi+\frac{1}{2}(\lambda_x^2x_0^2+\lambda_y^2y_0^2+\lambda_z^2z_0^2)\psi+U\left[\int K(\mathbf{r_0}-\mathbf{r}')|\psi(\mathbf{r}')|^2d\mathbf{r}'\right]\psi, \tag{2.1}$$

where $U=4\pi Na/a_0$. The particular shape of the kernel function depends strongly on the energy of the interaction and the geometry of the molecules involved. In general it is very difficult to know its precise shape except for very simple cases [28].

In this paper we are interested in basic general qualitative results, and we do not try to model the specific details of the interaction for any particular atom. This is why we will concentrate on the simplest case where the kernel depends on one parameter ϵ, related to the kernel ''size,'' which characterizes the range of interactions, such that

$$\lim_{\epsilon\to 0} K_\epsilon(\mathbf{r})=\delta(\mathbf{r}). \tag{2.2}$$

We will concentrate on spherically symmetric interaction kernels which model spherically symmetric molecules. The main implication of this fact is that the kernel depends only on the distance between two atoms, $K(|\mathbf{r}-\mathbf{r}'|)$. Other possibilities were proposed in the literature [25].

The question we will address in what follows is the possibility of blow-up with regular kernels K_ϵ, and the dynamics of processes related to concentration of the solution of Eq. (2.1). This is a very intricate mathematical problem, which even in the simplest δ-interaction case (cubic nonlinearity) has not been completely solved; only a few estimates on collapse conditions exist. Thus we will not provide mathematically rigorous proofs, but only join numerical simulations and approximate analytical techniques to understand the problem.

III. STRONGLY NONLOCAL LIMIT

A. Qualitative arguments

Let us start with some arguments showing that nonlocality of a *rather general type* can prevent collapse. Although it is not essential, the algebra is simplified by assuming cylindrically ($\lambda_x=\lambda_y=1$) or spherically symmetric ($\lambda_x=\lambda_y=\lambda_z=1$) traps. We note that there exist at least two integrals of motion of Eq. (2.1),

$$N=\int|\psi|^2\,d\mathbf{r}, \tag{3.1a}$$

$$H=\frac{1}{2}\int\,(|\nabla\psi|^2+J(\mathbf{r})|\psi|^2+\rho^2|\psi|^2)\,d\mathbf{r},\qquad(3.1b)$$

where $\rho=|\mathbf{r}|$ and

$$J(\mathbf{r})=U\int\,K(\mathbf{r}-\mathbf{r}')|\psi(\mathbf{r}')|^2\,d\mathbf{r}',\qquad(3.2)$$

which have the usual meaning of the intensity (in the case of Bose-Einstein condensation applications, it is interpreted as the number of particles) and the energy of the wave ($i\psi_\tau=\delta H/\delta\overline{\psi}$). Let us now consider the situation ''near'' the collapse, assuming that the wave function is strongly localized. More precisely, it will be assumed that the localization region of the solution l is much less than the range of the nonlocal interactions ϵ, $l\ll\epsilon$ [29].

First we concentrate on the case when the kernel function is nonsingular at $\mathbf{r}\to 0$. Then in the leading approximation one can approximate

$$J(\mathbf{r})=UNK(\mathbf{r})+O(l/\epsilon).\qquad(3.3)$$

Making the natural supposition (not essential for the final result) that if collapse occurs it occurs at $\mathbf{r}=0$ (i.e., at the minimum of the confining potential), one finds the following linear equation for the wave function:

$$i\frac{\partial\psi}{\partial\tau}=-\frac{1}{2}\nabla_0^2\psi+UNK(0)\psi.\qquad(3.4)$$

In Eq. (3.4) we have neglected the confining potential, since it is of order of $l^2\nu\ll 1$.

As it is evident, Eq. (3.4) does not display collapse. Moreover, the dispersion will lead to spreading out of any initially localized wave packet.

On the other hand, if the width $\langle\rho^2\rangle$ is much larger than the range of interactions $l\gg\epsilon$, the δ-function limit is applicable, and the dominant nonlinearity will lead to collapse. Thus we conclude that there must exist *two different tendencies*: a spreading for $l\ll\epsilon$ and a narrowing for $l\gg\epsilon$ of the wave packet. This must lead to oscillations of the wave packet width between different scales, the smaller one being of the order of the interaction range, a fact which we analyze below. As a matter of fact the limit $l\ll\epsilon$ opposite to the limit of local interactions, and can be interpreted as the case of infinite range interactions, $K(\mathbf{r})\equiv K(0)$.

The above qualitative arguments can be generalized to the case of a singular kernel, which we represent in the form

$$K(\mathbf{r}-\mathbf{r}')=\frac{\tilde{K}(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

where $\tilde{K}(0)\neq 0$. Then Eq. (3.2) can be approximated by

$$J(\mathbf{r})\approx U\tilde{K}(0)\int\frac{|\psi(\mathbf{r}')|^2}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'.\qquad(3.5)$$

Next we observe that since the functions in the integrand of the right hand side of this equation are positive, there exists a function on τ, $R(\tau)$, making sense of effective radius of the wave function, such that

$$\int\,\frac{|\psi(\mathbf{r}')|^2}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'=\frac{N}{R(\tau)}\qquad(3.6)$$

and $R(\tau)>0$ for any τ *before* collapse occurs (if it occurs). On the other hand, one can use the inequality

$$\int\,\frac{|\psi(\mathbf{r}')|^2}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\leqslant 2N^{1/2}\left(\int\,(\nabla|\psi|)^2d\mathbf{r}\right)^{1/2}$$

proven in Ref. [26], which means that for a localized wave function one can estimate $R(\tau)\sim l$, where l is before a size of localization region. On the other hand, in the limit $l\to 0$, the kinetic term $\nabla_0^2\psi\sim l^{-2}$ becomes dominant, which means that the dispersion dominates the nonlinearity which is $\sim l^{-1}$. Balance between these two effects takes place at some $l_{eff}>0$. Thus in the regime of strong localization (i.e., the quasicollapse regime) the equation describing the dynamics of the wave packet reads

$$i\frac{\partial\psi}{\partial\tau}=-\frac{1}{2}\nabla_0^2\psi+\frac{UN\tilde{K}(0)}{l_{eff}}\psi.\qquad(3.7)$$

[cf. Eq. (3.4)]. Thus all above arguments about the ''near collapse'' behavior of the wave function can be applied to this last case, as well.

B. Radially symmetric problem

Let us consider now a particular model when $K\equiv K(\rho-\rho')$ (we call it the radially symmetric problem). Then it is also possible to argue that the collapse does not occur using the more conventional language of momenta [21,30]. To this end let us define the mean squared width of the wave packet,

$$\langle\rho^2\rangle=2\pi(n-1)\int_0^\infty|\psi(\rho)|^2\rho^{n+1}d\rho,\qquad(3.8)$$

where $n=2$ and 3 is the spatial dimension. Then it is a straightforward algebra to obtain

$$\frac{d^2}{d\tau^2}\langle\rho^2\rangle=4H-4\langle\rho^2\rangle-4\pi(n-1)$$
$$\times\int_0^\infty|\psi|^2\left[J(\rho)+\frac{1}{2}\rho J'(\rho)\right]\rho^{n-1}d\rho.\qquad(3.9)$$

In the limit of an infinite range of interactions, one finds [in this limit $J(\rho)=UNK(0)$]

$$\frac{d^2}{d\tau^2}\langle\rho^2\rangle=4H-4\langle\rho^2\rangle-2UN^2K(0).\qquad(3.10)$$

The solution of Eq. (3.10) reads

$$\langle\rho^2\rangle=\rho_0^2\sin(2\tau+\phi_0)+C,\qquad(3.11)$$

where $C=H-UN^2K(0)/2$, and ρ_0 and ϕ_0 are real constants. It follows from the energy conservation law [Eq. (3.1b)] that at $l\to 0$ one has $2H\geqslant UN^2K(0)+\langle\rho^2\rangle$, i.e., $C\geqslant\langle\rho^2\rangle/2$. Combining the last estimate with Eq. (3.11), one concludes that $C \geqslant \rho_0^2 \sin(2\tau+\phi_0)$. In other words $\langle\rho^2\rangle$ is always positive $\langle\rho^2\rangle \geqslant 0$. This result rules out the possibility that there could exist a collapse in the sense that all the solution became concentrated on one particular point (e.g., a δ-like singularity).

IV. ANALYTICAL RESULTS FOR THE GENERAL CASE

A. Exact results for the center of mass

It was pointed out in previous works [31,32] that the center of mass, defined by

$$\langle \mathbf{r}_0 \rangle = \int \mathbf{r} |\psi|^2 d\mathbf{r}, \tag{4.1}$$

performs harmonic oscillations no matter what the nonlinear interaction is. This result is also valid for the nonlocal interaction case under very general conditions, as follows from the extension of Erhenfest theorem to our problem:

$$\frac{d^2}{d\tau^2}\langle x_0\rangle + \lambda_x^2 \langle x_0\rangle = 0, \tag{4.2a}$$

$$\frac{d^2}{d\tau^2}\langle y_0\rangle + \lambda_y^2 \langle y_0\rangle = 0, \tag{4.2b}$$

$$\frac{d^2}{d\tau^2}\langle z_0\rangle + \lambda_z^2 \langle z_0\rangle = 0. \tag{4.2c}$$

To obtain additional exact information on this problem, one possibility is to use the moment method [33–35] in radial symmetry. Moreover, it can be seen that it does not provide exact results for our case. Even the use of the uniform divergence approximation [34] is not possible, and the only possibility is to restrict to the classical time-dependent variational techniques, which we consider in Sec. IV B.

B. Time dependent variational formalism

Following the standard procedure used extensively in the framework of collapse problems for nonlinear Schrödinger equations [36], we first identify a Lagrangian density for problem (1.1), which is

$$\mathcal{L} = \frac{i\hbar}{2}\left(\Psi \frac{\partial \Psi^*}{\partial t} - \Psi^* \frac{\partial \Psi}{\partial t}\right) + \frac{\hbar^2}{2m}|\nabla\Psi|^2 + V(r)|\Psi|^2 + \frac{U}{2}\int K(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r})|^2 |\Psi(\mathbf{r}')|^2 \, d\mathbf{r}', \tag{4.3}$$

where the asterisk denotes complex conjugation. That is, instead of working with the Gross-Pitaevskii equation, we can treat the action,

$$S = \int \mathcal{L} \, d^3 r \, dt = \int_{t_i}^{t_f} L(t)\, dt, \tag{4.4}$$

where t_i and t_f are initial and final times, and study its invariance properties and extrema, which are in turn solutions of Eq. (1.1).

To simplify the problem, we restrict the shape of the function Ψ to a convenient family of trial functions and study the time evolution of the parameters that define that family. A natural choice, which corresponds to the exact solution in the linear limit ($U=0$), ensures a good asymptotic behavior, and has provided good results in our previous works on related systems [20,31], is an n-dimensional Gaussian-like function

$$\Psi(x,y,z,t) = A \prod_\eta \exp\left\{ \frac{-[\eta-\eta_{CM}]^2}{2w_\eta^2} + i\eta\alpha_\eta + i\eta^2\beta_\eta \right\}, \tag{4.5}$$

where A (amplitude), w_η (width), α_η (slope speed), β_η (square root of the curvature radius), and η_0 (center of the cloud) are free parameters. We are considering n as a free parameter that can be set to 2 or 3 depending on the dimensionality of the problem considered. In the applications of these models to Bose-Einstein condensation phenomena, many systems are three dimensional; however, in certain situations a two-dimensional condensate can be considered a good theoretical model [37], and is easier to compare with numerical simulations of Eq. (1.1). The procedure for deriving equations for the parameters has been described in previous works [20,31] and will not be repeated here.

To go on with the analysis, a particular shape must be chosen for the kernel function. For simplicity, we will consider a simple n-dimensional Gaussian kernel of the form

$$K(\mathbf{r}) = \left(\frac{1}{2\pi\epsilon^2}\right)^{n/2} e^{-\mathbf{r}^2/2\epsilon^2}. \tag{4.6}$$

The main equations obtained when computing the evolution equations for these kinds of systems are those related to the width. To do this, it is useful to introduce a set of rescaled variables for time, $\tau = \nu t$, and the widths $w_\eta = a_0 v_\eta$ ($\eta = x,y,z$). For an n-dimensional condensate the equations are found to be

$$\frac{d^2 v_k}{d\tau^2} + \lambda_k^2 v_k = \frac{1}{v_k^3} + \frac{P v_k}{v_k^2+\delta^2} \prod_\eta \frac{1}{(v_\eta^2+\delta^2)^{1/2}}, \tag{4.7}$$

where $P = \sqrt{2/\pi} N a / a_0$ (the strength of the atom-atom interaction) and $\delta = \epsilon / a_0$. The remaining parameters α_η and β_η obey separate equations, but essentially can be computed independently once $v_k(t)$ are known.

System (4.7) is generated by the Hamiltonian

$$H = \frac{1}{2}\sum_\eta \dot{v}_\eta^2 + \frac{1}{2}\sum_\eta \left(\lambda_\eta^2 v_\eta^2 + \frac{1}{v_\eta^2}\right) + P\prod_\eta \frac{1}{(v_\eta^2+\delta^2)^{1/2}}. \tag{4.8}$$

In terms of system (4.7), collapse corresponds to the behavior when $v_k \to 0$. As a matter of fact the explicit expression of the Hamiltonian already shows that collapse is prevented by nonlocal interactions (i.e., by nonzero δ) in the framework of this simple model. Indeed one can easily see that H has a lower bound $H \geqslant P/\delta^n$.

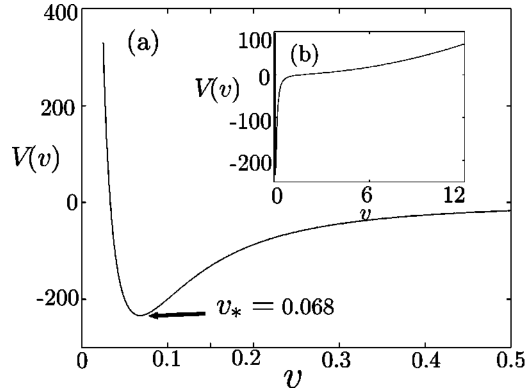

FIG. 1. Potential $V(v)$ for $P=-10$ and $\delta=0.1$. (a) Detail of the small scale. (b) Large scale.

C. 2D case: Equilibrium points and minimum width

In the present section we will concentrate on two-dimensional (2D) condensates for the sake of comparison with numerical simulations of Eq. (2.1), which will be presented in Sec. V. In this case the equations are

$$\frac{d^2 v_x}{d\tau^2}+\lambda_x^2 v_x=\frac{1}{v_x^3}+\frac{P v_x}{(v_x^2+\delta^2)^{3/2}(v_y^2+\delta^2)^{1/2}}, \quad (4.9a)$$

$$\frac{d^2 v_y}{d\tau^2}+\lambda_y^2 v_y=\frac{1}{v_y^3}+\frac{P v_y}{(v_y^2+\delta^2)^{3/2}(v_x^2+\delta^2)^{1/2}}. \quad (4.9b)$$

Let us consider the case when $v_x=v_y=v$, which corresponds to a cylindrical symmetry around the center of the wave function. When the solution has the symmetry of the external potential (i.e., $\eta_{CM}=0$) this corresponds to the usual cylindrically symmetric case, since the wave function amplitude depends only on $\mathbf{r}$. In this case the equations are simpler:

$$\frac{d^2 v}{d\tau^2}+v=\frac{1}{v^3}+\frac{P v}{(v^2+\delta^2)^2}. \quad (4.10)$$

This equation can be obtained from the potential

$$V(v)=\frac{1}{2}v^2+\frac{1}{2v^2}+\frac{P}{2(v^2+\delta^2)}, \quad (4.11)$$

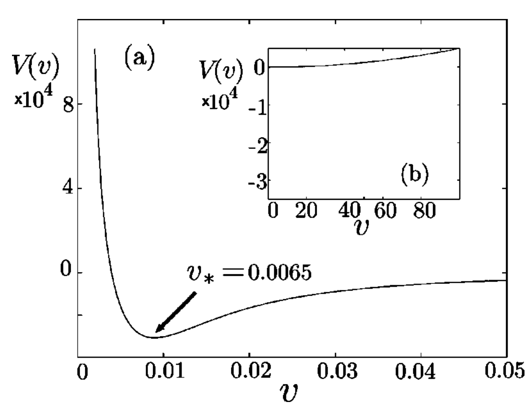

FIG. 2. Potential $V(v)$ for $P=-10$ and $\delta=0.01$. (a) Detail of the small scale. (b) Large scale.

which is plotted in Figs. 1 and 2 for some particular parameter values. It can be seen that even when P is negative, corresponding to negative scattering length, the potential is repulsive at the origin, so that no blowup is possible.

The case of interest in applications is that of small δ values. In this limit the potential has two scales. For $v=O(\delta)$ the parabolic term can be neglected, since the last two terms are dominant. For $v=O(\infty)-O(\infty\delta)$ the last term can be neglected (at least when P is not large), and the potential is dominated by the parabolic term. The existence of two different length scales in the potential is also clear in Figs. 1 and 2.

The equilibrium points v_* are the solutions of the algebraic equations

$$v=\frac{1}{v^3}+\frac{Pv}{(v^2+\delta^2)^2}. \quad (4.12)$$

The $\delta=0$ case was analyzed in Ref. [20]; now the situation is quite different, since collapse is not possible. In a generic case one can show that there exists only one positive root of Eq. (4.12). To this end we introduce a new variable z, $v/\delta=e^z$, and rewrite Eq. (4.12) in the form

$$\sinh[2(z+z_0)]\cosh^2 z-\frac{1}{4}Pe^{2z_0}=0, \quad (4.13)$$

where $\delta=e^{z_0}$. The left hand side of this equation is a monotonic function of z and thus there exists only one real root of Eq. (4.13). Moreover the root is finite for $\delta\neq 0$, which means that the collapse ($v=0$) is not possible. This result is in agreement with the considerations of Sec. III.

Let us concentrate on the small δ case. To do this we define a new variable $q=(v/\delta)^2>0$, so that Eq. (4.12) becomes

$$\delta^4=\frac{1}{q^2}+\frac{P}{(q+1)^2}. \quad (4.14)$$

Since we are going to deal with small δ values, we can neglect the left hand side [38] and, after some algebra, obtain the only equilibrium point as $q_*=1/\sqrt{|P|}-1$, provided $|P|>1$, which implies that

$$v_*\simeq\frac{\delta}{\sqrt{\sqrt{|P|}-1}}, \quad (4.15)$$

and is consistent with our previous assumptions on the linear term. The fact that the equilibrium point depends linearly on δ is interesting and provides a first estimate of the order of magnitude of the turning point of the potential which is then roughly $O(\delta)$ for a wide range of initial energies. We would like to point out that this estimate is a lower bound for the minimum width, since it could happen (as it does) that there is only part of the solution of Eq. (1.1) which tends to collapse, and then the contribution of the noncollapsing part will make the width finite. In that case our estimate would correspond roughly to the width of the collapsing peak.

If $|P|\approx 1$, Eq. (4.15) cannot be applied, since then the denominator can be small. In this case one obtains the law $v_*=(2^{1/2}\delta)^{1/3}$.

It is also possible to compute the frequency of the small oscillations around the potential minimum, which defines a time scale that could be of the order of oscillations found in the condensate dynamics due to the competition between nonlocal dispersion and trapping forces:

$$\Omega=\sqrt{1+\frac{3}{v_*^3}+P\frac{(3v_*^2-\delta^2)}{(v_*^2+\delta^2)^3}}. \tag{4.16}$$

An important feature of this formula is that the frequency grows as $v_*\to 0$, i.e., as the range of interactions δ, goes to zero. Later we will check the validity of these predictions.

D. 3D case: Equilibrium points and minimum width

Considering again the simplifying assumption that $v_x=v_y=v_z=v$, the dynamical equations in this case are

$$\frac{d^2v}{d\tau^2}+v=\frac{1}{v^3}+\frac{Pv}{(v^2+\delta^2)^{5/2}}, \tag{4.17}$$

and the potential

$$V(v)=\frac{1}{2}v^2+\frac{1}{2v^2}+\frac{P}{3(v^2+\delta^2)^{3/2}}. \tag{4.18}$$

As before, collapse can be ruled out, since the singularity has been removed. However the equilibrium point satisfies a more complicated equation

$$v=\frac{1}{v^3}+\frac{Pv}{(v^2+\delta^2)^{5/2}}. \tag{4.19}$$

Again defining $q=v^2/\delta^2$, the reduced equation is

$$\delta^4=\frac{1}{q^2}+\frac{P}{\delta(q+1)^{5/2}}, \tag{4.20}$$

and the δ dependence of q is nontrivial. Without making any approximations Eq. (4.20) can be written as

$$\delta^2(q+1)^5(\delta^4q^2-1)^2-P^2q^4=0, \tag{4.21}$$

whose solutions for each (P,δ) pair provide the right equilibria v_*. Now using the z variables as in Sec. IV C, one can prove that there exists only one positive root of Eq. (4.21).

It is possible to investigate the orders of the different terms in Eq. (4.20), and the only simplifying assumption is that $q\ll 1$, since we expect now that collapse is stronger than before as is usual in three-dimensional problems. Using this assumption we find that $\delta/q^2=-P$, and thus

$$v_*\simeq\frac{\delta^{5/4}}{|P|^{1/4}}, \tag{4.22}$$

which is smaller than the two-dimensional equilibrium width. This is an indication that even though it seems plausible that collapse does not take place, the compression of the width due to frustrated collapse process is stronger than that of the two-dimensional case.

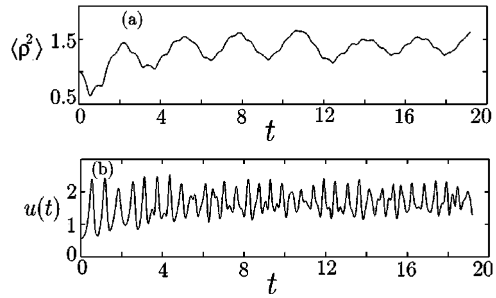

FIG. 3. Width (a) and amplitude (b) oscillations in a supercritical condensate with $U=-20$ and $\delta^2=0.05$. The frustrated collapse events appear as peaks in the condensate amplitude (maximum height).

V. NUMERICAL RESULTS

The analyses of Secs. III and IV share two common conclusions: (i) There should be a limit on the minimum width of the wave packet, predicted to be of the order of the interaction range by the Gaussian ansatz approach; and (ii) there must exist oscillations on the wave packet. In order to test these results and other predictions related to the dynamics of the condensate which we obtained during our approximate variational analysis, we have integrated numerically Eq. (2.1). In our numerical simulations we start with Gaussian initial data, and then compute the solution using a symmetrized second order in time Fourier pseudospectral method. Typical grid sizes range from 128×128 to 512×512 points. Simulation times in adimensional units were of the order of 50–100 (integration step $\Delta t=0.01$), which allows us to capture all the relevant dynamical features.

In our numerical simulations we studied the region of U values contained between the linear case ($U=0$) and a large value of $U=100$, which is about ten times above the threshold for collapse in the local case [39]. In practice we worked with $\delta^2\in[0.005,0.1]$.

The first conclusion of our analysis is that up to the precision of the computation we can conclude that *there is no collapse in all the situations analyzed*. Results of a typical simulation are shown in detail in Fig. 3. Two things are clear: first, the wave packet width oscillates with a minimum width of order $O(1)$; second, the maximum amplitude of the condensate performs oscillations with a dominant frequency.

One could expect that, according to Eq. (4.15), the minimum width of the wave packet was of order δ. However, that would be the case if the entire wave function were involved in the frustrated collapse events, which is not the case. As it happens, in collapse in the local nonlinear Schrödinger equation only part of the wave function takes part in the squeezing dynamics, the remaining part being responsible for the finite size width. In our case this is very clear in Fig. 4 where the two contributions to the solution are clearly seen, one being of order $O(1)$ and the other corresponding to a smaller scale l_c. It is also interesting how the low amplitude extended ''inert part'' oscillates according to the trap frequency, while the peak dynamics is ruled out by the nonlocal

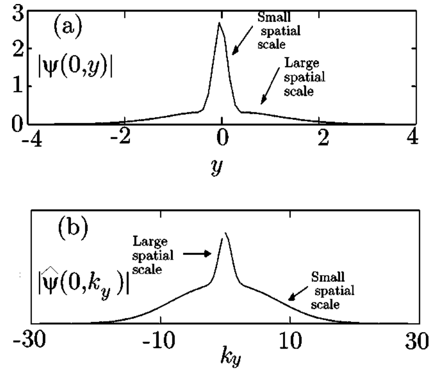

FIG. 4. (a) Trasversal section of the condensate amplitude $|\psi(0,y)|$ during a frustrated collapse event for $t \simeq 0.4$. The two contributions to the solution are very clear. The spatial spectrum (b) also has signatures of the existence of two scales.

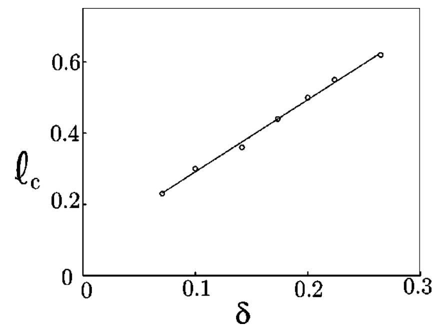

FIG. 6. Minimum width of the ''collapsing'' part l_c (the width of the small scale peak) as a function of δ for $U=-20$. The circles represent the values obtained by the numerical simulations of Eq. (2.1) and the solid line is a least squares interpolant, which proves the linear dependence on δ.

which again supports the variational analysis and provides additional insight on the problem [40].

dynamics, a phenomenon which is clear in the width oscillations of Fig. 3, where the two frequencies are present in the dynamics. In fact, the local oscillations were predicted in Sec. III, and also appear as oscillations in the potential well in the variational formalism. They correspond to ''frustrated collapse'' events, since if the nonlocality were not present the concentration dynamics would not stop at scale l_c but would continue up to infinity.

It is remarkable that even when the variational method does not take into account the existence of two scales, at least some of its predictions match reasonably well with the simulated dynamics of Eq. (2.1). For example, Fig. 5 shows the frequency of the amplitude oscillations of the condensate as a function of δ (circles). The variational estimates (solid line) are in qualitative agreement. This result cannot be improved in the framework of a simple variational analysis, since it corresponds to the Gaussian ansatz, and is derived for the oscillations near the bottom of the potential well.

Another relevant prediction of the variational analysis is the size of the small scale l_c generated during the quasicollapse event, which should be of order δ in two dimensions. To check this we have estimated the width of the collapsing peak using the small scale of the spatial representation of the ψ plot and the long scale of the spatial spectra for different values of δ. Our results are summarized in Fig. 6, were it can be seen that the width of the peak depends linearly on δ,

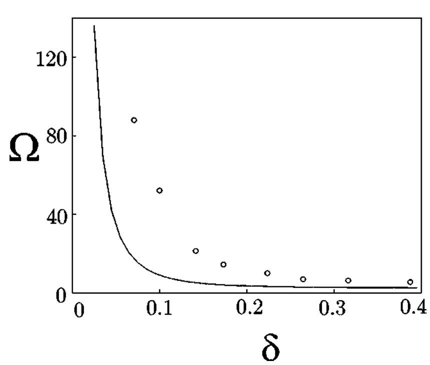

FIG. 5. Oscillation frequency of the condensate (circles) for $U=-20$ and different δ values against the variational estimate (solid line).

VI. CONCLUSIONS

In this paper we have studied a nonlocal nonlinear Schrödinger equation including a parabolic external potential. We have studied several characteristics of the dynamics of the ''quasicollapse'' processes. The analytical tools combine an exact analysis of the highly nonlocal limit, moment analysis, and collective coordinate analysis. Even though we do not present rigorous proofs, the predictions of all methods match consistently in support of our predictions.

Some of the analytical predictions were based upon the assumption of cylindrical or spherical symmetry of the wave function, although it is easy to generalize the results to other symmetries. From numerical solutions one sees that radially symmetric solutions are stable, and no asymmetric instabilities grow when one starts with symmetric initial data. In fact, in the context of local NLS equation collapse phenomena seem to be radially symmetric proceses [42].

The analytical predictions were tested with a numerical scheme. Up to the range in which the numerical simulations can be trusted, we find a verification of all of our analytical predictions. Other relevant features of the dynamics such as the splitting of the solution on two different parts are found in the framework of numerical simulations.

The fact that there is no collapse in this system points out that the Hamiltonian is bounded below, as shown in Ref. [26] for several relevant instances of $K(\mathbf{r})$ in the system without external forces [i.e. $V(\mathbf{r})=0$)]. Thus one may construct a ground state for this system so that stationary states could probably be obtained. This fact could find applications in the framework of ultracold gases, where it could imply the existence of large stable negative scattering length condensates.

Our study was restricted to symmetric kernels. It is also of significant interest to consider models with essentially anysotropic nonlocal kernels. In the case when nonlocal interactions exist along all three (in the 3D case) or two (in the 2D case) directions, the qualitative arguments presented in Sec. III still hold, and nonlocal interactions should prevent collapse. The most interesting (at least from a mathematical point of view) situation appears when nonlocality is present only in one or two (in the 3D case) directions, while in other directions the kernel is local. This problem seems to be nontrivial, and must be studied separately.

From a fundamental point of view our analysis is, to our knowledge, the first systematic study of the behavior of the effect of nonlocal interacting Bose-Einstein condensates, taking into account different aspects: variational analysis, moment approximations, and numerical techniques. We hope this study will provide a new understanding of the complex phenomena involved in nonlocal nonlinear wave equations.

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