Analog of the Wigner-Moyal equation for the electromagnetic field

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The evolution equation for the Wigner distribution of the classical electromagnetic field is derived for a nonstationary and inhomogeneous optical medium, which is formally similar to the Wigner-Moyal equation for a quantum system. The geometric optics approximation is discussed in detail, and the conservation equation for the number of photons is justified. The influence of dispersion is also considered.

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I. INTRODUCTION

The Wigner function was introduced in order to represent the quantum state of a system in the corresponding classical phase space [1]. This is a very important concept which can be used to establish a link between the wave and particle manifestations of the quantum fields, and it is currently used in quantum optics [2,3]. It is also well known that the space and time evolution of the Wigner distribution is described by the Wigner-Moyal equation [4], which reduces, in the classical limit, to the one-particle Liouville equation.

An interesting aspect of the electromagnetic field is that its wave and particle properties can be completely described in purely classical terms: the electromagnetic waves are described by Maxwell's equations, and the photon trajectories are described by the ray equations of the geometric optics approximation. It is the aim of the present work to establish a link between these two kinds of classical descriptions by using the Wigner function of the electromagnetic field, and by deriving the corresponding evolution equation.

Quite recently, this approach was used for the particular case of waves propagating in a nonstationary plasma [5,6], where the Wigner function was used to define in general terms the photon occupation number, or number of photons $N_k(\vec{r},t)$, for modes propagating with wave vector $\vec{k}$, at a position $\vec{r}$ and time t, and an evolution equation was established. Actually, in Ref. [6], the Wigner-Moyal equation was already derived, but it was stated in a quite implicit way, and only for the plasma case.

Here we generalize this work in order to consider an arbitrary optical medium and to explicitly derive a general form of the Wigner-Moyal equation for the classical electromagnetic field. We also show that, in the geometric optics approximation, this equation reduces to the conservation equation for the number of photons N_k, which is formally identical to the one-particle Liouville equation, in analogy with the above mentioned results of the quantum theory.

This paper is organized in the following way. In Sec. II, we derive the Wigner-Moyal equation for the electromagnetic waves propagating in a nondispersive and nondissipative medium. In Sec. III, a simplified form of the Wigner function is introduced, which is valid for a spectrum of linear waves. The evolution equation for the number of photons is derived and the geometric optics approximation is discussed. In Sec. IV, the derivation of Sec. II is generalized for the case of dispersive media. Finally, in Sec. V, the results are summarized.

II. NONDISPERSIVE MEDIUM

We first consider a nondispersive medium, in order to clearly state our approach. We also assume that the medium is isotropic and nondissipative. In the absence of charge and current distributions, we have, from Maxwell's equations,

$$\nabla^2\vec{E}-\nabla(\nabla\cdot\vec{E})-\mu_0\frac{\partial^2}{\partial t^2}\vec{D}=0, \tag{1}$$

where $\vec{D}=\epsilon_0\epsilon\vec{E}$ is the displacement vector. We know that $\epsilon=1+\chi$, where χ is the susceptibility of the medium. Assuming, for simplicity, that the fields are transverse ($\nabla\cdot\vec{E}=0$), we can write

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\chi\vec{E}). \tag{2}$$

In general, the transverse field approximation is not valid for an arbitrary inhomogeneity. However, it is well known that for weak inhomogeneities, where the properties of the medium vary on a scale much larger that the characteristic field wavelengths, such an approximation is valid, and it is commonly used in optical media and plasmas.

Now we use the notation $\vec{E}_i\equiv\vec{E}(\vec{r}_i,t_i)$ and $\chi_i\equiv\chi(\vec{r}_i,t_i)$, for $i=1$ and 2, and we can write

$$\left(\nabla_1^2-\frac{1}{c^2}\frac{\partial^2}{\partial t_1^2}\right)\vec{E}_1=\frac{1}{c^2}\frac{\partial^2}{\partial t_1^2}\chi_1\vec{E}_1, \tag{3}$$

$$\left(\nabla_2^2-\frac{1}{c^2}\frac{\partial^2}{\partial t_2^2}\right)\vec{E}_2=\frac{1}{c^2}\frac{\partial^2}{\partial t_2^2}\chi_2\vec{E}_2. \tag{4}$$

Let us multiply the first of these equations by $\vec{E}_2^*$, and the complex conjugate of the second one by $\vec{E}_1$. Noting that, in the absence of losses, the refractive index is always real, $\chi_i=\chi_i^*$, we obtain, after subtracting the resulting two equations,

$$\left[(\nabla_1^2-\nabla_2^2)-\frac{1}{c^2}\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial t_2^2}\right)\right]C_{12}=\frac{1}{c^2}\left(\frac{\partial^2}{\partial t_1^2}\chi_1-\frac{\partial^2}{\partial t_2^2}\chi_2\right)C_{12}, \quad (5)$$

with

$$C_{12}=\vec{E}_1\cdot\vec{E}_2^*. \quad (6)$$

It should be noted that, for $\vec{r}_1=\vec{r}_2$ and $t_1=t_2$, this quantity reduces to the square of the electromagnetic field amplitude: $C_{12}(\vec{r}_1=\vec{r}_2,t_1=t_2)=|E(\vec{r},t)|^2$. For convenience, we introduce new space and time variables, such that

$$\vec{r}=\tfrac{1}{2}(\vec{r}_1+\vec{r}_2), \quad \vec{s}=\vec{r}_1-\vec{r}_2 \quad (7)$$

and

$$t=\tfrac{1}{2}(t_1+t_2), \quad \tau=t_1-t_2. \quad (8)$$

Using these variable transformations, we can easily realize that

$$\left(\frac{\partial^2}{\partial t_1^2}\chi_1-\frac{\partial^2}{\partial t_2^2}\chi_2\right)=\left(\frac{1}{4}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\tau^2}\right)(\chi_1-\chi_2)+\frac{\partial^2}{\partial t\partial\tau}(\chi_1+\chi_2). \quad (9)$$

This expression can be simplified by noting that τ is a fast time scale, and t is a slow time scale, as will become more obvious below. Furthermore, we can assume that the susceptibility χ is a slowly varying function of time, and that its dependence on the fast time variable τ is negligible. Using $(\chi_1+\chi_2)\simeq 2\chi$, we can then write this equation as

$$2\left(\nabla\cdot\nabla_s-\frac{\epsilon}{c^2}\frac{\partial^2}{\partial t\partial\tau}\right)C_{12}\simeq\frac{1}{c^2}(\chi_1-\chi_2)\frac{\partial^2}{\partial\tau^2}C_{12}+\frac{2}{c^2}\frac{\partial\chi}{\partial t}\frac{\partial}{\partial\tau}C_{12}. \quad (10)$$

We know that, by making a Taylor expansion of a function of time $f(t+\tau)$ around $f(t)$, we can obtain

$$f(t+\tau)=f(t)+\sum_{m=1}^{\infty}\frac{1}{m!}\tau^m\frac{\partial^m}{\partial t^m}f(t)\simeq f(t)+\tau\frac{\partial f(t)}{\partial t}+\cdots. \quad (11)$$

This can be written in a more elegant and more compact form, by using an exponential operator, as

$$f(t+\tau)=\exp\left(\tau\frac{\partial}{\partial t}\right)f(t). \quad (12)$$

A power series development of this exponential operator clearly shows that this is equivalent to Eq. (11). Similarly, a function of the coordinates $f(\vec{r}+\vec{s})$ can be expanded around $f(\vec{r})$ as

$$f(\vec{r}+\vec{s})=\exp(\vec{s}\cdot\nabla)f(\vec{r}). \quad (13)$$

This means that, by performing a double (space and time) Taylor expansion of the susceptibilities χ_1 and χ_2 around $\vec{r}$ and t, we obtain

$$\chi_1=\chi(\vec{r}+\vec{s}/2,t+\tau/2)=\exp\left(\frac{\vec{s}}{2}\cdot\nabla+\frac{\tau}{2}\frac{\partial}{\partial t}\right)\chi(\vec{r},t) \quad (14)$$

and

$$\chi_2=\chi(\vec{r}-\vec{s}/2,t-\tau/2)=\exp\left(-\frac{\vec{s}}{2}\cdot\nabla-\frac{\tau}{2}\frac{\partial}{\partial t}\right)\chi(\vec{r},t). \quad (15)$$

The difference between the two values of the susceptibility of the medium can now be written as

$$(\chi_1-\chi_2)=2\sinh\left(\frac{\vec{s}}{2}\cdot\nabla+\frac{\tau}{2}\frac{\partial}{\partial t}\right)\chi=2\sum_{l=0}^{\infty}\frac{1}{(2l+1)!}\left[\frac{\vec{s}}{2}\cdot\nabla+\frac{\tau}{2}\frac{\partial}{\partial t}\right]^{2l+1}\chi. \quad (16)$$

At this point it is useful to introduce the double Fourier transformation of C_{12}:

$$C_{12}\equiv C(\vec{r},\vec{s},t,\tau)=\int\frac{d\vec{k}}{(2\pi)^3}\int\frac{d\omega}{2\pi}F(\vec{r},t;\omega,\vec{k})e^{i\vec{k}\cdot\vec{s}-i\omega\tau}. \quad (17)$$

The corresponding inverse transformation is related with the electric field as shown:

$$F(\vec{r},t;\omega,\vec{k})=\int d\vec{s}\int d\tau C(\vec{r},\vec{s},t,\tau)e^{-i\vec{k}\cdot\vec{s}+i\omega\tau}=\int d\vec{s}\int d\tau\vec{E}\left(\vec{r}+\frac{\vec{s}}{2},t+\frac{\tau}{2}\right)\cdot\times\vec{E}^*\left(\vec{r}-\frac{\vec{s}}{2},t-\frac{\tau}{2}\right)e^{-i\vec{k}\cdot\vec{s}+i\omega\tau}. \quad (18)$$

This quantity is formally quite similar to the Wigner function for a quantum system [1]. Therefore we can call it the double (or, space and time) *Wigner function for the electric field*. It should be noted, however, that the usual Wigner function for quantum systems is not a double (space and time) but a single (space) or reduced quasidistribution. A single Wigner function for the electromagnetic field will also be defined below.

Replacing this definition in Eq. (10), we obtain

$$\left(\frac{\partial}{\partial t}+\frac{c^2\vec{k}}{\omega\epsilon}\cdot\nabla\right)F+\frac{\partial\ln\epsilon}{\partial t}F=i\frac{\omega}{2\epsilon}(\chi_1-\chi_2)F, \quad (19)$$

where $(\chi_1-\chi_2)$ is determined by Eq. (16). But, from the definition of F, we conclude that

$$\frac{\partial^m}{\partial \vec{k}^m} = (-i\vec{s})^m F, \quad \frac{\partial^m}{\partial \omega^m} = (i\tau)^m F. \tag{20}$$

This means that we can write, on the right-hand side of Eq. (19),

$$(\chi_1 - \chi_2)F = 2\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!}\left[\frac{1}{2}\frac{\partial}{\partial \vec{k}}\cdot\nabla - \frac{1}{2}\frac{\partial}{\partial\omega}\frac{\partial}{\partial t}\right]^{2l+1}\chi F. \tag{21}$$

Replacing this result in Eq. (19) we finally obtain

$$\left(\epsilon\frac{\partial}{\partial t} + \frac{c^2\vec{k}}{\omega}\cdot\nabla\right)F + \left(\frac{\partial\epsilon}{\partial t}\right)F = -\omega(\epsilon\sin\Lambda F), \tag{22}$$

where Λ is a differential operator, which acts both backwards on ϵ and forward on F. It can be defined by

$$\Lambda = \frac{1}{2}^{\leftarrow}\left[\frac{\partial}{\partial\vec{r}}\cdot\frac{\partial}{\partial\vec{k}} - \frac{\partial}{\partial t}\frac{\partial}{\partial\omega}\right]^{\rightarrow}. \tag{23}$$

The right and left arrows are used to indicate that, in both terms, the first differential operator acts backwardly on ϵ and the second one acts forwardly on F. The sine differential operator in Eq. (22) is, in fact, an infinite series of differential operators, according to

$$\sin\Lambda = \sum_{n=0}^{\infty}\frac{(-1)^l}{(2l+1)!}\Lambda^{2l+1}. \tag{24}$$

The result stated in Eq. (22) shows that, at the cost of such unusual operators, from Maxwell's equations we were able derive a closed evolution equation for the Wigner function F of the electric field. This is valid in quite general conditions, apart from our basic assumptions that the medium should be nondispersive and that the dielectric constant should only evolve on a slow space and time scale. Its relation with the geometric optics approximation will become apparent in Sec. III.

Equation (22) is formally quite similar to the Wigner-Moyal equation for quantum systems [1,4], except for the term on the time derivative of the refractive index, which has no equivalent in the quantum mechanical problem. For this reason this equation can be called the Wigner-Moyal equation for the electromagnetic field.

III. KINETIC EQUATION FOR PHOTONS

It is now useful to introduce a few simplifying assumptions. The first one is associated with the character of the electromagnetic spectrum. We can assume that such a spectrum is determined by a linear superposition of waves. For each spectral component, the value of the frequency ω has to satisfy the linear dispersion relation of the medium:

$$\omega = \omega_k = kc/\sqrt{\epsilon}. \tag{25}$$

The corresponding group velocity is

$$\vec{v}_k = \frac{\partial\omega_k}{\partial\vec{k}} = \frac{c}{\sqrt{\epsilon}}\frac{\vec{k}}{k} = \frac{c^2}{\omega_k\epsilon}\vec{k}. \tag{26}$$

In this case of a linear wave spectrum, the Wigner function F simplifies to

$$F \equiv F(\vec{r},t;\omega,\vec{k}) = F_k(\vec{r},t)\,\delta(\omega-\omega_k). \tag{27}$$

Replacing this in Eq. (17), and noting that the reduced Wigner function F_k is independent of ω, and consequently that

$$\frac{\partial^m F}{\partial\omega^m} = F_k\frac{\partial^m}{\partial\omega^m}\delta(\omega-\omega_k) = (-1)^m\delta(\omega-\omega_k)\frac{\partial^m F_k}{\partial\omega^m} = 0, \tag{28}$$

we can write the Wigner-Moyal equation in a simplified form:

$$\left(\frac{\partial}{\partial t} + \vec{v}_k\cdot\nabla\right)F_k + \frac{\partial\ln\epsilon}{\partial t}F_k = -\frac{\omega_k}{\epsilon}[\epsilon\sin\Lambda_k F_k]. \tag{29}$$

Here, Λ_k is a reduced differential operator defined by

$$\Lambda_k = \frac{1}{2}^{\leftarrow}\frac{\partial}{\partial\vec{r}}\cdot\frac{\partial}{\partial\vec{k}}^{\rightarrow}. \tag{30}$$

Because, in the Wigner-Moyal equation, the sine operators are usually too complicated to be explicitly calculated in specific problems, it can be useful to simply retain the first term in development (24):

$$\sin\Lambda_k \simeq \Lambda_k. \tag{31}$$

This is only valid for a slowly varying medium, where the gradients contained in the operator Λ_k are very small. In such a case, we are close to the conditions where the geometric optics approximation is valid. The Wigner-Moyal equation reduces to

$$\left(\frac{\partial}{\partial t} + \vec{v}_k\cdot\nabla\right)F_k + \left(\frac{\partial\ln\epsilon}{\partial t}\right)F_k \simeq -\frac{\omega_k}{2\epsilon}\left(\frac{\partial\epsilon}{\partial\vec{r}}\cdot\frac{\partial F_k}{\partial\vec{k}}\right). \tag{32}$$

On the other hand, if we neglect the logarithmic derivative in this equation, we note that it implies that a triple equality exists, namely,

$$dt = \frac{d\vec{r}}{\vec{v}_k} = \frac{d\vec{k}}{(\omega_k/2\epsilon)(\partial\epsilon/\partial\vec{r})}. \tag{33}$$

This is equivalent to stating that

$$\frac{d\vec{r}}{dt} = \vec{v}_k = \frac{\partial\omega_k}{\partial\vec{k}}, \tag{34}$$

$$\frac{d\vec{k}}{dt} = \frac{\omega_k}{2\epsilon}\frac{\partial\epsilon}{\partial\vec{r}} = \frac{kc}{2\epsilon^{3/2}}\frac{\partial\epsilon}{\partial\vec{r}} = -\frac{\partial\omega_k}{\partial\vec{r}}. \tag{35}$$

Here we recover the ray equations of the geometric optics approximation, written in Hamiltonian form. Clearly, $\vec{r}$ and $\vec{k}$

are the canonical variables, and the frequency ω_k is the Hamiltonian function. They are nothing but the characteristic equations of the simplified version of the Wigner-Moyal equation, which can then be written as

$$\frac{d}{dt}F_k \equiv \left(\frac{\partial}{\partial t} + \vec{v}_k \cdot \nabla + \frac{d\vec{k}}{dt}\cdot\frac{\partial}{\partial \vec{k}}\right)F_k \simeq 0. \tag{36}$$

This equation states the conservation of the Wigner function F_k, and it is valid when the logarithmic time derivative, as well as the higher order derivatives associated with the diffraction terms $l>0$ in the development of the sine operator $\sin\Lambda_k$, can be neglected.

Furthermore, by replacing Eq. (27) in the definition of C_{12}, we obtain

$$\begin{aligned} C_{12} &\equiv C(\vec{r},\vec{s},t,\tau) \\ &= e^{-i\omega_k\tau}\int F_k(\vec{r},t)e^{i\vec{k}\cdot\vec{s}}\frac{d\vec{k}}{(2\pi)^3} \\ &= e^{-i\omega_k\tau}C(\vec{r},\vec{s},t,\tau=0). \end{aligned} \tag{37}$$

According to Eq. (18), this simply means that we can define $F_k(\vec{r},t)$ as the space Wigner function for the electric field:

$$\begin{aligned} F_k(\vec{r},t) &= \int C(\vec{r},\vec{s},t,\tau=0)e^{-i\vec{k}\cdot\vec{s}}d\vec{s} \\ &= \int \vec{E}(\vec{r}+\vec{s}/2,t)\cdot\vec{E}^*(\vec{r}-\vec{s}/2,t)e^{-i\vec{k}\cdot\vec{s}}d\vec{s}. \end{aligned} \tag{38}$$

It is now useful to introduce the concept of the number of photons $N_k(\vec{r},t)$, defining it in terms of the reduced Wigner function, as

$$N_k(\vec{r},t) = \frac{\epsilon_0}{8\hbar}\left(\frac{\partial R}{\partial \omega}\right)_{\omega_k}F_k(\vec{r},t), \tag{39}$$

where $R=0$ is the dispersion relation of the medium. Such a definition was introduced in our recent work [5,6]. As it states, it can be applied to arbitrary forms of wave fields (plane, spherical or cylindrical waves). In particular, if we take the simple case of plane waves, such that $\vec{E}(\vec{r},t) = \vec{E}_0\exp(i\vec{k}_0\cdot\vec{r} - i\omega_0 t)$, this reduces to

$$N_k(\vec{r},t) = \frac{\epsilon_0}{8\hbar}\frac{\partial R}{\partial \omega}|E_0|^2\delta(\vec{k}-\vec{k}_0). \tag{40}$$

This is just the definition commonly found in the literature [7,8], which is not very useful to describe, e.g., short laser pulses. For the case considered here of a nondispersive medium, the dispersion relation $R=0$ can be written as

$$R \equiv R(\omega,\vec{k}) = \epsilon - c^2k^2/\omega^2 = 0. \tag{41}$$

The expression for the number of photons [Eq. (39)] is then reduced to

$$N_k(\vec{r},t) = \frac{\epsilon_0}{4\hbar}\frac{\epsilon}{\omega_k}F_k(\vec{r},t). \tag{42}$$

We can now return to the Wigner-Moyal equation (36), but including the logarithmic derivative of the refractive index, and rewrite it as

$$\frac{d}{dt}F_k = -\left(\frac{\partial \ln \epsilon}{\partial t}\right)F_k, \tag{43}$$

where the total derivative is determined by Eq. (36). On the other hand, if we take the total time derivative of the number of photons [Eq. (42)], and note that

$$\frac{d\omega_k}{dt} = \frac{\partial \omega_k}{\partial t} = -\frac{\omega_k}{2}\left(\frac{\partial \ln \epsilon}{\partial t}\right), \tag{44}$$

we can then obtain

$$\frac{d}{dt}N_k = \left[\left(\frac{1}{2}\frac{\partial}{\partial t} + \vec{v}_k\cdot\nabla\right)\ln\epsilon\right]N_k. \tag{45}$$

Neglecting the slow variations of the refractive index appearing on the right hand side, we can finally state an equation of conservation, for the number of photons, in the form

$$\frac{dN_k}{dt} \equiv \left(\frac{\partial}{\partial t} + \vec{v}_k\cdot\nabla + \frac{d\vec{k}}{dt}\cdot\frac{\partial}{\partial\vec{k}}\right)N_k = 0. \tag{46}$$

This equation simply states that the number of photons N_k is conserved. Of course, such a statement could be made by using simple physical arguments. But the present derivation has the advantage of using a precise and general definition for N_k. On the other hand, we understand from it that the conservation equation for the number of photons is only valid when the higher order terms contained in the sine operator of the Wigner-Moyal equation can be neglected. This means that these terms represent diffraction corrections to the geometric optics approximation.

IV. DISPERSIVE MEDIUM

The above derivation is conceptually quite interesting, because it establishes a clear link between the exact Maxwell's equations and a kinetic equation for photons. However, its range of validity is not very wide, because we have neglected dispersion. The generalization to the case of a dispersive medium is considered in this section. For simplicity, we still neglect the losses in the medium, which can easily be included in the calculations, as discussed at the end of this section. First of all, if the electromagnetic radiation propagates in a dispersive medium, our starting equation (1) is replaced by

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0\frac{\partial^2}{\partial t^2}\vec{P}, \tag{47}$$

where $\vec{P} = \epsilon_0\vec{E} + \vec{D}$ is the polarization vector. In general terms, this is related to the electric field $\vec{E}$ by the integral

$$\vec{P}(\vec{r},t) = \epsilon_0\int d\vec{r}'\int dt'\chi(\vec{r},t,\vec{r}',t')\vec{E}(\vec{r}-\vec{r}',t-t'). \tag{48}$$

Returning to the procedure followed in Sec. II, we can see that Eqs. (3) and (4) should be replaced by

$$\left(\nabla_i^2-\frac{1}{c^2}\frac{\partial^2}{\partial t_i^2}\right)\vec{E}_i=\mu_0\frac{\partial^2}{\partial t_i^2}\vec{P}_i \tag{49}$$

for $i=1$ and 2.

Again, from this we can derive an evolution equation for the quantity $C_{12}=\vec{E}_1\cdot\vec{E}_2^*$. The result is

$$\left[(\nabla_1^2-\nabla_2^2)-\frac{1}{c^2}\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial t_2^2}\right)\right]C_{12}=\mu_0\left[\frac{\partial^2}{\partial t_1^2}(\vec{P}_1\cdot\vec{E}_2^*)-\frac{\partial^2}{\partial t_2^2}(\vec{P}_2^*\cdot\vec{E}_1)\right]. \tag{50}$$

Let us now introduce the space and time variables defined by Eqs. (7) and (8). This equation becomes

$$2\left(\nabla\cdot\nabla_s-\frac{1}{c^2}\frac{\partial^2}{\partial t\partial\tau}\right)C_{12}=\mu_0\left(\frac{1}{4}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\tau^2}\right)(\vec{P}_1\cdot\vec{E}_2^*-\vec{P}_2^*\cdot\vec{E}_1)+\mu_0\frac{\partial^2}{\partial t\partial\tau}(\vec{P}_1\cdot\vec{E}_2^*+\vec{P}_2^*\cdot\vec{E}_1). \tag{51}$$

At this point, we can introduce the Fourier transformation

$$\vec{E}_i\equiv\vec{E}(\vec{r}_i,t_i)=\int\frac{d\omega_i}{2\pi}\int\frac{d\vec{k}_i}{(2pi)^3}\vec{E}(\omega_i,\vec{k}_i)e^{i\vec{k}_i\cdot\vec{r}_i-i\omega_it_i}. \tag{52}$$

A similar transformation for the polarization vector is defined by

$$\vec{P}_i\equiv\vec{P}(\vec{r}_i,t_i)=\int\frac{d\omega_i}{2\pi}\int\frac{d\vec{k}_i}{(2pi)^3}\vec{P}(\vec{r}_i,t_i;\omega_i,\vec{k}_i)e^{i\vec{k}_i\cdot\vec{r}_i-i\omega_it_i}, \tag{53}$$

where we have

$$\vec{P}(\vec{r}_i,t_i;\omega_i,\vec{k}_i)=\epsilon_0\chi(\vec{r}_i,t_i;\omega_i,\vec{k}_i)\vec{E}(\omega_i,\vec{k}_i). \tag{54}$$

The susceptibility $\chi(\vec{r},t;\omega,\vec{k})$, appearing in this expression, is assumed to be a slowly varying function of space and time. We can rewrite the quantity C_{12} in terms of the Fourier components of the electric field. But, because this would lead to quite cumbersome expressions, we prefer to introduce new frequency and wave vector variables, such that

$$\vec{q}=\vec{k}_1+\vec{k}_2,\quad \vec{k}=\tfrac{1}{2}(\vec{k}_1-\vec{k}_2) \tag{55}$$

and

$$\Omega=\omega_1+\omega_2,\quad \omega=\tfrac{1}{2}(\omega_1-\omega_2). \tag{56}$$

As in the case of the space and time variable transformations [Eqs. (7) and (8)], the Jacobian of the new transformations is equal to 1: $d\omega_1 d\omega_2=d\Omega d\omega$ and $d\vec{k}_1 d\vec{k}_2=d\vec{q}d\vec{k}$. In terms of these new variables, the quantity C_{12} becomes formally identical to Eq. (17), as it should, with the quantity $F(\vec{r},t;\omega,\vec{k})$ now defined as

$$F(\vec{r},t;\omega,\vec{k})=\int\frac{d\Omega}{2\pi}\int\frac{d\vec{q}}{(2\pi)^3}J(\vec{q},\vec{k},\Omega,\omega)e^{i\vec{q}\cdot\vec{r}-i\Omega t} \tag{57}$$

and

$$J(\vec{q},\vec{k},\Omega,\omega)=\vec{E}(\omega+\Omega/2,\vec{k}+\vec{q}/2)\cdot\vec{E}(-\omega+\Omega/2,-\vec{k}+\vec{q}/2). \tag{58}$$

Returning to Eq. (51), and retaining on its right hand side only the dominant term, the one proportional to $\partial^2/\partial\tau^2$, we can write

$$2\left[\nabla\cdot\nabla_s-\frac{1}{c^2}\frac{\partial^2}{\partial t\partial\tau}\right]C_{12}=-\frac{1}{c^2}\int\frac{d\Omega}{2\pi}\times\int\frac{d\vec{q}}{(2\pi)^3}(\eta_+-\eta_-)J(\vec{q},\vec{k},\Omega,\omega)\times e^{i\vec{q}\cdot\vec{r}-i\Omega t}e^{i\vec{k}\cdot\vec{s}-i\omega\tau}, \tag{59}$$

where, in order to simplify the expression, we have introduced the quantities

$$\eta_\pm=\left(\omega\pm\frac{\Omega}{2}\right)^2\chi\left(\vec{r}\pm\frac{\vec{s}}{2},t\pm\frac{\tau}{2};\omega\pm\frac{\Omega}{2},\vec{k}\pm\frac{\vec{q}}{2}\right). \tag{60}$$

Here we should note that $|\omega|\gg|\Omega|$, because ω is associated with the fast time scale τ, whereas the frequency Ω is associated with the slow time scale t. In the same way, we can assume that $|\vec{k}|\gg|\vec{q}|$. Developing these quantities around the values $(\omega,\vec{k})$ and $(\vec{r},t)$, we obtain

$$\eta_\pm\simeq\eta_0\pm\frac{\Omega}{2}\frac{\partial\eta_0}{\partial\omega}\pm\frac{\vec{q}}{2}\cdot\frac{\partial\eta_0}{\partial\vec{k}}\pm\frac{\tau}{2}\frac{\partial\eta_0}{\partial t}\pm\frac{\vec{s}}{2}\cdot\frac{\partial\eta_0}{\partial\vec{r}}+\cdots, \tag{61}$$

where we have considered that

$$\eta_0=\omega^2\chi(\vec{r},t;\omega,\vec{k}). \tag{62}$$

This means that, in Eq. (59), we can use

$$\eta_+-\eta_-=\left(\Omega\frac{\partial\eta_0}{\partial\omega}+\vec{q}\cdot\frac{\partial\eta_0}{\partial\vec{k}}\right)+\left(\tau\frac{\partial\eta_0}{\partial t}+\vec{s}\cdot\frac{\partial\eta_0}{\partial\vec{r}}\right). \tag{63}$$

But we also note that the quantity η_0 and its derivatives are independent of Ω and $\vec{q}$. This means that, in Eq. (59), they can be taken out of the integrals. This allows us to make the following replacements, in the same equation:

$$\int\frac{d\Omega}{2\pi}\int\frac{d\vec{q}}{(2\pi)^3}\Omega J(\vec{q},\vec{k},\Omega,\omega)e^{i\vec{q}\cdot\vec{r}-i\Omega t}=i\frac{\partial}{\partial t}F(\vec{r},t;\omega,\vec{k}) \tag{64}$$

and

$$\int \frac{d\Omega}{2\pi}\int \frac{d\vec{q}}{(2\pi)^3}\vec{q}J(\vec{q},\vec{k},\Omega,\omega)e^{i\vec{q}\cdot\vec{r}-i\Omega t}=-i\nabla F(\vec{r},t;\omega,\vec{k}). \tag{65}$$

The result is

$$2\left[\nabla\cdot\nabla_s-\frac{1}{c^2}\frac{\partial^2}{\partial t\partial\tau}\right]C_{12} =-\frac{1}{c^2}\int\frac{d\omega}{2\pi}\int\frac{d\vec{k}}{(2\pi)^3} \times\left[i\left(\frac{\partial\eta_0}{\partial\omega}\frac{\partial}{\partial t}-\frac{\partial\eta_0}{\partial\vec{k}}\cdot\nabla\right)F(\vec{r},t;\omega,\vec{k}) +\left(\tau\frac{\partial\eta_0}{\partial t}+\vec{s}\cdot\nabla\eta_0\right)F(\vec{r},t;\omega,\vec{k})\right]e^{i\vec{k}\cdot\vec{s}-i\omega\tau}. \tag{66}$$

We can now replace $C_{12}\equiv C(\vec{r},t,\vec{s},\tau)$ by its Fourier integral, as defined by Eq. (57), and obtain

$$2i\left(\vec{k}\cdot\nabla+\frac{\omega}{c^2}\frac{\partial}{\partial t}\right)F(\vec{r},t;\omega,\vec{k}) =-\frac{i}{c^2}\left(\frac{\partial\eta_0}{\partial\omega}\frac{\partial}{\partial t}-\frac{\partial\eta_0}{\partial\vec{k}}\cdot\nabla\right)F(\vec{r},t;\omega,\vec{k}) -\frac{1}{c^2}\int d\vec{s}\int d\tau\int\frac{d\omega'}{2\pi}\int\frac{d\vec{k}'}{(2\pi)^3} \times\left(\tau\frac{\partial\eta_0}{\partial t}+\vec{s}\cdot\nabla\eta_0\right)C(\vec{r},t,\vec{s},\tau) \times e^{i(\vec{k}'-\vec{k})\cdot\vec{s}}e^{-i(\omega'-\omega)\tau}. \tag{67}$$

But it is also clear that we can write

$$\int \tau e^{-i(\omega'-\omega)\tau}d\tau=i\frac{\partial}{\partial\omega'}\int e^{-i(\omega'-\omega)\tau}d\tau =2\pi i\,\delta(\omega'-\omega)\frac{\partial}{\partial\omega'} \tag{68}$$

and

$$\int \vec{s}e^{i(\vec{k}'-\vec{k}\cdot\vec{s})}d\vec{s}=-(2\pi)^3 i\,\delta(\vec{k}'-\vec{k})\frac{\partial}{\partial\vec{k}'}. \tag{69}$$

This means that we can finally transform Eq. (67) into a closed differential equation for the Wigner function $F\equiv F(\vec{r},t,\omega,\vec{k})$, which takes the form

$$2\omega\left(\frac{\partial}{\partial t}+\frac{c^2\vec{k}}{\omega}\cdot\nabla\right)F=-\left(\frac{\partial\eta_0}{\partial\omega}\frac{\partial F}{\partial t}-\frac{\partial\eta_0}{\partial\vec{k}}\cdot\nabla F\right) +\left(\frac{\partial F}{\partial\omega}\frac{\partial\eta_0}{\partial t}-\frac{\partial F}{\partial\vec{k}}\cdot\nabla\eta_0\right). \tag{70}$$

After rearranging the terms in this equation, we can rewrite it in a more suitable form,

$$\left(\frac{\partial}{\partial t}+\vec{v}_g\cdot\nabla\right)F=-\frac{2}{2\omega+\partial\eta_0/\partial\omega}(\eta_0\Lambda F), \tag{71}$$

where Λ is the differential operator defined by Eq. (23), and $\vec{v}_g$ is the group velocity defined by

$$\vec{v}_g=\frac{2c^2\vec{k}-\omega^2\partial\epsilon/\partial\vec{k}}{2\omega\epsilon+\omega^2\partial\epsilon/\partial\omega}, \tag{72}$$

with $\epsilon=1+\chi=1+\eta/\omega^2$.

It can easily be seen that, if we use the full development of $\eta_\pm$ around $(\vec{r},t)$, instead of the first terms, we can obtain the operator $\sin\Lambda$ instead of Λ, which is a characteristic feature of the Wigner-Moyal equation. The present approach therefore generalizes the above derivation of this equation to the case of a dispersive medium. Obviously, for a nondispersive medium, such that $\partial\eta_0/\partial\omega=0$, this would reduce to the result of Sec. II.

Let us assume that the electromagnetic wave spectrum is made of a superposition of linear waves, such that we can use Eq. (49): $F=F_k\delta(\omega-\omega_k)$. Then, Eq. (71) becomes

$$\left(\frac{\partial}{\partial t}+\vec{v}_k\cdot\frac{\partial}{\partial\vec{k}}+\frac{1}{(\partial\omega^2\epsilon/\partial\omega)_{\omega_k}}\frac{\partial\eta_k}{\partial\vec{r}}\cdot\frac{\partial}{\partial\vec{k}}\right)F_k=0, \tag{73}$$

where $\vec{v}_k=(\partial\omega/\partial\vec{k})_{\omega_k}$ and $\eta_k=\omega_k^2\chi(\vec{r},\vec{k},t)$.

As an example of a dispersive medium, we can consider an isotropic plasma, where we have $\eta_k=-\omega_p^2$, and ω_p is the electron plasma frequency. In this case, the gradient of η_k appearing in the last term of this equation reduces to the gradient of the electron plasma density, or equivalently, to the gradient of the square of the plasma frequency. We then have

$$\left(\frac{\partial}{\partial t}+\vec{v}_k\cdot\frac{\partial}{\partial\vec{k}}-\frac{1}{2\omega_k}\frac{\partial\omega_p^2}{\partial\vec{r}}\cdot\frac{\partial}{\partial\vec{k}}\right)F_k=0, \tag{74}$$

where $\omega_k=\sqrt{k^2c^2+\omega_p^2(\vec{r},t)}$. This is equivalent to stating that the reduced Wigner function F_k is conserved,

$$\frac{d}{dt}F_k\equiv\left(\frac{\partial}{\partial t}+\vec{v}_k\cdot\frac{\partial}{\partial\vec{k}}+\frac{d\vec{k}}{dt}\cdot\frac{\partial}{\partial\vec{k}}\right)F_k=0, \tag{75}$$

because we know, from the photon ray equations, that,

$$\frac{d\vec{k}}{dt}=-\frac{\partial\omega_k}{\partial\vec{r}}=-\frac{1}{2\omega_k}\frac{\partial\omega_p^2}{\partial\vec{r}}. \tag{76}$$

From these reduced forms of the Wigner-Moyal equation for a dispersive medium, we can then justify the use of the equation of conservation for the number of photons $N_k(\vec{r},t)$ [Eq. (46)], which can also be called the kinetic equation for photons propagating in slowly varying dispersive media.

It is well know that, due to causality, propagation in a dispersive medium always implies some wave dissipation. Our calculations are therefore only approximate, and can only be assumed to be valid if the relevant mode frequencies ω_k are much larger than the linear mode damping coefficients: $\gamma_k \ll \omega_k$. Such an approximation breaks down near a resonance of the optical medium. In such a case, on the right hand side of Eq. (75), we have to replace zero by the small quantity $-2\gamma_k F_k$.

Another limitation of the present calculation is that we have only retained a single polarization mode where, in general, there are two (for dielectric media) and three (for plasmas) independent polarizations. Each one would be described by a Wigner-Moyal equation like Eq. (75), to which we could also add linear coupling terms due to the inhomogeneities of the medium. We know that such a coupling is negligible, except when the refractive index of the different polarization states are nearly equal. Similarly, nonlinear mode coupling terms could be added to Eq. (75) by using perturbation methods [7,8].

Let us also briefly comment on the extension of our calculations to anisotropic media. In this case, the scalar quantity C_{12} would have to be generalized to a second-rank tensor $C_{ij}=E_iE_j$, but the results would be comparable. This and other generalizations of the present work will be discussed in a future work.

V. CONCLUSIONS

It was shown in this work that an evolution equation for the Wigner function of the classical electromagnetic field in a nonstationary and inhomogeneous medium can be derived. The simple case of wave propagation in a nondispersive medium was considered in detail, and its generalization to a dispersive medium was also considered. For the sake of clarity, the discussion was restricted to isotropic and nondissipative media, but the inclusion of anisotropy and dissipation in our calculations is straightforward.

The Wigner-Moyal equation derived here can be seen as a general transport equation for the number of photons. This work shows that, in general, the number of photons is not conserved. The validity conditions of the conservation of the number of photons, and of the geometric optics approximation, are clarified, and the first order corrections associated with diffraction are identified.

This work extends the Wigner-Moyal approach to the case of purely classical fields, and establishes the link between kinetic equation for the classical particles (photons) and the corresponding wave field equations. In that sense it can be extended to other classical fields, and can eventually be used to derive kinetic equations for other classical particles, such as (purely electrostatic) plasmons and phonons.

Finally, it should be noted that the classical Wigner function for the electromagnetic field is sometimes used to characterize ultrashort laser pulses with a time-dependent spectrum, as measured in optical experiments [9]. In contrast, an evolution equation of this quantity seems to have been ignored. The Wigner-Moyal equation, in its various versions described here, can eventually be used to understand the space-time evolution of such short pulses in an optical medium.

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