Synchronization of laser oscillators, associative memory, and optical neurocomputing

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We investigate here possible neurocomputational features of networks of laser oscillators. Our approach is similar to classical optical neurocomputing where artificial neurons are lasers and connection matrices are holographic media. However, we consider oscillatory neurons communicating via phases rather than amplitudes. Memorized patterns correspond to synchronized states where the neurons oscillate with equal frequencies and with prescribed phase relations. The mechanism of recognition is related to phase locking.

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I. INTRODUCTION

A new computing paradigm, neurocomputing, is emerging from modeling artificial neural networks. A typical neurocomputer is a network of simple analog units, artificial neurons, that process information in parallel. Instead of performing a general purpose computation via execution of a program, the neurocomputer performs pattern recognition and associative recall via self-organization of neurons [1].

Using optics instead of electronics provides some advantages for neurocomputing, mainly because optical signals can go through each other without significant distortion. Most optical neurocomputer architectures (see, e.g., [2–5] and references therein) consist of a lattice of neuronal elements interconnected via a holographic medium; see Fig. 1. The neurons interact via light intensities. This corresponds to amplitude modulation (AM) encoding when each neuron is a monochromatic laser. The phases do not play any role.

In this paper we explore the possibility that the laser oscillators interact via phases, which corresponds to the phase modulation (PM) encoding. This approach is in the spirit of FM interaction theory [6–10], which is motivated by the theoretical and experimental observations that cortical neurons are sensitive to the fine temporal structure (timing or phase) of the incoming pulse train.

In the next section we consider a simple model that describes some aspects of behavior of a network of identical coupled lasers. It can be treated as the reduction of Maxwell-Bloch equations after adiabatic elimination of the polarization. In subsequent sections we simplify the model even further to glimpse its neurocomputational features.

We stress that we study pattern recognition via phase modulation in a *simple model* of laser networks. Our study is purely theoretical, and it might still be a long way before one can demonstrate *experimentally* that laser networks could have such neurocomputational properties.

II. THE MODEL

We assume that the dimensionless rate equations for the complex electric field E_i and the excess carrier number N_i of the ith laser are

$$E_i' = (1+\mathrm{i}\alpha)N_iE_i + \mathrm{i}\omega E_i + \sum_{j=1}^{n} c_{ij}E_j, \tag{1}$$

$$N_i' = \mu[P - N_i - (1+2N_i)|E_i|^2], \tag{2}$$

where $' = d/ds$, $s = t\tau_p^{-1}$ is the time measured in units of the photon lifetime τ_p, $\mu = \tau_p/\tau_s$ is the ratio of photon to carrier time scales, where τ_s is the carrier lifetime, P is the pumping above threshold, α is the linewidth enhancement factor, ω is normalized optical frequency, and c_{ij} are complex connection coefficients. Since the equations are invariant under the translation $E_i \mapsto e^{\mathrm{i}\omega t}E_i$, the parameter ω may be assumed to be zero.

It is convenient to use polar coordinates

$$E_i = r_i e^{\mathrm{i}\varphi_i} \quad \text{and} \quad c_{ij} = s_{ij}e^{\mathrm{i}\psi_{ij}}$$

to rewrite the model (1) and (2) in the form

$$\varphi_i' = \alpha N_i + \omega + \sum_{j=1}^{n} s_{ij}\frac{r_j}{r_i}\sin(\varphi_j + \psi_{ij} - \varphi_i), \tag{3}$$

$$r_i' = N_i r_i + \sum_{j=1}^{n} s_{ij} r_j \cos(\varphi_j + \psi_{ij} - \varphi_i), \tag{4}$$

$$N_i' = \mu[P - N_i - (1+2N_i)|r_i|^2]. \tag{5}$$

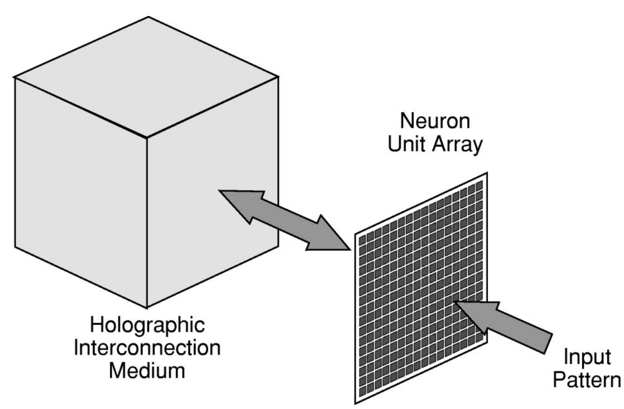

FIG. 1. Conceptual architecture of a typical optical neurocomputer.

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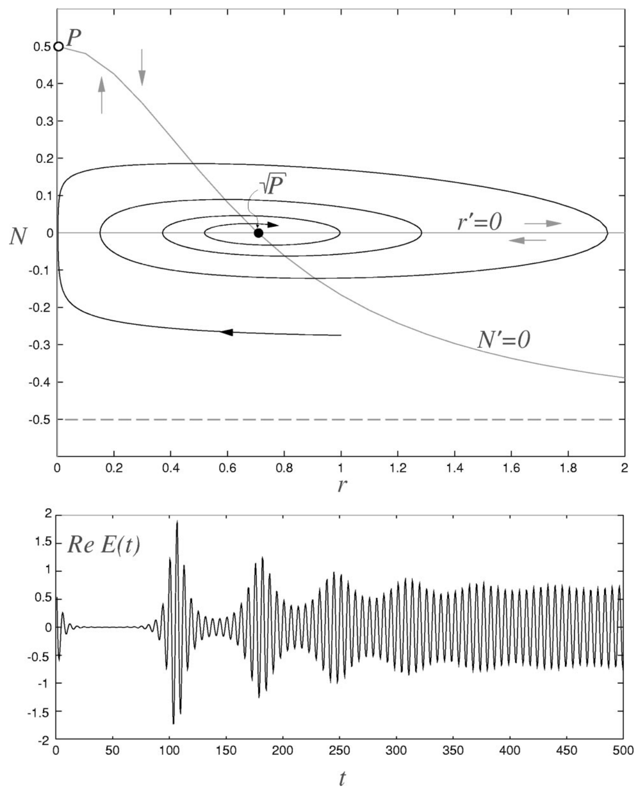

FIG. 2. Top: Phase portrait, fast and slow nullclines, and a typical trajectory of the relaxation system $r'=Nr$, $N'=\mu[P-N-(1+2N)r^2]$. Bottom: Corresponding complex electric field $E=re^{i\varphi}$.

Much insight can be gained by considering the first equation separately from the other two. When the lasers are unconnected $(s_{ij}=0)$, the equations decouple, and one can study the (r_i,N_i) dynamics. When the pumping is above the threshold $(P>0)$, we have that

$$(r_i(t),N_i(t))\to(\sqrt{P},0),$$

see Fig. 2, and the phase $\varphi_i(t)\to\omega t+\varphi_i^0$, where φ_i^0 is determined by the initial conditions.

When the lasers are connected $(s_{ij}>0)$, they may exhibit quite complicated dynamics, even when there are only two of them; see, e.g., [11–13].

III. NEUROCOMPUTATIONAL PROPERTIES

First, we consider the model (1) and (2) for $\alpha=0$. If all $r_i(t)\to r_0$, then the phase model (3) has Kuramoto's form [14]

$$\varphi_i'=\omega+\sum_{j=1}^{n} s_{ij}\sin(\varphi_j+\psi_{ij}-\varphi_i) \tag{6}$$

whose neurocomputational properties are well known [6,8]: It behaves like a Hopfield network when $s_{ij}=s_{ji}$ and $\psi_{ij}=-\psi_{ji}$ for all i and j. This is equivalent to the requirement that the matrix of connections $C=(c_{ij})$ in Eq. (1) be self-adjoint. Indeed, we can denote $\varphi_i=\omega t+\phi_i$ and verify that the phase deviations form the gradient system

$$\phi_i'=-\frac{\partial U}{\partial \phi_i},$$

where

$$U(\phi_1,\ldots,\phi_n)=-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij}\cos(\varphi_j+\psi_{ij}-\varphi_i)$$

is a potential function [6]. As a result, we know that the vector of phase deviations $\phi=(\phi_1,\ldots,\phi_n)\in\mathbb{T}^n$ always converges to an equilibrium on the n-torus $\mathbb{T}^n$. Therefore, the vector of original phases $\varphi=\omega t+\phi$ always converges to a limit cycle attractor having frequency ω and phase relations defined by the vector ϕ. There could be many such attractors corresponding to many memorized images, which are encoded into the connection matrix $C=(c_{ij})$. An example is given in Sec. IV.

A distinguishing feature of associative memory in natural and artificial neural systems is that it persists even in the presence of relatively large distortions of the synaptic connections. Thus, one can expect that neurocomputational behavior of the phase model (3) may be similar to that of Kuramoto's system (6) even when the $r_i(t)$ converge to different values, or do not converge at all. This expectation is supported by numerous simulations, which show that attractors of Eq. (3) lie in small neighborhoods of those of Eq. (6).

When the linewidth enhancement factor α is not zero, the frequency of the oscillatory electric field depends on the excess carrier number N_i. This may change the synchronization properties of the phase model (3) on the time scale larger than $1/\mu$ unless the connection matrix satisfies certain conditions, as we show in the next section.

IV. ILLUSTRATION

We next illustrate some neurocomputational properties of the model (1) and (2).

A. Hebbian learning rule

Suppose we are given a set of m complex vectors to be memorized,

$$\xi^k=(\xi_1^k,\xi_2^k,\ldots,\xi_n^k)\in\mathbb{C}^n, \quad |\xi_i^k|=1,$$

where $k=1,\ldots,m$; see Fig. 3. The argument (angle) difference between any ξ_i^k and ξ_j^k denotes the desired phase difference between the ith and jth oscillators. For example, the relation $\xi_i^k=\xi_j^k$ means that the ith and the jth oscillators should be in-phase $(\varphi_i=\varphi_j)$, and $\xi_i^k=-\xi_j^k$ means they should be antiphase $(\varphi_i=\varphi_j+\pi)$. Notice that the problem of mirror images does not exist in oscillatory neural networks, since both ξ^k and $-\xi^k$ result in the same phase relations. We use the complex Hebbian learning rule [15,16]

$$c_{ij}=\frac{\varepsilon}{n}\sum_{k=1}^{m}\xi_i^k\bar{\xi}_j^k, \tag{7}$$

which produces self-adjoint connection matrix. Here $\varepsilon>0$ denotes the strength of connections between the lasers.

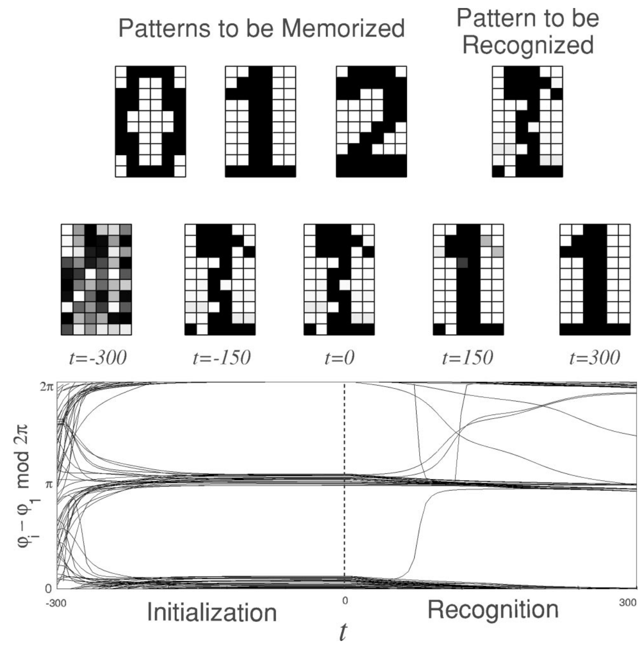

FIG. 3. Illustration of associative memory and recall by the network of lasers. Shown is simulation of the model (1) and (2) with parameters $\omega=1$, $\alpha=0$, $P=0.1$, and $\mu=0.01$. The interconnection coefficients are defined by the Hebbian rule (7) with $\varepsilon=0.01$. Initialization stage: $t\in[-300,0]$. Recognition stage: $t\in[0,300]$.

If c_{ij} were synaptic coefficients for a Hopfield-Grossberg-like neural network, then the network would have $2m$ attractors in some small neighborhoods of ξ^k and their mirror images $-\xi^k$, provided that $m<n/8$ and the vectors ξ^k are nearly orthogonal. Numerous simulations confirm similar estimates for oscillatory neural networks [17].

B. Initialization and recognition

In the standard Hopfield-Grossberg paradigm, the image to be recognized is the initial state of the network. In our case it is the distribution of phases. It is easy to set initial values in a computer simulation, but it might be quite difficult in real laser systems since one does not have direct access to the phase of a laser.

There are several possible mechanisms of setting the laser phases. For example, one can switch the lasers off, i.e., bring them below threshold by changing the parameter P in Eq. (2), then inject a seed having appropriate phase, and then switch the lasers on, i.e., bring them above the threshold. Alternatively, one can entrain the network to a relatively strong external input having appropriate phase relation. Let us elaborate. Suppose the vector to be recognized is $\xi^0\in\mathbb{C}^n$, but the network phases initially have certain random values; see Fig. 3. We wish to entrain the network to the periodic input associated with ξ^0, so that the phases converge to the desired values. In Fig. 3 we force the network with the periodic signal $\epsilon\xi^0 e^{i\omega t}$, where ϵ is much larger than the strength of connections ε in Eq. (7). The lasers lock to the signal. After the initialization stage is complete (time $t=0$ in Fig. 3) and the lasers have phase relations defined by the vector ξ^0, we remove the forcing signal and restore the connections between the lasers. Now the recognition starts from the input image ξ^0, and the network converges to the appropriate limit cycle attractor. The lasers oscillate with equal frequencies and certain phase deviations corresponding to the memorized pattern. To see the phase relations between the lasers, we subtract $\varphi_1(t)$ from all $\varphi_i(t), i=1,\dots,n$, and plot the resulting vectors in Fig. 3.

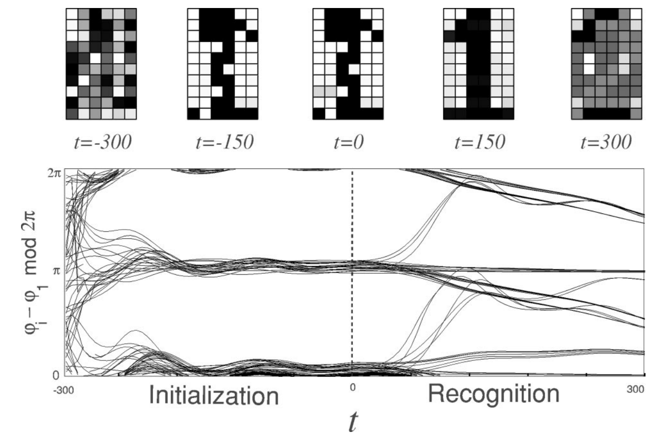

FIG. 4. Illustration of associative memory and recall by the network of semiconductor lasers with a nonzero linewidth enhancement factor ($\alpha=1$). All other coefficients and parameters are as in Fig. 3. Notice that successful recognition occurs on a short time scale. The phases eventually diverge from the memorized configuration due to their sensitivity to the values of N_i; see Eq. (3).

In Fig. 4 we drop the assumption $\alpha=0$ and repeat the initialization and recognition procedures. The initial behavior is similar to that depicted in Fig. 3: They lock to the forcing signal representing the input pattern to be recognized, and then they quickly converge to an appropriate locking state representing one of the memorized images (see the phase pattern at $t=150$). Here, however, the similarity may end. The slowly evolving terms αN_i in Eq. (3) may pull the oscillator frequencies apart, and they eventually may unlock (see the phase pattern at $t=300$). This phenomenon does not occur when the memorized vectors ξ^k are pairwise orthogo-

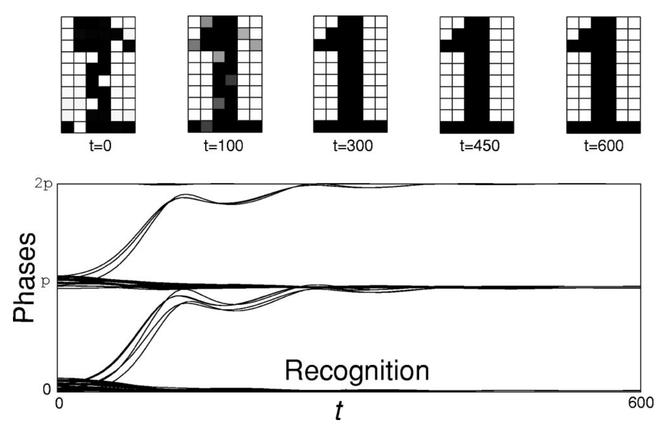

FIG. 5. Illustration of associative memory and recall by the network of lasers having nonzero linewidth enhancement factor and orthogonal memorized images. All coefficients and parameters are as in Fig. 4 except that the vectors ξ^k corresponding to the memorized patterns ''0'' and ''2'' are substituted by random vectors that are nearly orthogonal to the vector corresponding to the memorized pattern ''1.'' Notice that successful recognition occurs on a long time scale.

nal. Indeed, these vectors become eigenvectors of the connection matrix defined by Eq. (7) with the eigenvalue ε. Simple calculations show that all $N_i(t) \rightarrow -\varepsilon$ in this case. In particular, they all have identical values. Hence, all oscillators have equal frequencies and stable locking is possible, as we illustrate in Fig. 5.

V. DISCUSSION

The goal of this paper is to illustrate possible neurocomputational properties of phase-sensitive devices such as laser networks described by the model (1) and (2), not to provide a detailed theory of their dynamics. We hope that this paper will stimulate new research activity in the two fields of neural networks and optical computing.

We now list some important issues that have yet to be addressed.

(i) *Nonidentical lasers*. We considered the simplest case of identical lasers. All of the results persist when the lasers are nearly identical relative to the strength of connection ε in Eq. (7); see [6]. When heterogeneity is much larger than ε, the lasers might not synchronize, and may possibly exhibit quite complicated activity. An intermediate case has yet to be studied.

(ii) *Noise*. Emission of photons is a stochastic process, which is the major source of noise in laser oscillators. It can contribute to frequency instabilities and disrupt phase locking during the initialization and recognition stages.

(iii) *Scaling issues*. Our analysis is applicable when the number of laser oscillators, n, is finite. It is not clear what would happen when $n \rightarrow \infty$.

(iv) *Delayed interactions*. We have assumed that the size of the neurocomputer is small and comparable with the laser wavelength, so that we can neglect the interaction time delays [18]. Currently we investigate the effect of delays. For this we use the Lang-Kobayashi model [19], which is similar to Eq. (1) and (2) except that the connections have the form $c_{ij}E_j(t-\tau_{ij})$.

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