Theoretical continuous equation derived from the microscopic dynamics for growing interfaces in quenched media

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We present an analytical continuous equation for the Tang and Leschhorn model [Phys. Rev. A **45**, R8309 (1992)] derived from their microscopic rules using a regularization procedure. As well in this approach, the nonlinear term $(\boldsymbol{\nabla} h)^2$ arises naturally from the microscopic dynamics even if the continuous equation is not the Kardar-Parisi-Zhang equation [Phys. Rev. Lett. **56**, 889 (1986)] with quenched noise (QKPZ). Our equation is similar to a QKPZ equation but with multiplicative quenched and thermal noise. The numerical integration of our equation reproduces all the scaling exponents of the directed percolation depinning model.

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The investigation of rough surfaces and interfaces has attracted much attention for decades due to its importance in many fields, such as the motion of liquids in porous media, growth of bacterial colonies, crystal growth, etc. When a fluid wets a porous medium, a nonequilibrium self-affine rough interface is generated. The interface has been characterized through scaling of the interfacial width $w=\langle [h_i - \langle h_i \rangle]^2 \rangle^{1/2}$ with time t and lateral size L. The result is the determination of two exponents β and α called dynamical and roughness exponents, respectively. The interfacial width $w \sim L^{\alpha}$ for $t \gg t^*$ and $w \sim t^{\beta}$ for $t \ll t^*$, where $t^* = L^{\alpha/\beta}$ is the crossover time between these two regimes. Much effort has been done to understand the leading mechanisms of these processes and to try to explain how the dynamics affects the scaling exponents [1]. The formation of interfaces is determinated by several factors, it is very difficult to discriminate theoretically all of them. The knowledge of the dynamical nonlinearities, the disorder of the media, and the theoretical model representing experimental results are difficult to overcome due the complex nature of the growth. The disorder affects the motion of the interface and leads to its roughness. Two main kinds of disorder have been proposed: the ''annealed'' noise that depends only on time and the ''quenched'' disorder due to the inhomogeneity of the media where the moving phase is propagating. The discrete models provided a useful approach to obtain the exponents that allows its classification in universality classes. By extensively studying these models, one can obtain the scaling behaviors and the corresponding universality classes and then associate the continuous stochastic equations with the given discrete growth models.

The most used method of establishing the correspondence between a continuous growth equation and a discrete model is to numerically simulate the model and compare the obtained scaling exponents with those of the corresponding continuous equation. In this context, attempts are being made to classify quenched disorder models in terms of universality classes based on an equation of motion such as

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \mathcal{F} + \nu \nabla^2 h + \frac{\lambda}{2} (\boldsymbol{\nabla} h)^2 + \xi(\mathbf{x},h) + \eta(\mathbf{x},t), \quad (1)$$

where $\mathcal{F}$ is the driving force responsible for the advance of the interface, $\xi(\mathbf{x},h)$ is the quenched disorder or pinning forces, and $\eta(\mathbf{x},t)$ is the thermal noise. The noises are white. Equation (1) is Kardar-Parisi-Zhang equation [2] with quenched noise (QKPZ). When $\lambda \rightarrow 0$, the quenched Edwards-Wilkinson equation [3] is recovered. In absence of quenched noise, their thermal versions are recovered, named TKPZ and TEW equations, respectively. Much effort has been made in order to classify discrete models and experiments in universality classes depending on the value of the coefficient λ associated with the nonlinearity of the QKPZ. Numerical [4,5] and analytical [6] studies indicate λ is relevant at the depinning transition for discrete models in anisotropic media. These results only show that the nonlinear term exists but they do not confirm that these models are represented by the QKPZ. However, the exponents obtained by numerical simulation of Eq. (1), without thermal noise [7], agree very well with those of the model in anisotropic media.

A powerful method of establishing the correspondence between a continuous growth equation and a discrete model is to derive the continuous equation from a given discrete model analytically. Among them, a systematic method proposed by Vvedensky *et al.* [8], where the continuous equations can be constructed directly from the growth rules of the discrete model based on the master-equation description, has been applied to the derivation of growth equations for some discrete models [8–11] with thermal noise. This method has proved to be useful to derive continuous equations from the master equations with the advantage that the sources of the terms of the Langevin equation can be identified and the parameters related to the microscopic dynamics. However, it

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is easier to achieve the same results using a microscopic equation based on rules for the evolution of the height. The derivation of continuous equations from discrete models is an interesting subject that has not been addressed in the context of growth in the presence of quenched media.

The aim of this work is to obtain the continuous equation from the microscopic dynamics of a variant of the directed percolation depinning (DPD) model [12,13] in order to establish if it is related in some way to the QKPZ equation. The main goal of our paper is to have obtained analytically the differential equation that describes the dynamics of the Tang and and Leschhorn model [12]. To our knowledge this is the first time that a Langevin equation has been obtained from the microscopic dynamics in quenched media.

As we shall show below, the dynamics of the height is strongly affected by a *multiplicative quenched noise*. We chose this model because it presents the principal features of some experiments like the imbibition of a viscous fluid in a porous media driven by capillary forces [13,14]. In the TL model, the interface growth takes place in a square lattice of edge L with cells of size a that represents the mean size of a pore. Consider each cell $\mathbf{r}$ is assigned a random pinning force $g(\mathbf{r})$ uniformly distributed in the interval $[0,1]$. For a given applied pressure $p>0$, we can divide the cells into two groups, those with $g(\mathbf{r})\leqslant p$ (free or active cells), and those with $g(\mathbf{r})>p$ (blocked or inactive cells). Denoting by q the density of inactive cells on the lattice, we have $q=1-p$ for $0<p<1$ and $q=0$ for $p\geqslant 1$. In this model the critical pressure is $p_c=0.461$. Periodic boundary conditions are used. We consider the evolution of the height of the i-th site in this model. Let us denote by $h_i(t)$ the height of the i-th generic site at time t. The set $\{h_i, i=1,\dots,N\}$, where $N=L/a$, defines the interface between wet and dry cells. Given a site, chosen between N, say the site j, the height in the site i is increased by a with probability (i) 1 if $j=i\pm 1$ and $h_{i\pm 1}\geqslant h_i+2a$ and $h_i<h_{i\pm 2}$, (ii) 1/2 if $j=i\pm 1$ and $h_{i\pm 1}\geqslant h_i+2a$ and $h_i=h_{i\pm 2}$, (iii) 1 if $j=i$ and $h_i<\min(h_{i-1},h_{i+1})+2a$ and $F_i(h_i+a)=1$. Otherwise no growth happens. $F_i(h_i+a)=\Theta(p-g_i(h_i+a))$ is called the activity function [15] and $\Theta(x)$ is the unit step function defined as $\Theta(x)=1$ for $x\geqslant 0$ and equals to 0 otherwise, p is the microscopic driving force- and $g_i(h_i+a)$ is the quenched noise just above the interface distributed in the interval $[0,1]$. Notice that the activity function F is the competition between the driving force and the quenched noise, so F is also a ''noise.'' Provided that the system size is large and that the intrinsic fluctuations are not too large [8], the evolution equation for the height in a site i, in a short lapse τ, is

$$\frac{\partial h_i}{\partial t}=\frac{a}{\tau}G_i+\eta_i, \qquad (2)$$

where τ is the mean lapse between successive election of any site and G_i [15] contains the microscopic growing rules for the evolution of the height at this site due to that a site j is chosen at time t. Here η_i is a Gaussian ''thermal'' noise with zero mean and covariance

$$\langle \eta_i(t)\,\eta_j(t')\rangle=\frac{a^2}{\tau}\,G_i\delta_{ij}\delta(t-t'). \qquad (3)$$

Notice that in the notation of [8] the transition rate from a configuration H to another H' is $W(H,H')=(1/\tau)\Sigma_k\delta(h_k',h_k+a)G_k\Pi_{j\neq k}\delta(h_j',h_j)$. So the first moment is $(1/\tau)\Sigma_{H'}(h_k'-h_k)W(H,H')=(a/\tau)G_k$. As a consequence of the fact that subsequent configurations differ only in the height at one site all the moments are diagonal and proportional to the first moment [10].

For this model [15,16],

$$G_i(h_{i-1},h_i,h_{i+1})=W_{i+1}+W_{i-1}+F_i(|h_i|+1)W_i, \qquad (4)$$

where $|h_i|\equiv[h_i/a]$ denotes the integer part of h_i in units of a. This definition is unnecessary if a is taken as one, as in the discrete model. We shall show below that in the continuous limit, it is necessary for an analytic extension of the activity function. In Eq. (4),

$$W_{i\pm 1}=\tfrac{1}{2}\,[1-\Theta(H_{i\pm 2}^{i})+\Theta(H_i^{i\pm 2})]\,\Theta(H_i^{i\pm 1}-2),$$
$$W_i=1-\Theta(U_i-2), \qquad (5)$$

where $H_r^s=(h_s-h_r)/a$ and $U_i=(1/a)[h_i-\min(h_{i+1},h_{i-1})]$. Notice that all the heights are in units of a in order to keep the arguments of the step function without units. For $W_{i\pm 1}$ [15] the δ Kronecker function has been taken as

$$\delta(x,y)=\Theta(x-y)+\Theta(y-x)-1. \qquad (6)$$

Using the fact that $\min(x,y)=\frac{1}{2}\{(x+y)-(x-y)[\Theta(x-y)-\Theta(y-x)]\}$ and with a more compact notation

$$U_i=\tfrac{1}{2}\{H_{i+1}^{i}+H_{i-1}^{i}+H_{i-1}^{i+1}\,[\Theta(H_{i-1}^{i+1})-\Theta(H_{i+1}^{i-1})]\}. \qquad (7)$$

The representation of the step function can be expanded as $\Theta(x)=\Sigma_{k=0}^{\infty}c_kx^k$ providing that x is smooth. Our focus is on properties of the surface on large length scales, so we kept the expansion of the step function to first order in his argument. The next step is to regularize the height defining an interpolating function. This is done by expanding the height $h_{i+l}\equiv h(x_i+x_l)$ around $x_i=ia$. Retaining only the leading terms in the expansion, the adimensional difference of heights is

$$H_{i+m}^{i+l}=(l-m)\,\partial_x h]_{x_i}+\tfrac{1}{2}(l^2-m^2)\,\partial_x^2 h]_{x_i}\,a+O(a^2), \qquad (8)$$

where $\partial_x^j h=\partial^j h/\partial x^j$.

Notice that in any discrete model there is in principle an infinite number of nonlinearities, but at long wavelengths the higher-order derivatives can be neglected using scaling arguments, since one expects affine interfaces over a long range of scales, and then one is usually concerned with the form of the relevant terms.

Replacing Eq. (8) in Eqs. (5) and (7), using the expansion of the step function and retaining the leading terms to order $O(a)$, Eq. (2) can be written as

$$\frac{\partial h(x_i,t)}{\partial t}=\frac{a}{\tau}\,[W(x_i+a)+W(x_i-a)$$
$$+W(x_i)\,F(x_i,|h(x_i)|+1)]+\eta(x_i,t), \qquad (9)$$

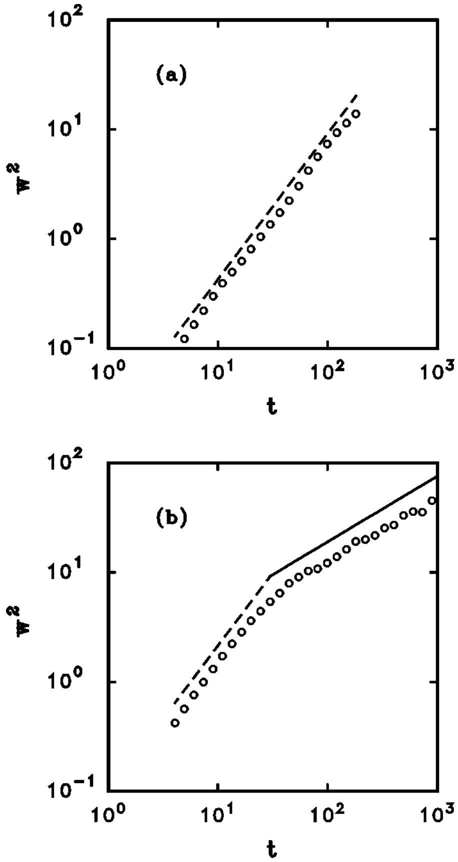

FIG. 1. Log-log plot of the square roughness w^2 vs time for $C=1.3$. In (a) $p=0.1$, for this value of C the critical pressure is $p_c \simeq 0.1$. The circles show the results obtained from the numerical integration of Eq. (13). The dashed line is used as a guide and as exponent $2\beta=1.34$. In (b) $p=0.3$, the dashed line has slope $2\beta=1.34$ and the solid line has slope $2\beta_m=0.66$. The numerical integration has been done with $L=1024$ and over 30 independent samples.

with

$$W(x+a)+W(x-a)=(c_0-2c_1)+4c_1^2\,(\partial_x h)^2 + ac_1\left[\tfrac{1}{2}+4(c_0-2c_1)\right]\,\partial_x^2 h, \tag{10}$$

$$W(x)=1-(c_0-2c_1)-4c_1^2\,(\partial_x h)^2+\tfrac{1}{2}ac_1\,\partial_x^2 h. \tag{11}$$

Notice that the argument of $F=\Theta[p-g(x_i,|h(x_i)|+1)]$ is not smooth, so its expansion is meaningless. In order to extend the definition of the activity function F to the continuous, we construct an interpolation function

$$\tilde{F}[x_i,h(x_i)]=F(x_i,|h_i|)+\frac{\delta h}{a}[F(x_i,|h_i|+1)-F(x_i,|h_i|)] + O(\delta h^2), \tag{12}$$

with $0\leqslant \delta h \leqslant a$ that measures the departure of the height from the low pore. Then $\tilde{F}$ is a smooth function taking continuous values in the interval $[0,1]$. With this definition we ensure that the characteristic size of the correlation between pores is of the order of the pore size. In real materials there always exists a typical size of the inhomogeneities in the disordered media that plays the role of the lattice constant a.

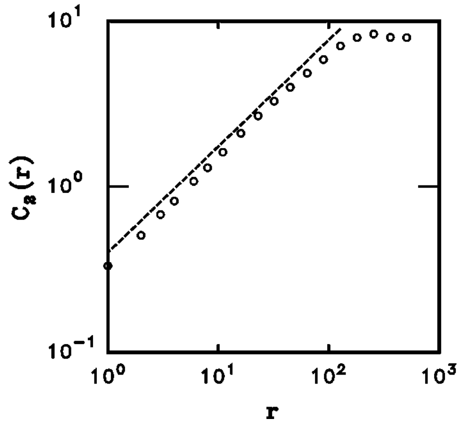

FIG. 2. Log-log plot of $C_2(r)$ as a function of r for $p=0.1$ and $C=1.3$. The dashed line that is used as a guide has slope $2\alpha=1.28$.

The final step is a coarse-grained spatial average of the variables in order to obtain smooth continuous functions at a macroscopic level. In this way we obtain the stochastic continuous equation for this model,

$$\frac{\partial h}{\partial t}=\mu(\tilde{F})+\nu(\tilde{F})\partial_x^2 h+\lambda(\tilde{F})(\partial_x h)^2+\eta(x,t), \tag{13}$$

where

$$\mu(\tilde{F})=[(c_0-2c_1)(1-\tilde{F})+\tilde{F}]\frac{a}{\tau}, \tag{14}$$

$$\nu(\tilde{F})=c_1\left[\tfrac{1}{2}\,(1+\tilde{F})+4(c_0-2c_1)\right]\frac{a^2}{\tau}, \tag{15}$$

$$\lambda(\tilde{F})=4\,c_1^2\,(1-\tilde{F})\frac{a}{\tau}. \tag{16}$$

and $\tilde{F}\equiv F(x,h)$ as was defined in Eq. (12). Notice that $\mu(\tilde{F})$ is now the effective competition between the driving force and the quenched noise. The coefficients μ, ν, and λ take different values at each point of the interface.

Equation (13) shows that the nonlinearity *arises naturally* as a consequence of the microscopic model. Our result is in agreement with those of Réka *et al.* [5] who obtained numerically a parabolic shape of the local velocity as a function of the gradient for the DPD model near above the criticality for different reduced forces (p/p_c-1). In order to interpret the expressions of μ,ν,λ it is necessary to introduce a continuous representation of the Θ function. The best choice is the shifted hyperbolic tangent [10], defined as $\Theta(x)=\{1+\tanh[C(x+b)]\}/2$, where b is the shift and C is a parameter that allows us to recover the Θ in the limit $C\rightarrow\infty$. We choose $b=1/2$. The reason for our choice is that it allows us to define the δ function as Eq. (6). The coefficients fulfill

$$c_0=\frac{1}{2}\left[1+\tanh\left(\frac{C}{2}\right)\right] \text{ and } c_1=\frac{C}{2}\cosh^{-2}\left(\frac{C}{2}\right). \tag{17}$$

When the conditions (17) are satisfied, the coefficient ν is always positive. The coefficient λ is greater or equal to zero independently of the representation of the step function. In the limit $p\rightarrow 0$, the TKPZ equation is recovered taking $\tilde{F}=0$. Far above the criticality $(p\rightarrow 1)$, the interface moves without stopping (the effective force is positive and the nonlinear term becomes negligible), so the dynamics is close to the one described by the TEW equation as we see from Eq. (13) taking $\tilde{F}=1$. However, the fact that TEW and TKPZ limits are recovered is a specific characteristic of this particular model.

In all cases studied so far (see for example [10]), the continuous equations are restricted to some values of C. In our case, for the numerical integration, we choose the value of C taking into account that in the continuous model, near the criticality, where $\tilde{F}$ must be close to zero (notice that in the discrete model, as we approach to the critical value, F is mostly zero because the interface gets pinned by long chains of inactive sites [15,16]), it is necessary that μ mostly becomes negative in order to brake the advance of the interface. It is precisely this restriction for C that gives a physical meaning to our continuous equation. In Figs. 1 we show the temporal scaling behavior of the roughness w obtained from the numerical integration of Eq. (13). At the criticality, a slope $\beta=0.67\pm0.05$ was obtained. Above the threshold we recover a crossover between the exponent $\beta=0.67\pm0.05$ and the $\beta_m\simeq 1/3$ as was obtained by Leschhorn [7] by means of the numerical integration of the QKPZ equation and by his automaton version. In Fig. 2 we show the scaling behavior of the correlation function $C_2(r,t)=\langle[h_{i+r}(t)-h_i(t)]^2\rangle^{1/2}$. The exponent obtained was $\alpha=0.641\pm0.07$ in agreement with the DPD models. Figure 3 shows a log-log plot of the global interface velocity v as a function of $p-p_c$. A velocity exponent $\theta\simeq 0.642$ close to the DPD one was obtained. The numerical integration was made in short lattices using a discretized version of the continuous equation (13). The results in large systems and the details of the integration will be published elsewhere. Notice that even if the exponents from our equation are very similar than the one obtained from the QKPZ one, our equation is very different. The main difference is that the coefficients of the nonlinear and the Laplacian terms in our equation are strongly affected by the local characteristic of the substratum.

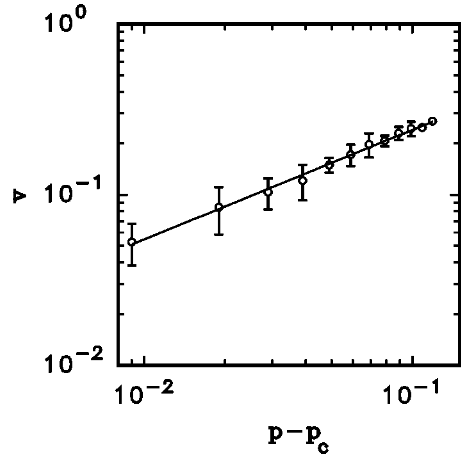

FIG. 3. Log-log plot of global interface velocity v as a function of $p-p_c$ for $p_c=0.11$ and $C=1.3$. The slope of the line is $\theta\simeq 0.642$.

How could our results be used in order to explain the role played by the disordered media in the experiments? In the experiments the advancement of the interface is determinated by the coupled effect of the random distribution of the capillary sizes, the surface tension, and the local properties of the flow, so it is not surprising that all these effects give rise to a multiplicative noise in any evolution equation that intends to represent an experimental growth with disordered media.

Summarizing, we derive the continuous equation from the microscopic one for the TL model. Our equation allows us to explain that the lateral growth contribution is mainly responsible for the roughness near the criticality. In our work, the nonlinear term arises naturally as a consequence of the microscopic dynamics. The numerical integration of our equation reproduces very accurately all the scaling exponents of the DPD model. These results show that Eq. (13) describes the TL model with the advantage that it has been analytically deduced from the microscopic rules. Despite that the behavior of our equation is equivalent to the behavior of the QKPZ one, it is formally different. To our knowledge, this is the first time that an analytical continuous equation derived from a microscopic model does not match the phenomenological equation that was hoped to describe the model. Finally, we hope that this framework can be used in other growing models with quenched noise.

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