Upper bounds on convective heat transport in a rotating fluid layer of infinite Prandtl number: Case of intermediate Taylor numbers

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By means of the Howard-Busse method of the optimum theory of turbulence we obtain upper bounds on the convective heat transport in a heated from below layer of fluid of infinite Prandtl number rotating with a constant angular velocity about the vertical axis. We consider the region of intermediate Taylor numbers: $\alpha_1^4 \ll \mathrm{Ta} \ll \alpha_1^6$ where α_1 is the wave number connected to the $1-\alpha$-solution of the variational problem. The studied optimum fields possess a three-layer or four-layer structure: in addition to the internal, intermediate, and boundary layers, Ekman layers could arise between the intermediate and boundary ones. For the discussed interval of Taylor numbers the intermediate layers do not expand in the direction of the internal layers. We present an asymptotic theory for the case of the fluid layer with rigid lower boundary and stress-free upper boundary. We use an improved solution of the Euler-Lagrange equations of the variational problem for the intermediate sublayer of the optimum field. This solution leads also to correction of the thicknesses of the boundary layers and to lowering of the upper bounds on the convective heat transport for the cases of fluid layer with stress-free or with rigid boundaries. Thus the known upper bounds for these cases can be treated as upper bounds on the upper bounds on the convective heat transport. For the case of the fluid layer with stress-free boundaries the four-layer optimum fields leads to bounds on the convective heat transport which change from $R^{1/3}$ at the lower boundary of their interval of validity to values slightly large than $R^{2/7}$ near the upper boundary of the interval of validity. Finally we discuss the area of application of the obtained bounds with respect to the Taylor number Ta and Rayleigh number R.

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I. INTRODUCTION

The methods of the optimum theory of turbulence are among the few tools for obtaining rigorous estimates of the turbulent quantities directly from the Navier-Stokes equations. Because of the lack of knowledge of the turbulent solutions of the Navier-Stokes equations and because the full numerical simulations of the turbulence flows with very large Rayleigh or Reynolds numbers which are out of reach today, we use the methods of the turbulence theory in order to obtain expressions for the mean properties of the turbulent flows. The Navier-Stokes equations are nonlinear and thus a sequence of coupled equations arise because of the fact that the equation for the nth statistical moment of the flow quantities depends on the $(n+1)$th moment. One way to deal with this situation is to use closure schemes which replace the above-mentioned infinite system with a finite one in such a way that the solution of the finite system of equations becomes close to the real flow. Despite this closeness it is not definite that the solution obtained in such a way is a solution of the Navier-Stokes equations. The optimum theory of turbulence is based on another approach. By means of its methods we derive upper bounds on the turbulent quantities using integral constraints which are a part of an infinite system of moment equations. Using a finite number of these integral constraints we enlarge the class of fields among which the upper-bound solution of the corresponding variational problem is sought. Thus it is ensured that all solutions of the Navier-Stokes equations are contained in the obtained manifold of fields, and moreover the energy balance of the real flow is retained. We can further restrict the class of admissible fields by taking into an account additional integral constraints. Thus the obtained bounds could be lowered and we obtain a sequence of problems whose solutions converge to the solution of the problem with full Navier-Stokes equations as constraints.

In the simplest problems of the optimum theory of turbulence we use the lowest possible number of integral constraints, normally the first two in the case of thermal convection. The use of more integral constraints complicates the variational problem in such a way that in most cases it can be solved only numerically. The simplest variational problems allow us to obtain asymptotic analytical upper bounds for the turbulent quantities when the control parameters (Rayleigh number, Reynolds number, Taylor number, etc.) have large values. The corresponding variational functionals lead to Euler-Lagrange equations which contain as a particular case the onset of the thermal convection for the studied system and whose solutions allow us to obtain power laws (eventually with logarithmic corrections) for the case of large values of the control parameters. It must be expected that the experimental data are well below these upper bounds obtained by using only several of the infinite number of integral constraints. In some cases however, for example, for the case of a thermal convection in a porous medium [1,2] the obtained numerical bounds are surprisingly close to the experimental values of the corresponding quantities. And in principle adding more constraints to the corresponding variational problem could lead to improved bounds.

There exist two methods of the optimum theory of turbulence. Malkus [3,4] suggested that the convecting fluid should transport a maximum amount of heat. This hypothesis is not valid in general but the ideas of Malkus stimulated Howard [5] to obtain upper bounds on the heat flux through a horizontally infinite layer of fluid by means of a variational problem subject to some constraints. Busse [7] introduced the multi-α-solutions of the variational problem. The Howard-Busse method was further developed by Chan [8] and applied to many cases of fluid flows and thermal convection [9–19].

Doering and Constantin [20] proposed another method for obtaining bounds on the quantities connected to the fluid flow, based on the idea for a decomposition of the velocity fields into a steady background field which carries the inhomogeneous boundary conditions, and a homogeneous fluctuations field. If an appropriate background field is constructed (it has to satisfy certain spectral constraints) one easily obtains an upper bound on the corresponding turbulent quantity. The Doering-Constantin method and its modification, proposed by Nicodemus, Grossmann, and Holthaus [21] have found many applications in the last several years [22–31]. The relationship between the Howard-Busse and Doering-Constantin methods as well as formulation of variational problems for the Navier-Stokes equations are discussed in Refs. [32–35]. The optimum theory of turbulence was applied also in plasma physics for obtaining upper bounds on the heat transport due to the ion-temperature gradient, on the energy dissipation in a turbulent pinch, etc. [36–42].

The turbulent thermal convection under the action of rotation is important for the studying of the earth's atmosphere and oceans as well as for the dynamics of solar and planetary atmospheres. Thus it is the subject of extensive theoretical and experimental investigations [43–70]. In this article we shall derive upper bounds on the convective heat transport in the horizontal layer, rotating about a vertical axis, of fluid for the case of moderate rotation rates, i.e., for such values of the Taylor number for which the rotation does not influence the internal layers of the fields which are solutions of the Euler-Lagrange equations of the corresponding variational problem. The problem for obtaining an upper bound on the heat transport in a fluid layer, heated from below, rotating about a vertical axis has been discussed from the point of view of the Howard-Busse method in Ref. [71] for the case of stress-free boundaries and in Ref. [72] for the case of rigid boundaries. The first discussion of the problem from the point of view of the Doering-Constantin method is presented in Ref. [73]. The structure of the article is as follows. In Sec. II we formulate the variational problem using tho two integral constraints, obtained from Boussinesq equations, continuity equation, and the assumption of infinite Prandtl number. Then we derive the corresponding Euler-Lagrange equations. The solutions of these equations are referred further as optimum fields. In Sec. III we discuss the possible structures of the optimum fields and select the range of rotation rates we shall investigate. In Sec. IV we derive the upper bound on the convective heat transport for the case of a fluid layer with rigid lower boundary and stress-free upper boundary. In Sec. V we obtain upper bounds on the convective heat transport in a rotating layer with stress-free boundaries. We use modified solutions of the Euler-Lagrange equations of the variational problem for the intermediate layers of the optimum fields and consider the cases of three-layer and four-layer optimum fields. In Sec. VI we use again the above mentioned improved solution of the Euler-Lagrange equations and obtain upper bound on the convective heat transport in a rotating layer with rigid boundaries on the basis of four-layer optimum fields. The last section is devoted to a discussion of obtained results and their application area. In the appendix we present the equations of the quasilinear approximation which is connected with the simplest variational problems of the optimum theory of turbulence.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

Let us consider a horizontal layer of fluid, heated from below, which rotates about the vertical axis with a constant angular velocity Ω. We shall discuss the idealized situation of an infinite layer and as a model we consider the Boussinesq approximation to the equations of the fluid flow [6]. We denote the layer thickness as d, the thermometric conductivity and kinematic viscosity of the fluid as κ and ν, the acceleration of the gravity as g, the temperature difference between the upper and lower fluid boundary as ΔT, and the density of the fluid as ρ. Taking d as a unit for length, κ/d as unit for velocity, d^2/κ as unit for time, and $\rho\nu\kappa/d^2$ as unit for pressure, we obtain the dimensionless form of the Boussinesq equations

$$\frac{1}{P}\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\frac{1}{E}\nabla p+\nabla^2\mathbf{u}+RT\mathbf{k}+\frac{2}{E}\mathbf{u}\times\mathbf{k}, \quad (1)$$

$$\frac{\partial\Theta}{\partial t}+\mathbf{u}\cdot\nabla\Theta=\nabla^2\Theta, \quad (2)$$

$$\nabla\cdot\mathbf{u}=0, \quad (3)$$

with rigid boundary conditions at $z=-1/2$: $u_3=\partial u_3/\partial z=T=0$, and stress-free boundary conditions at $z=1/2$: $u_3=\partial^2 u_3/\partial z^2=T=0$. $P=\nu/\kappa$ is the Prandtl number, $E=\nu/(\Omega d^2)$ is the Ekman number, $R=(\gamma g\Delta T d^3)/(\kappa\nu)$ is the Rayleigh number, γ is the coefficient of thermal expansion, p is the pressure, and $\mathbf{k}$ is the unit vector in the direction opposite to the gravity. The quantity Θ in Eq. (2) is the total temperature field and T is the deviation of the temperature field from its horizontal mean,

$$\Theta=\overline{\Theta}+T. \quad (4)$$

Below we shall use also the Taylor number, $\mathrm{Ta}=(2/E)^2$, and averages of the quantities over the planes $z=$const (denoted as $\overline{q}$) and over the fluid layer (denoted as $\langle q\rangle$). Denoting the horizontal size of the fluid layer as L and the limes when $L\to\infty$ as lim we define

$$\overline{q}=\lim\frac{1}{4L^2}\int_{-l}^{l}\int_{-L}^{L}dx\,dy\,q(x,y,z,t), \quad (5)$$

$$\langle q\rangle=\lim\frac{1}{4L^2}\int_{-L}^{L}\int_{-L}^{L}\int_{-1/2}^{1/2}dx\,dy\,dz\,q(x,y,z,t). \quad (6)$$

We shall formulate a variational problem using two moment equations obtained on the basis of the Boussinesq equations. We shall assume that all necessary horizontal averages of the functions describing the flow exist, that the horizontal averages of the fluctuation quantities vanish, and that the flow is statistically steady in time and homogeneous in the horizontal averages. Our goal is to obtain an upper bound on the convective heat transport through the fluid layer, i.e., on the Nusselt number

$$\mathrm{Nu}=1+\langle u_3 T\rangle. \tag{7}$$

We introduce Eq. (4) in the Boussinesq equations, multiply Eq. (1) by the velocity **u** and the average over the fluid layer. Thus we obtain the relationship (known also as a power integral in the optimum theory of turbulence)

$$\langle|\nabla \mathbf{u}|^2\rangle=R\langle u_3 T\rangle. \tag{8}$$

Another power integral can be obtained by a multiplication of Eq. (2) by T and by averaging the result over the fluid layer. The obtained relationship contains the term $\langle u_3 T(\partial\overline{\Theta}/\partial z)\rangle$. We transform this term by a horizontal averaging of the heat equation and integrating the obtained result with respect to z. Thus we obtain the relationship

$$\langle|\nabla T|^2\rangle=\langle u_3 T\rangle^2-\langle\overline{u_3 T}^2\rangle+\langle u_3 T\rangle. \tag{9}$$

The assumption that the Prandtl number is infinite allows us to include additional restrictions on the manifold of fields from which we shall extract the upper bounds on the convective heat transport. The above assumption simplifies the mathematical analysis and has been used in Refs. [71–73]. The problem for the dependence of the upper bounds on the convective heat transport on the Prandtl number is quite interesting. Our investigations [74] show that the upper bound on the convective heat transport in the case of a horizontal fluid layer, heated from below and rotating about a vertical axis, depends weakly on the Prandtl number when the Prandtl number is about 7 and larger. This defines the region of Prandtl numbers for which the approximation of the infinite Prandtl number, used here, is valid.

When the Prandtl number is infinite, the Navier-Stokes equation becomes linear and we can include it as a constraint in the variational problem. We shall take into account the equation of continuity by the general representation of a solenoidal field **u** in terms of a poloidal and a toroidal component

$$\mathbf{u}=\nabla\times(\nabla\times\mathbf{k}\phi)+\nabla\times\mathbf{k}\psi. \tag{10}$$

We introduce Eq. (10) into the Navier-Stokes equation ($P=\infty$) and perform the rescalings

$$\mathbf{u}=\langle u_3 T\rangle^{1/2}R^{1/2}\mathbf{v},\quad T=\langle u_3 T\rangle^{1/2}R^{-1/2}\theta. \tag{11}$$

Let us denote the z-component of the rescaled velocity field v as w. Taking the z-component of the horizontal curl and z-component of the double curl of the result we obtain the relationships

$$\nabla^2 f+\frac{2}{E}\frac{\partial w}{\partial z}=0, \tag{12}$$

$$\nabla^4 w+\nabla_1^2\theta-\frac{2}{E}\frac{\partial f}{\partial z}=0, \tag{13}$$

where $f=-\nabla_1\psi$ is the vertical component of the vorticity. After the rescaling the power integral Eq. (9) becomes

$$\langle u_3 T\rangle=\frac{\langle wT\rangle-(1/R)\langle|\nabla\theta|^2\rangle}{\langle(\langle w\theta\rangle-\overline{w\theta})^2\rangle}. \tag{14}$$

We impose the condition: $\langle w\theta\rangle=1$ and write the variational problem as follows:

Find the maximum $F(R,\mathrm{Ta})$ of the variational functional

$$\begin{aligned}\mathcal{F}(w,\theta,f,R,\mathrm{Ta})=&\frac{1-(1/R)\langle|\nabla\theta|^2\rangle}{\langle(1-\overline{w\theta}^2\rangle}+2\lambda^*\langle w\theta-1\rangle\\&+2\left\langle p^*\left(\nabla^2 f+\frac{2}{E}\frac{\partial w}{\partial z}\right)\right\rangle\\&+2\left\langle q^*\left(\nabla^4 w+\nabla_1^2\theta-\frac{2}{E}\frac{\partial f}{\partial z}\right)\right\rangle,\end{aligned} \tag{15}$$

among all fields w,θ,f subject to the boundary conditions: $w=\theta=\partial w/\partial z=f=0$ at $z=-1/2$, and $w=\theta=\partial^2 w/\partial^2 z=\partial f/\partial z=0$ at $z=1/2$. p^*,q^*,λ^* are Lagrange multipliers. The functional, Eq. (15), is obtained on the basis of the power integral, Eq. (9). It can be easily checked that once w and θ are determined from the corresponding Euler-Lagrange equations then the other power integral, Eq. (8), is automatically satisfied.

After the elimination of the Lagrange multipliers the Euler-Lagrange equations for the above variational problem become

$$\begin{aligned}&\frac{1}{RF}\nabla^2\left(\nabla^6+\mathrm{Ta}\frac{\partial^2}{\partial z^2}\right)\theta+\left(\nabla^6+\mathrm{Ta}\frac{\partial^2}{\partial z^2}\right)\left[w\left(1-\overline{w\theta}-\frac{\lambda}{F_1}\right)\right]\\&\quad-\nabla_1^2\nabla^2\left[\theta\left(1-\overline{w\theta}+\frac{\lambda}{F_1}\right)\right]=0,\end{aligned} \tag{16}$$

in addition to Eqs. (12) and (13). We can exclude the vorticity in one of the Eqs. (12) and (13) and thus obtain

$$\left(\nabla^6+\mathrm{Ta}\frac{\partial^2}{\partial z^2}\right)w+\nabla_1^2\nabla^2\theta=0, \tag{17}$$

$$\nabla^2 f+\frac{2}{E}\frac{\partial w}{\partial z}=0. \tag{18}$$

In the cases discussed below, we shall use this one of the equivalent systems of equations (12), (13), (16), or (16), (17), (18) which is more convenient for description of the corresponding case.

The kind of the nonlinearity present in the obtained Euler-Lagrange equations allows solutions in which the horizontal dependence is separated from the vertical dependence. Thus in general we can write the solutions of the Euler-Lagrange equations as Fourier series (multi-α-solutions of Busse [7])

$$w=\sum_{i=1}^{N} w_n(z)\phi_n(x,y), \quad \theta=\sum_{i=1}^{N} \theta_n(z)\phi_n(x,y), \tag{19}$$

$$f=\sum_{i=1}^{N} f_n(z)\phi_n(x,y),$$

where $N=1,2,3,\ldots$, $\overline{\phi_n\phi_m}=\delta_{nm}$, δ_{nm} is the Kronecker delta-symbol, and $\nabla_1\phi_n=-\alpha_n\phi_n$. The equations corresponding to the $1-\alpha$-solution of the variational problem [$N=1$ in Eq. (19)] are

$$\left(\frac{d^2}{dz^2}-\alpha_1^2\right)^3 w_1+\mathrm{Ta}\frac{d^2w_1}{dz^2}-\alpha_1^2\left(\frac{d^2}{dz^2}-\alpha_1^2\right)\theta_1=0, \tag{20}$$

$$\frac{1}{RF_1}\left(\frac{d^2}{dz^2}-\alpha_1^2\right)\left[\left(\frac{d^2}{dz^2}-\alpha_1^2\right)^3+\mathrm{Ta}\frac{d^2}{dz^2}\right]\theta_1 +\left[\left(\frac{d^2}{dz^2}-\alpha_1^2\right)^3+\mathrm{Ta}\frac{d^2}{dz^2}\right]\left[w_1\left(1-w_1\theta_1+\frac{\lambda}{F}\right)\right] -\alpha_1^2\left(\frac{d^2}{dz^2}-\alpha_1^2\right)\left[\theta_1\left(1-w_1\theta_1+\frac{\lambda}{F_1}\right)\right]=0, \tag{21}$$

$$\left(\frac{d^2}{dz^2}-\alpha_1^2\right)f_1+\frac{2}{E}\,\frac{dw_1}{dz}=0, \tag{22}$$

where

$$\frac{1}{2}\leqslant\lambda=\frac{1}{2}\left(2-\frac{1}{R}\langle|\nabla\theta|^2\rangle\right)\leqslant 1. \tag{23}$$

In order to obtain the quasilinear approximation (see the appendix) from the Euler equations of the variational problem we shall rearrange Eq. (16) as follows:

$$\left(\nabla^6+\mathrm{Ta}\frac{\partial^2}{\partial z^2}\right)\left[\frac{\nabla^2\theta}{RF}+\left(1-\overline{w\theta}+\frac{\lambda}{F_1}\right)w\right] =\nabla^2\nabla_1^2\left[\theta\left(1-\overline{w\theta}+\frac{\lambda}{F_1}\right)\right]. \tag{24}$$

When the Rayleigh number is large enough the terms of the right-hand side of Eq. (24) can be neglected. Thus we obtain Eq. (A8) taking into account that $F=\mathrm{Nu}-1$. Introducing the multi-α-solutions we obtain from Eq. (24) for the case $N=1$

$$\frac{1}{R(\mathrm{Nu}-1)}\left(\frac{d^2}{dz^2}-\alpha_1^2\right)\theta_1=\left[w_1\theta_1-1-\frac{\lambda}{\mathrm{Nu}-1}\right]w_1. \tag{25}$$

We note that the close relation between the Euler-Lagrange equations of the optimum theory of turbulence and the equations of the quasilinear approximation exists for the relative simple variational functionals based only on the first power integrals of the equations of the fluid motion. If we take into account more power integrals the Euler-Lagrange equations of the variational problem become more complicated than the equations of the quasilinear approximation.

III. STRUCTURE OF THE OPTIMUM FIELDS AND INTERVALS OF TAYLOR NUMBERS

The presence of rotation complicates the problem for the construction of the optimum fields. First of all, the flow fields which are solutions of the Navier-Stokes equations can develop Ekman layers, and second, when the Rayleigh number is fixed and the Taylor number increases, the thicknesses of the layers of the flow fields can change. We shall incorporate these two points in the process of construction of the fields which satisfy the Euler-Lagrange equations obtained in the previous section. In the case without rotation the optimum fields have three-layer structure from the middle plane of the fluid layer to one of its boundaries. The optimum fields have an internal layer which fills almost the entire fluid layer, except the small regions near the boundaries, where an intermediate layer ensures the transition between the internal layer and the boundary layer in which the optimum fields have appropriate behavior in order to satisfy the corresponding boundary conditions. In the internal and intermediate layers we have the relationship $w_1\theta_1=1$ which is broken in the boundary layer. The terms containing derivatives are negligible in the internal layers and dominant in the boundary layers. The presence of the rotation leads to the possibility of arising of additional Ekman layers between the intermediate and boundary layers. Here we have again two possibilities. The first one is that the internal layers are not influenced by the rotation which is the case for some intermediate interval of Taylor numbers. When the Taylor number increases further even the internal layers of the optimum fields begin to feel the rotation and the intermediate layers expand in the direction of the internal layers. Thus the four-layer optimum fields tend again to three-layer ones and the rotation leads to decreasing of the bound on the Nusselt number. The studied system possesses three parameters which can be changed: the Rayleigh number, the Taylor number, and the wave number connected with the optimum field corresponding to the $1-\alpha$-solution of the variational problem. Let us fix the Rayleigh number at some large enough value. In this region $\lambda \ll F_1$ and assuming that in the intermediate layers the terms containing derivatives in Eq. (20) are small compared with the other terms, we obtain the equation

$$-\alpha_1^6 w_1+\mathrm{Ta}\,\frac{d^2w_1}{dz^2}+\alpha_1^4\theta=0, \tag{26}$$

in which we keep the term containing the Taylor number in order to investigate the influence of the rotation. This term is zero without rotation. The increasing of the Taylor number leads to an increase of the influence of the rotation and when $\mathrm{Ta}\propto\alpha_1^4$ the term is considerably large and must be taken into an account. From this value of the Taylor number the rotation begins to influence the intermediate layers of the optimum fields. The term containing rotation in Eq. (26) becomes dominant and this is the case when $\mathrm{Ta}\propto\alpha_1^6$ and the rotation begins to also influence the internal layers of the optimum field. Thus we have the following possibilities with respect to the Taylor number and the wave number of the optimum field:

(1) Ta=0: no rotation. The optimum fields have three-layer structure which is symmetric for the case of a fluid layer with two rigid boundaries [8] and for the case of a fluid layer with two stress-free boundaries [16]. The optimum fields become asymmetric for the case of a fluid layer with rigid lower boundary and stress-free upper boundary [18,19].

(2) $\mathrm{Ta} \ll \alpha_1^4$: The rotation is weak enough and the layer structure of the optimum fields is the same as in the case without rotation.

(3) $\mathrm{Ta} \propto O(\alpha_1^4)$: All layers of the optimum fields except the internal ones feel the rotation. This region is quite interesting because here the optimum fields could have different structure, i.e., we can consider three-layer optimum fields or we can impose an additional requirement that the optimum fields must possess an Ekman layer in addition to internal, intermediate, and boundary ones.

(4) $\alpha_1^4 \ll \mathrm{Ta} \ll \alpha_1^6$: For the case of a fluid layer with two stress-free boundaries we shall obtain bounds on the convective heat transport on the basis of three-layer and four-layer optimum fields. For the case of a fluid layer with two rigid boundaries we discuss the four-layer structure of the optimum fields. We note here that the Euler-Lagrange equations of the variational problem considered in this article do not allow solutions describing three-layer optimum fields for the discussed interval of Taylor numbers in the case of rigid boundary conditions. For the case of a fluid layer with rigid lower boundary and stress-free upper boundary we shall investigate optimum fields with four layers.

(5) $\mathrm{Ta} \propto O(\alpha_1^6)$: The intermediate layers begin their expansion in the direction of the internal layers.

(6) $\mathrm{Ta} \gg \alpha_1^6$: The intermediate layers expand in direction of the internal layers. The increasing of the Taylor number leads to decreasing of the Nusselt number. As the Nusselt number is connected with the thickness of the boundary layers, a process of a thickening of the boundary layers begins and the Ekman layers become the thinnest ones for the optimum fields. The further increasing of the Taylor number leads to Nu=1, i.e., the heat is transported only by thermal conduction.

In this article we shall present a theory for the case $\alpha_1^4 \ll \mathrm{Ta} \ll \alpha_1^6$. We shall refer to these Taylor numbers as intermediate ones. We shall treat these as large Taylor numbers for which the rotation influences the internal layers of the optimum fields (i.e., $\mathrm{Ta} \gg \alpha_1^6$).

IV. THE CASE OF FLUID LAYER WITH RIGID LOWER BOUNDARY AND STRESS-FREE UPPER BOUNDARY

In the interval $\alpha_1^4 \ll \mathrm{Ta} \ll \alpha_1^6$ the coordinate remains z for the upper and lower internal sublayers of the optimum field. Moreover we have $w_1\theta_1=1$ and $f_1=0$. We remember that $\lambda/F_1 \ll 1$ and assuming also $\nabla^2\theta/(RF_1)\ll 1$ we obtain the solutions of the Euler-Lagrange equations: $w_1=\tilde{w}_1/\alpha$; $\theta_1=\tilde{\theta}_1\alpha$; $\tilde{w}_1=\tilde{\theta}_1=1$.

The Euler-Lagrange equations for the lower and upper intermediate layers are Eq. (26) and

$$\alpha_1^2 f_1 = \mathrm{Ta}^{1/2}\frac{dw_1}{dz}, \tag{27}$$

$$w_1\theta_1 = 1. \tag{28}$$

Let $w_1=\check{w}_1/\alpha_1$, $\theta=\alpha_1\check{\theta}_1$, and the coordinates for the upper and lower intermediate layers be as follows: $\xi_l-\xi_{0l}=\alpha_1^3\,\mathrm{Ta}^{-1/2}(z+1/2)$; $\xi_u-\xi_{0u}=\alpha^3\,\mathrm{Ta}^{-1/2}(1/2-z)$. Here $\xi_{l0},\xi_{u,0}$ are parameters which can be determined by a matching to the corresponding Ekman layers. Thus we have to solve the equation

$$\check{w}_1\left(\frac{d^2}{d\xi_{u,l}^2}-1\right)\check{w}_1+1=0. \tag{29}$$

For our purposes we need only an approximate solution of Eq. (29) when $\xi_{u,l}\to 0$. This solution must satisfy the requirement $\check{w}_1\to 0$, Eq. (29), and the first integral of Eq. (29)

$$\left(\frac{d\check{w}_1}{d\xi_{u,l}}\right)^2-\check{w}_1^2+\ln(\check{w}_1^2)+\mathrm{const}=0, \tag{30}$$

where const is a constant of integration. The solutions presented in Refs. [71] and [72] do not satisfy the above first integral. The solution which satisfies Eq. (30) along with the other requirements is

$$\check{w}_1(\xi_{u,l})=\xi_{u,l}\sqrt{\ln\left(\frac{1}{\xi_{u,l}^2}\right)-\ln\ln\left(\frac{1}{\xi_{u,l}^2}\right)}. \tag{31}$$

This solution leads to changes in the thicknesses of the sublayers of the optimum field and to changes in the upper bounds on the convective heat transport for the cases of fluid layer with two stress-free and with two rigid boundaries (see Secs. V and VI). We shall match Eq. (31) to the solutions of the upper and lower Ekman layers.

In the Ekman layers we have approximately $w_1\theta_1=1$ and the corresponding Euler-Lagrange equations can be written in the form

$$\frac{d^4w_1}{dz^4}-\mathrm{Ta}^{1/2}\frac{df_1}{dz}=0, \tag{32}$$

$$\frac{d^2f_1}{dz^2}+\mathrm{Ta}^{1/2}\frac{dw_1}{dz}=0. \tag{33}$$

For the lower Ekman layer we introduce the coordinate $\phi_l=(1/\sqrt{(2)}\mathrm{Ta}^{1/4}(1/2+z)$ and the boundary conditions are: $w_1=f_1=dw_1/d\phi_l=0$ when $\phi_l=0$. The solutions of the Euler-Lagrange equations are

$$w_1=c_l\sqrt{2}-2c_l\exp(-\phi_l)\cos(\phi_l-\pi/4), \tag{34}$$

$$\mathrm{Ta}^{-1/4}f_1=2c_l-2c_l\,\exp(-\phi_l)\cos(\phi_l). \tag{35}$$

c_l is a constant which will be determined by the matching below.

For the upper Ekman layer we introduce the coordinate $\phi_u = [1/\sqrt{(2)}]\mathrm{Ta}^{1/4}(1/2 - z)$. The boundary conditions when $\phi_u = 0$ are $w_1 = df_1/dz = d^2 w_1/dz^2 = 0$ and the solutions of the Euler-Lagrange equations are

$$w_1 = c_u(1 - \exp(-\phi_u)\cos(\phi_u)), \tag{36}$$

$$f_1 = \frac{1}{\sqrt{2}} c_u \mathrm{Ta}^{1/4}[\exp(-\phi_u)(\cos(\phi_u) - \sin(\phi_u)) + 2\phi_u] + k\,\mathrm{Ta}^{1/2}, \tag{37}$$

where k is a constant of integration determined by the boundary conditions.

We shall obtain expression for c_u and c_l by matching the solutions between the corresponding intermediate and Ekman layers. The matching of the solutions for w_1 and f_1 between the lower intermediate and Ekman layers leads us to the relationships

$$c_l = \frac{1}{2\,\mathrm{Ta}^{1/4}} \sqrt{\ln\left(\frac{2\,\mathrm{Ta}^{1/2}}{\alpha_1^2}\right) - \ln\ln\left(\frac{2\,\mathrm{Ta}^{1/2}}{\alpha_1^2}\right)}, \tag{38}$$

$$\xi_{l,0} = \frac{\alpha}{\sqrt{2}\,\mathrm{Ta}^{1/4}}, \tag{39}$$

where $\xi_{l,0}$ is the coordinate of the matching point. The matching of the solutions for w_1 and f_1 between the upper intermediate and Ekman layers at the point $\xi_{u,0}$ leads to the relationships

$$c_u = \frac{\alpha_1}{\mathrm{Ta}^{1/2}} \sqrt{\ln\left(\frac{\mathrm{Ta}}{\alpha_1^4}\right) - \ln\ln\left(\frac{\mathrm{Ta}}{\alpha_1^4}\right)}, \tag{40}$$

$$\xi_{u,0} = \frac{\alpha_1^2}{\mathrm{Ta}^{1/2}}. \tag{41}$$

In the Euler-Lagrange equations for the upper and lower boundary layers dominant terms are those containing the highest derivatives. Thus we obtain

$$\frac{d^4 w_1}{dz^4} = 0, \tag{42}$$

$$\frac{1}{RF_1}\frac{d^2\theta_1}{dz^2} + w_1(1 - w_1\theta_1) = 0, \tag{43}$$

$$\frac{d^2 f_1}{dz^2} = 0. \tag{44}$$

For the lower and upper boundary layers we introduce the coordinates $\eta_l = (1/\delta_l)(z + 1/2)$ and $\eta_u = (1/\delta_u)(1/2 - z)$. The solutions for w_1, θ_1, and f_1 in the boundary layers must match the corresponding solution for the Ekman layers. Performing the matching we obtain the relationships for the lower boundary layer

$$w_1 = (c_l\,\mathrm{Ta}^{1/2}\delta_l^2\eta_l^2)/\sqrt{2}, \tag{45}$$

$$f_1 = \sqrt{2}c_l\,\mathrm{Ta}^{1/2}\delta_l\eta_l. \tag{46}$$

Analogous for the upper boundary layer we obtain

$$w_1 = c_u\,\mathrm{Ta}^{1/4}\delta_u\eta_u/\sqrt{2}, \tag{47}$$

$$f_1 = k\,\mathrm{Ta}^{1/2} + \frac{\sqrt{2}}{2}c_u\left(\mathrm{Ta}^{1/4} - \frac{1}{2}\mathrm{Ta}^{3/4}\delta_u^2\eta_u^2\right). \tag{48}$$

We assume

$$2 = RF_1 c_u^2\,\mathrm{Ta}^{1/2}\delta_u^4, \tag{49}$$

$$2 = c_l^2\,\mathrm{Ta}\,RF_1\delta_l^6, \tag{50}$$

and thus Eq. (43) becomes

$$\frac{d^2\hat{\theta}_1}{d\eta_l^2} + \eta_l^2(1 - \eta_l^2\hat{\theta}_1) = 0 \tag{51}$$

for the lower boundary layer where

$$\theta_1 = \sqrt{2}\hat{\theta}_1/(c_l\,\mathrm{Ta}^{1/2}\delta_l^2), \tag{52}$$

and

$$\frac{d^2\hat{\theta}_1}{d\eta_u^2} + \eta_u(1 - \eta_u\hat{\theta}_1) = 0 \tag{53}$$

for the upper boundary layer where

$$\theta_1 = \sqrt{2}\hat{\theta}_1/(c_u\,\mathrm{Ta}^{1/4}\delta_u). \tag{54}$$

We have F_1, α_1, δ_u, and δ_l as unknown quantities and for them we have Eqs. (49), (50), and

$$F_1 = \frac{1 - (1/R)\langle|\nabla\theta|^2\rangle}{\langle(1 - \overline{w\theta^2})\rangle} = \frac{N}{Z}, \tag{55}$$

which must be rewritten in terms of the other unknown quantities. For the denominator Z we obtain

$$Z = \delta_l I_l + \delta_u I_u, \tag{56}$$

where

$$I_l = \int_0^\infty d\eta_l(1 - \eta_l^2\hat{\theta}_1)^2 \approx 0.9255, \tag{57}$$

and

$$I_u = \int_0^\infty d\eta_u(1 - \eta_u\hat{\theta}_1)^2 \approx 0.796\,35. \tag{58}$$

In order to obtain a relationship for the numerator N we take into an account that for the $1-\alpha$-solution,

$$\langle|\nabla\theta|^2\rangle = \langle\alpha_1^2\theta_1^2\rangle + \langle(d\theta_1/dz)^2\rangle. \tag{59}$$

For the first term of Eq. (59) we assume that the contributions from the intermediate, Ekman, and the boundary layers are small in comparison to the contribution from the internal layers which is approximately α_1^4. For the second term the contributions from the boundary layers are dominant. Evaluating them and introducing the integrals

$$J_l = \int_0^\infty \left(\frac{d\hat{\theta}_1}{d\eta_l} \right)^2 \approx 0.1851, \tag{60}$$

$$J_u = \int_0^\infty \left(\frac{d\hat{\theta}_1}{d\eta_u} \right)^2 \approx 0.2635, \tag{61}$$

we obtain the following expression for F_1:

$$F_1 = \frac{1-\alpha_1^4/R}{\delta_l D_l + \delta_u D_u}, \tag{62}$$

where $D_l = I_l + 2J_l$ and $D_u = I_u + 2J_u$. The approximate solution of the system of Eqs. (49), (50), and (62) is

$$\delta = 2^{7/15}(D_u+D_l)^{1/5}\,\mathrm{Ta}^{-1/10}R^{-1/5} \times [\ln(2\,\mathrm{Ta}^{1/2}) - \ln\ln(2\,\mathrm{Ta}^{1/2})]^{-1/5}, \tag{63}$$

where the thicknesses of the upper and lower boundary layers are of the same order. For α_1 and F_1 we obtain

$$\alpha_1 = 2^{-1/5}(D_u+D_l)^{1/5}\,\mathrm{Ta}^{2/5}R^{-1/5} \times [\ln(2\,\mathrm{Ta}^{1/2}) - \ln\ln(2\,\mathrm{Ta}^{1/2})]^{3/10} \times [\ln(\mathrm{Ta}) - \ln\ln(\mathrm{Ta})]^{-1/2}, \tag{64}$$

$$F_1 = 2^{-7/15}(D_u+D_l)^{-6/5}\,\mathrm{Ta}^{1/10}R^{1/5} \times [\ln(2\,\mathrm{Ta}^{1/2}) - \ln\ln(2\,\mathrm{Ta}^{1/2})]^{1/5}. \tag{65}$$

V. BOUNDS FOR THE CASE OF A FLUID LAYER WITH TWO STRESS-FREE BOUNDARIES

Because of the symmetric boundary conditions it is sufficient to consider the layers of the optimum fields from the midplane of the fluid layer to the lower boundary of the fluid layer. For the case of optimum fields with three-layer structure we have in the internal layer $w_1 = \tilde{w}_1/\alpha_1; \theta_1 = \alpha_1\tilde{\theta}_1$. The coordinate for the intermediate layer is $\xi = \alpha_1^3\,\mathrm{Ta}^{-1/2}(z+1/2)$ and the solution for w_1 we need for an analysis of the boundary layer is $w_1 = \check{w}_1/\alpha_1$ where

$$\check{w}_1 = \xi \sqrt{\ln\left(\frac{1}{\xi^2}\right) - \ln\ln\left(\frac{1}{\xi^2}\right)}. \tag{66}$$

For the boundary layer we have the coordinate $\eta = (\mathrm{Ta}^{1/2}/(\alpha_1\delta_1))(z+1/2)$ and rescale the field w_1 and θ_1 as follows: $w_1 = A\hat{w}_1; \theta_1 = \hat{\theta}_1/A$. A is determined by a matching between the solutions of the Euler-Lagrange equations for w_1 in the intermediate and in the boundary layer. The result is

$$A = \frac{\alpha_1^3\delta_1}{\mathrm{Ta}} \sqrt{\ln\left(\frac{\mathrm{Ta}^2}{\alpha_1^8\delta_1^2}\right) - \ln\ln\left(\frac{\mathrm{Ta}^2}{\alpha_1^8\delta_1^2}\right)}. \tag{67}$$

In addition $\hat{w}_1 = \eta$ which satisfies the boundary layer equation for w_1 and the stress-free boundary conditions. We assume that the relationship, Eq. (71), holds and obtain the following equation for $\hat{\theta}_1$:

$$\frac{d^2\hat{\theta}_1}{d\eta^2} + \hat{w}_1(1-\hat{w}_1\hat{\theta}_1) = 0, \tag{68}$$

with the boundary conditions $\hat{\theta}_1(0)=0;\ \ \hat{\theta}_1(\eta\to\infty) = 1/\eta$. The solution is

$$\hat{\theta}_1 = \frac{1}{2}\eta\int_0^1 dt(1-t^2)^{1/4}\exp\left(-\frac{1}{2}\eta^2 t\right). \tag{69}$$

We obtain the dependencies of the optimum convective heat transport, thickness of the boundary layers and wave number by means of the system of equations

$$F_1 = \frac{\mathrm{Ta}^{1/2}}{2\alpha_1\delta_1 D}\left(1 - \frac{\alpha_1^4}{R}\right), \tag{70}$$

$$RF_1\alpha_1^8\delta_1^4\,\mathrm{Ta}^{-3}\left[\ln\left(\frac{\mathrm{Ta}^2}{\alpha_1^8\delta_1^2}\right) - \ln\ln\left(\frac{\mathrm{Ta}^2}{\alpha_1^8\delta_1^2}\right)\right] = 1, \tag{71}$$

$$\frac{\partial F_1}{\partial\alpha_1} = 0, \tag{72}$$

where $D = I + J$, $I = \int_0^\infty d\eta(1-\eta\hat{\theta}_1)^2$, $J = \int_0^\infty d\eta(d\hat{\theta}_1/d\eta)^2$. From Eqs. (70) and (71) we obtain the solutions for the quasilinear approximation of the solved problem

$$\delta_1 = (2D)^{1/3}\alpha_1^{-7/3}\,\mathrm{Ta}^{5/6}(R-\alpha_1^4)^{-1/3} \times \left[\ln\left(\frac{\mathrm{Ta}^2}{\alpha_1^8\delta_1^2}\right) - \ln\ln\left(\frac{\mathrm{Ta}^2}{\alpha_1^8\delta_1^2}\right)\right]^{-1/3}, \tag{73}$$

$$F_1 = (2D)^{-4/3}\,\mathrm{Ta}^{-1/3}R^{-1}(R-\alpha_1^4)^{4/3}\alpha_1^{-4/3} \times \left[\ln\left(\frac{\mathrm{Ta}^2}{\alpha_1^8\delta_1^2}\right) - \ln\ln\left(\frac{\mathrm{Ta}^2}{\alpha_1^8\delta_1^2}\right)\right]^{1/3}. \tag{74}$$

Using Eq. (72) the approximate solution of the system Eqs. (70)–(72) is

$$\alpha_1 = \left(\frac{R}{5}\right)^{1/4}, \tag{75}$$

$$\delta_1 = (3D)^{1/3}\left(\frac{R}{5}\right)^{-11/12}\mathrm{Ta}^{5/6}\left[\ln\left(\frac{5\,\mathrm{Ta}^2}{R}\right)\right]^{-1/3} \times \left(1 + \frac{\ln\{\ln[(5\,\mathrm{Ta}^2/R)^{1/6}]\}}{3\ln[(5\,\mathrm{Ta}^2/R)^{1/6}]}\right)^{1/3}, \tag{76}$$

$$F_1=2\cdot 3^{-1/3}\cdot 5^{-5/3}D^{-4/3}R^{2/3}\,\mathrm{Ta}^{-1/3}\left[\ln\left(\frac{5\,\mathrm{Ta}^2}{R}\right)\right]^{1/3}\times\left(1+\frac{\ln\{\ln[(5\,\mathrm{Ta}^2/R)^{1/6}]\}}{3\ln[(5\,\mathrm{Ta}^2/R)^{1/6}]}\right)^{-1/3}. \tag{77}$$

The last terms in Eqs. (76) and (77) are the corrections to the thickness of the boundary layers and to the upper bound on the convective heat transport in comparison to the corresponding quantities obtained in Ref. [71]. Thus the solution, Eq. (66), leads to thicker boundary layers and to lower bounds on the convective heat transport and the corresponding bound obtained in Ref. [71] can be treated as upper bound on the upper bound on the Nusselt number.

For the case of optimum fields with four-layer structure we shall discuss again the layers of the optimum fields starting from the midplane of the fluid layer in direction of the lower boundary of the fluid layer. For the internal layer the solutions of the Euler-Lagrange equations are: $w_1=\tilde{w}_1/\alpha_1$; $\theta_1=\tilde{\theta}_1\alpha_1$; $\tilde{w}_1=\tilde{\theta}_1=1$. For the intermediate layer the coordinate is :$\xi=\alpha_1/(\sqrt{2}\,\mathrm{Ta}^{1/4})+\alpha_1^3\,\mathrm{Ta}^{1/2}(z+1/2)$. The approximate solution for w_1 when ξ is small is $w_1=\check{w}_1/\alpha$ where $\check{w}_1$ has the same form as Eq. (66). The coordinate for the Ekman layer is: $\phi=(\mathrm{Ta}^{1/4}/\sqrt{2})(z+1/2)$ and the solutions of the Euler-Lagrange equations in this layer are

$$w_1=c(1-e^{-\phi}\cos(\phi)), \tag{78}$$

$$f_1=k\,\mathrm{Ta}^{1/2}-\sqrt{2}c\,\mathrm{Ta}^{1/4}\left[\phi+\frac{1}{2}e^{-\phi}(\cos(\phi)-\sin(\phi))\right], \tag{79}$$

where the constant of integration k can be determined from the boundary conditions and

$$c=\frac{\alpha_1}{\mathrm{Ta}^{1/2}}\sqrt{\ln\left(\frac{\mathrm{Ta}}{\alpha_1^4}\right)-\ln\ln\left(\frac{\mathrm{Ta}}{\alpha_1^4}\right)}. \tag{80}$$

The coordinate for the boundary layer is $\eta=(1/\delta)(z+1/2)$ and performing a matching of the solutions of the Euler-Lagrange equations between the Ekman and the boundary layers we obtain

$$w_1=\frac{c\,\mathrm{Ta}^{1/4}\delta}{\sqrt{2}}\,\eta, \tag{81}$$

$$\theta_1=\frac{\sqrt{2}}{c\,\mathrm{Ta}^{1/4}\delta}\hat{\theta}_1, \tag{82}$$

where $\hat{\theta}_1$ is a solution of the equation

$$\frac{d^2\hat{\theta}_1}{d\eta^2}+\eta(1-\eta\hat{\theta}_1)=0, \tag{83}$$

which satisfies the stress-free boundary conditions. Thus the system of equations we have to solve is

$$F_1=(2D)^{-1}\delta_1^{-1}(1-\alpha_1^4/R), \tag{84}$$

$$RF_1\delta_1^4\frac{\alpha_1^2}{\mathrm{Ta}^{1/2}}\left[\ln\left(\frac{\mathrm{Ta}}{\alpha_1^4}\right)-\ln\ln\left(\frac{\mathrm{Ta}}{\alpha_1^4}\right)\right]=2, \tag{85}$$

where $D=I+J$ and I,J have the same form as for the case of the theory based on the three-layer optimum field. The solution of the last system of equations is

$$\delta_1=(4D)^{1/3}(R-\alpha_1^4)^{-1/3}\,\mathrm{Ta}^{1/6}\alpha_1^{-2/3}\times\left[\ln\left(\frac{\mathrm{Ta}}{\alpha_1^4}\right)-\ln\ln\left(\frac{\mathrm{Ta}}{\alpha_1^4}\right)\right]^{-1/3}, \tag{86}$$

$$F_1=2^{-5/3}D^{-4/3}R^{1/3}(1-\alpha_1^4/R)^{4/3}\,\mathrm{Ta}^{-1/6}\alpha_1^{2/3}\times\left[\ln\left(\frac{\mathrm{Ta}}{\alpha_1^4}\right)-\ln\ln\left(\frac{\mathrm{Ta}}{\alpha_1^4}\right)\right]^{1/3}. \tag{87}$$

F_1 has a maximum when α_1 is made as large as possible, i.e., when $\alpha_1\propto R^{1/4}$. Thus the upper bound F^* on F_1 is

$$F^*\propto R^{1/2}\,\mathrm{Ta}^{-1/6}[\ln(\mathrm{Ta}/R)-\ln\ln(\mathrm{Ta}/R)]^{1/3}. \tag{88}$$

VI. BOUND FOR THE CASE OF A FLUID LAYER WITH TWO RIGID BOUNDARIES

It is sufficient to discuss the sublayers of the optimum fields from the midplane of the fluid layer in the direction of the lower boundary of the fluid layer. The solutions of the Euler-Lagrange equations for the internal layer are $w_1=1/\alpha_1$, $\theta_1=\alpha_1$. In the intermediate layer the coordinate is $\xi=[\alpha_1/(\sqrt{2}\,\mathrm{Ta}^{1/4})]+\alpha_1^3\,\mathrm{Ta}^{-1/2}(z+1/2)$. The solution for w_1 in this layer when ξ is small is as for the cases discussed in the previous section.

The coordinate in the Ekman layer is $\phi=(\mathrm{Ta}^{1/4}/\sqrt{2})(z+1/2)$ and the solutions of the Euler-Lagrange equations are as follows:

$$w_1=c\sqrt{2}-2ce^{-\phi}\cos(\phi-\pi/4), \tag{89}$$

$$f_1=2c\,\mathrm{Ta}^{1/4}[1-e^{-\phi}\cos(\phi)], \tag{90}$$

where

$$c=\frac{1}{2\,\mathrm{Ta}^{1/4}}\sqrt{\ln\left(\frac{2\,\mathrm{Ta}^{1/2}}{\alpha_1^2}\right)-\ln\ln\left(\frac{2\,\mathrm{Ta}^{1/2}}{\alpha_1^2}\right)}. \tag{91}$$

The coordinate for the boundary layer is $\eta=(1/\delta)(z+1/2)$ and the matching of the solutions of the Euler-Lagrange equations between the Ekman and the boundary layers leads us to the solutions

$$w_1=\frac{c\,\mathrm{Ta}^{1/2}\delta^2}{\sqrt{2}}\,\eta^2, \tag{92}$$

$$\theta_1=\frac{\sqrt{2}}{c\,\mathrm{Ta}^{1/2}\delta^2}\hat{\theta}_1, \tag{93}$$

where $\hat{\theta}_1$ is a solution of the equation

$$\frac{d^2\hat{\theta}_1}{d\eta^2}+\eta^2(1-\eta^2\hat{\theta}_1)=0, \quad (94)$$

which satisfies the rigid boundary conditions for $\hat{\theta}_1$. For the thickness of the boundary layers and for the convective heat transport we have to solve the equations

$$F_1=(2D)^{-1}\delta_1^{-1}(1-\alpha_1^4/R), \quad (95)$$

$$RF_1\mathrm{Ta}^{1/2}\delta_1^6\left[\ln\left(\frac{2\,\mathrm{Ta}^{1/2}}{\alpha_1^2}\right)-\ln\ln\left(\frac{2\,\mathrm{Ta}^{1/2}}{\alpha_1^2}\right)\right]=8, \quad (96)$$

where $D=I+J$, $I=\int_0^\infty d\eta(1-\eta^2\hat{\theta}_1)^2$, and $J=\int_0^\infty d\eta(d\hat{\theta}_1/d\eta)^2$. The solution is

$$\delta_1=4^{2/5}D^{1/5}\,\mathrm{Ta}^{-1/10}R^{-1/5}(1-\alpha_1^4/R)^{-1/5}\times\left[\ln\left(\frac{2\,\mathrm{Ta}^{1/2}}{\alpha_1^2}\right)-\ln\ln\left(\frac{2\,\mathrm{Ta}^{1/2}}{\alpha_1^2}\right)\right]^{-1/5}, \quad (97)$$

$$F_1=2^{-9/5}D^{-6/5}\,\mathrm{Ta}^{1/10}R^{1/5}(1-\alpha_1^4/R)^{6/5}\times\left[\ln\left(\frac{2\,\mathrm{Ta}^{1/2}}{\alpha_1^2}\right)-\ln\ln\left(\frac{2\,\mathrm{Ta}^{1/2}}{\alpha_1^2}\right)\right]^{1/5}. \quad (98)$$

The last terms in Eqs. (97) and (98) are corrections to the boundary layer thickness and to the convective heat transport in comparison to the corresponding quantities obtained in Ref. [72]. Thus the upper bound obtained in Ref. [72] can be treated as upper bound on the upper bound on the convective heat transport. Using again the assumption $\alpha_1\propto\mathrm{Ta}^{1/6}$ we can obtain an upper bound on F_1:

$$F^{**}\propto\mathrm{Ta}^{1/10}R^{1/5}[\ln(\mathrm{Ta})-\ln\ln(\mathrm{Ta})]^{1/5}. \quad (99)$$

VII. DISCUSSION

The obtained bounds on the convective heat transport are valid in an interval of Taylor numbers. For the fluid layer with two-stress boundaries and for the case of three-layer optimum fields this interval is determined by the requirement

$$\alpha_1^4\ll\mathrm{Ta}\ll\alpha_1^6 \quad (100)$$

and by the requirement that the thickness of the boundary layers of the optimum fields must be much smaller than 1. Thus the interval of the Taylor numbers in which the obtained bounds are valid becomes

$$O(R)\ll\mathrm{Ta}\ll O(R^{11/10}). \quad (101)$$

The interval of validity of the bound on the heat transport obtained on the basis of four-layer optimum fields for the case of a fluid layer with two stress-free boundaries is

$$O(R)\ll\mathrm{Ta}\ll O(R^{4/3}(\ln R)^{4/3}). \quad (102)$$

In this integral of validity the upper bound on the convective heat transport changes its value from $F\propto R^{1/3}$ to values larger than $R^{5/18}(\ln R)^{1/3}$. It is interesting that the last bound is close to the power law $R^{2/7}$ when the Rayleigh numbers are high enough.

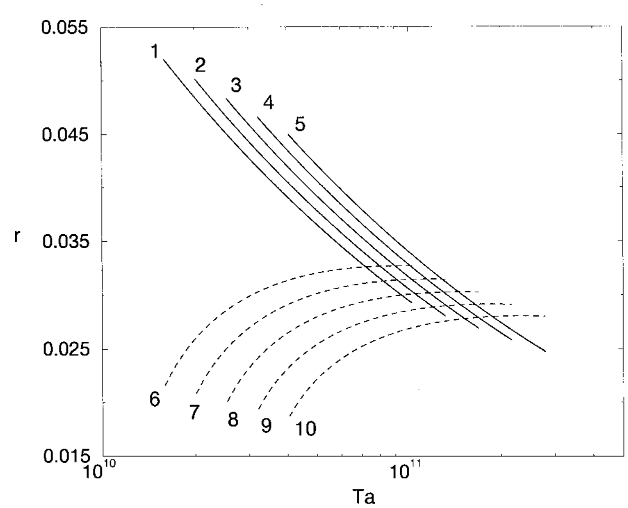

FIG. 1. Ratio r between the upper bounds obtained by three-layer and four-layer optimum fields in this article and the corresponding bound of Constantin, Hallstrom, and Putkaradze (case of a rotating layer with stress-free boundaries). Solid lines: ratio for the case of three-layer optimum fields; dashed lines: ratio for the case of four-layer optimum fields. Lines 1 and 6: $R=10^{10.1}$; lines 2 and 7: $R=10^{10.2}$; lines 3 and 8: $R=10^{10.3}$; lines 4 and 9: $R=10^{10.4}$; lines 5 and 10: $R=10^{10.5}$.

The interval of the validity of the bounds obtained for the case of a fluid layer with two rigid boundaries is determined by Eq. (100) and by the requirement that $w_1\theta_1\approx 1$ in the Ekman layers. Thus we obtain

$$O(R)\ll\mathrm{Ta}\ll O[(R\ln(R))^{4/3}]. \quad (103)$$

For the case of a fluid layer with rigid lower boundary and stress-free upper boundary we have as an additional requirement, $\alpha_1^4<R$. Thus the application area of the bound with respect to the Taylor number is

$$O(R)\ll\mathrm{Ta}\ll O(R^{9/8}). \quad (104)$$

The bounds on the Nusselt number obtained in Ref. [73] are

$$\mathrm{Nu}\leq\min[\sqrt{R}/6-1;1+E^2R^2/2+(7E^2+2E)R^2], \quad (105)$$

for the case of Dirichlet boundary conditions and

$$\mathrm{Nu}\leq\min[\sqrt{R}/6-1;1+R^2E^2/2], \quad (106)$$

for the case of periodic and stress-free boundary conditions where R and E are the Rayleigh and the Ekman numbers. These results are useful because they are valid for the whole region of values of Rayleigh and Ekman numbers where the thermal convection is present. The results obtained in this article are valid for selected regions of Rayleigh and Taylor numbers. The reason is that the assumptions concerning some terms of the Euler-Lagrange equations of the variational problem are applicable only for these selected regions. For the other regions of the Rayleigh and Taylor numbers the Euler-Lagrange equations are complex enough and can be solved only numerically.

Within their intervals of validity we can compare the upper bounds obtained here with the bounds of Constantin, Hallstrom, and Putkaradze [73]. Figure 1 shows the ratio between the bounds obtained in this article on the convective heat transport based on three-layer and four-layer fields and the corresponding bound from Ref. [73]. We would like to note the following:

(1) When the Rayleigh number is fixed the bounds obtained by the Howard-Busse method are valid in an interval of Taylor numbers, which is determined by the fixed value of the Rayleigh number. Thus the curves corresponding to different Rayleigh numbers have different length.

(2) We note that the bounds obtained in Ref. [73] are the first step in the direction of incorporating of the effect of the rotation in the Doering-Constantin bounds for the convective heat transport. The properly applied Doering-Constantin method should yield the same bounds as the bounds obtained above by means of the Howard-Busse method and in this case the ratio between the Howard-Busse and Doering-Constantin bounds should be equal to 1.

(3) The upper bound on the convective heat transport for the lower region of the interval of validity of the Howard-Busse bounds is this one obtained by means of the three-layer optimum fields. In the upper region of the interval of the validity of the bounds, the bound obtained by means of the four-layer optimum fields becomes larger than the three-layer fields bound. Thus the upper bound on the convective heat transport in this region is this one obtained by means of the four-layer optimum fields.

For the case of a fluid layer with rigid lower boundary and stress-free upper boundary we can consider also optimum fields which have an asymmetric structure: three sublayers from the midplane of the fluid layer in the direction of the upper stress-free boundary and four sublayers from the midplane of the fluid layer in direction of the lower rigid boundary. Such fields lead however to rapidly increasing boundary layer thickness with the Taylor number and thus the assumption that the boundary layer thicknesses are small is violated. In a future article we shall discuss this problem numerically in order to see whether this asymmetric variant of the optimum field arises as numerical solution of the Euler-Lagrange equations of the variational problem. The numerical discussion will clear also the relations between the three-layer and four-layer optimum fields. It could be expected that the numerical solutions of the Euler-Lagrange equations of the variational problem can describe three-layer fields when the rotation rate is small. These fields could develop Ekman layers with increasing Taylor number. It will be also of interest to study numerically the expansion of the intermediate layers of the optimum fields in the direction of the internal layers.

In this article we have investigated the $1-\alpha$-solution of the variational problem for the case of intermediate values of the Taylor number. This solution gives the upper bound on the convective heat transport in a finite range of Rayleigh number. In order to obtain the upper bounds for higher values of the Rayleigh number we have to consider the multi-α solutions of the variational problem. This as well as the development of the theory for the case of large Taylor numbers will be a subject of future research.

APPENDIX: THE QUASILINEAR APPROXIMATION

The connection between the simplest variational problem of the optimum theory of turbulence and the quasilinear approximation has been pointed out by Chan [8] and extensively used in Ref. [72]. Let us discuss steady solutions of the Navier-Stokes equations and subtract the horizontal average of the heat equation from the heat equation. Thus we obtain the relationship

$$\boldsymbol{\nabla}^2 T - u_3 \frac{d\overline{\Theta}}{dz} = \nabla \cdot (\mathbf{u}T) - \frac{d}{dz}\overline{u_3 T}. \tag{A1}$$

Within the quasilinear (mean-field) approximation we neglect the terms describing the interactions between the fluctuating quantities. Thus from Eq. (A1) we obtain

$$\nabla^2 T = u_3 \frac{d\overline{\Theta}}{dz}. \tag{A2}$$

From the horizontal average of the heat equation we obtain

$$\frac{d^2\overline{\Theta}}{dz^2} = \frac{d}{dz}\overline{u_3 T}. \tag{A3}$$

Integrating Eq. (A3) and taking into account the boundary conditions we obtain the relationship

$$\frac{d\overline{\Theta}}{dz} = \overline{u_3 T} - \langle u_3 T\rangle - 1. \tag{A4}$$

We rescale the quantities as follows: $T = \langle u_3 T\rangle^{1/2} R^{-1/2}\theta$, $u_3 = \langle u_3 T\rangle^{1/2} R^{1/2} w$; $\overline{\Theta} = \langle u_3 T\rangle \Theta^*$. Using this and the relationship between the convective heat transport and the Nusselt number $\langle u_3 T\rangle = \mathrm{Nu}-1$ we obtain from Eq. (A4)

$$\frac{d\Theta^*}{dz} = \overline{w\theta} - \langle w\theta\rangle - \frac{1}{\mathrm{Nu}-1}. \tag{A5}$$

This is one of the equations of the quasilinear approximation. Another equation of this approximation we obtain after setting $P=\infty$ in the Navier-Stokes equation. Then we rescale the result by the above rescaling relationships and take the z-component of the double curl of the resulting equation as well as the z-component of the curl of the resulting equation. Thus we obtain

$$\nabla^4 w + \nabla_1^2\theta - \frac{2}{E}\frac{\partial f}{\partial z} = 0, \tag{A6}$$

$$\nabla^2 f + \frac{2}{E}\frac{\partial w}{\partial z} = 0, \tag{A7}$$

where f is the rescaled vertical component of the vorticity. The last equation of the quasilinear approximation is the rescaled Eq. (A2),

$$\frac{1}{R(\mathrm{Nu}-1)}\nabla^2\theta = w\left[\overline{w\theta} - 1 - \frac{1}{\mathrm{Nu}-1}\right]. \tag{A8}$$

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