Quantum chaos border for quantum computing

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We study a generic model of quantum computer, composed of many qubits coupled by short-range interaction. Above a critical interqubit coupling strength, quantum chaos sets in, leading to quantum ergodicity of the computer eigenstates. In this regime the noninteracting qubit structure disappears, the eigenstates become complex, and the operability of the computer is destroyed. Despite the fact that the spacing between multiqubit states drops exponentially with the number of qubits n, we show that the quantum chaos border decreases only linearly with n. This opens a broad parameter region where the efficient operation of a quantum computer remains possible.

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Since the pioneering work of Feynman [1] and modern developments of efficient algorithms [2] and error-correcting codes [3,4], the realization of quantum computers became a challenge of modern physics [5]. Different experimental realizations have been proposed, including ion traps [6], nuclear magnetic resonance systems [7], nuclear spins with interaction controlled electronically [8,9], quantum dots [10], Cooper pair boxes [11], and optical lattices [12]. A key common feature of these experimental settings is the presence of interacting qubits (two-level systems). Here we analyze the effect of qubit interaction on operability of the quantum computer. The interaction is required since a quantum computer needs to perform two-qubit logical operation such as XOR [5]. We note that such a two-qubit gate has been experimentally realized [13].

In an isolated system of n uncoupled qubits, the dimension of the total Hilbert space N_H increases exponentially with n ($N_H=2^n$), while all eigenvalues of the Hamiltonian are included in an energy interval of size $\Delta E \sim n\Delta_0$, where Δ_0 is the average energy distance between the two states of one qubit. As a result, the average spacing Δ_n between adjacent energy levels of the Hamiltonian decreases exponentially with the number of qubits ($\Delta_n \sim n\Delta_0/N_H \ll \Delta_0$). When a coupling J between the qubits is added ($J<\Delta_0$), one still has $\Delta E \sim n\Delta_0$, N_H is unchanged, and the above estimate for Δ_n still holds. This general result for Δ_n is related to the exponentially large size N_H of the Hilbert space, which is one of the main reasons for the striking efficiency of quantum computing [1,2]. It implies that dense highly excited states are needed for the computation. However, when performing the computation one wants to operate with noninteracting multiqubit states $|\psi_i\rangle = |\alpha_1, \ldots, \alpha_n\rangle$ where $\alpha_k=0,1$ marks the polarization of each individual qubit. These quantum register states should remain well defined in the presence of interqubit coupling even if multiqubit levels are exponentially dense. Therefore the mixing of noninteracting multiqubit states induced by the interaction is crucial for the computer operability. In the field of quantum chaos [14,15] it is known that noninteracting states will be eventually mixed by the interaction and quantum ergodicity will set in: each quantum computer eigenstate will be composed of a large number of noninteracting multiqubit states $|\psi_i\rangle$ and the original quantum register states will be washed out. At first glance one would expect that such mixing happens when the coupling between qubits becomes comparable to the multiqubit spacing Δ_n. In such a case, the creation of quantum computers competitive with classical ones would be rather difficult: since hundreds of qubits are necessary, this would lead to absurdly strict restrictions on coupling strength. Indeed, for $n=1000$, the minimum number of qubits for which Shor's algorithm becomes useful [5], the multiqubit spacing becomes $\Delta_n \sim 10^3 \times 2^{-10^3}\Delta_0 \sim 10^{-298}$ K, where we used $\Delta_0 \sim 1$ K that corresponds to the typical one-qubit spacing in the experimental proposals [8,9]. It is clear that the residual interaction J between qubits in any experimental realization of the quantum computer will be larger than this. For example, in the proposal [9], the increase of effective electron mass by a factor of two, induced by the electrostatic gate potential, means that the spin-spin interaction is changed from $J \sim \Delta_0 \sim 1$ K (corresponding to a distance between donors of 200 Å and an effective Bohr radius of 30 Å in Eq. (2) of [9]) to the residual interaction $J \sim 10^{-5}$ K$\gg \Delta_n$.

However the problem is not so simple, since the interaction is always of two-body nature and not all of the multiqubit states are directly coupled. Actually the number of states directly coupled to such a quantum register state $|\psi_i\rangle$ increases not faster than quadratically with n. A similar problem appears in other physical many-body interacting systems such as nuclei, complex atoms, quantum dots, and quantum spin glasses [16–20]. It was realized that sufficiently strong interaction leads to quantum chaos and internal (dynamical) thermalization, where the eigenstates properties follow the predictions of random matrix theory (RMT) [14–18]. The quantum chaos border for this dynamical thermalization has been established only recently and it has been shown that the relevant coupling strength should be larger than the energy spacing between directly coupled states Δ_c [17,20]. Since Δ_c drops algebraically with n, it is exponentially larger than $\Delta_n \sim n2^{-n}\Delta_0$, and therefore a relatively large coupling strength is required for the emergence of quantum chaos and ergodicity. A similar border for interacting qubit systems would allow a reasonable regime of operability for quantum computers.

To investigate the emergence of quantum chaos in quantum computers, we chose a model of n qubits on a two-dimensional lattice with nearest-neighbor interqubit coupling. The Hamiltonian reads

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$$H=\sum_i \Gamma_i \sigma_i^z + \sum_{i<j} J_{ij}\sigma_i^x \sigma_j^x, \tag{1}$$

where the σ_i are the Pauli matrices for the qubit i and the second sum runs over nearest-neighbor qubit pairs with periodic boundary conditions applied. The energy spacing between the two states of a qubit is represented by Γ_i randomly and uniformly distributed in the interval $[\Delta_0-\delta/2,\Delta_0+\delta/2]$. The parameter δ gives the width of the distribution near the average value Δ_0 and varies from 0 to Δ_0. Here Γ_i can be viewed as the splitting of nuclear spin levels in a local magnetic field, as it is discussed in the experimental proposals [8,9]. The different values of Γ_i are needed to prepare a specific initial state by electromagnetic pulses in nuclear magnetic resonance. In this case the couplings J_{ij} will represent the hyperfine interaction between the spins, which is needed to build the quantum computer. Different physical mechanisms can generate these couplings, such as spin exciton exchange [8,9], dipole-dipole interaction, etc. For generality we chose J_{ij} randomly distributed in the interval $[-J,J]$. The Hamiltonian (1) can be considered as a generic quantum computer model, which catches the main physics of different experimental proposals. For example, a similar Hamiltonian appears in a quantum computer based on optical lattices [12,21]. We restrict ourselves to the case of static couplings that are always present as a residual interaction and are much larger than the multiqubit spacing Δ_n even for moderate values of n. In a sense Eq. (1) describes the hardware of the computer, while gates operation in time requires additional studies, which are possible only if the properties of the hardware are well understood.

As is well known in the field of quantum chaos, the transition to ergodic eigenstates is reflected in the level spacing statistics $P(s)$, which goes from the Poisson distribution $P_P(s)=\exp(-s)$ for nonergodic states to the Wigner-Dyson (WD) distribution $P_W(s)=(\pi s/2)\exp(-\pi s^2/4)$, corresponding to RMT, for ergodic states. Here s is the nearest level spacing measured in units of average spacing and $P(s)$ is the probability to find two adjacent levels whose spacing is in $[s,s+ds]$.

The majority of our data are displayed for the middle of the energy spectrum, where the transition starts, and which therefore sets the limit of operability of the quantum computer. The model (1) has two symmetry classes characterized by an odd or even number of qubits up, and the data are given for one symmetry class. In order to reduce statistical fluctuations, we use $5\leqslant N_D\leqslant 4\times10^4$ random realizations of Γ_i and J_{ij}, as is done usually in RMT [15]. Eigenvalues and eigenvectors are computed by exact diagonalization of the Hamiltonian matrix (1) for each realization. In this way the total number of spacings is $10^4<N_S\leqslant 1.6\times10^5$ ($N_S \propto N_D N_H$). An example of the transition in the spectral statistics is shown in Fig. 1.

To analyze the evolution of $P(s)$ with the coupling J, it is convenient to use the parameter $\eta=\int_0^{s_0}[P(s)-P_W(s)]ds/\int_0^{s_0}[P_P(s)-P_W(s)]ds$, where $s_0=0.4729\ldots$ is the intersection point of $P_P(s)$ and $P_W(s)$. In this way $P_P(s)$ corresponds to $\eta=1$, and $P_W(s)$ to $\eta=0$. As is usual in the field of quantum chaos, the variation of η characterizes the evolution of $P(s)$ [20]. The variation of η with respect to J/Δ_0 is presented in Fig. 2 for $\delta=\Delta_0$ showing that indeed η drops from 1 to 0 with increasing coupling strength. The transition appears to become sharper for larger system sizes. The typical J_c value near which the transition takes place corresponds to intermediate values of η. We chose the condition $\eta(J_c)=0.3$. The dependence of J_c on n is given in the Fig. 2. In analogy with other many-body systems discussed in [17,20], we expect that $J_c\approx\Delta_c \approx C\Delta_0/n$, where C is some numerical constant. Indeed, one multiqubit state is coupled to $2n$ other states in an energy interval of order $6\Delta_0$. This theoretical estimate is in agreement with the data of Fig. 2, with $C\approx3$. We stress that this

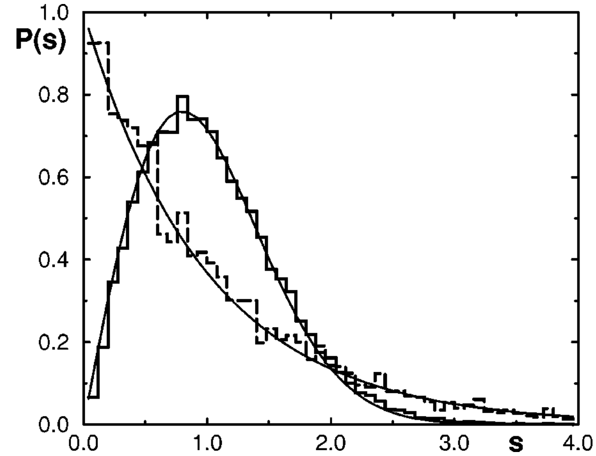

FIG. 1. Transition from Poisson to WD statistics in the model (1) for the states in the middle of the energy band ($\pm6.25\%$ around the center) for $n=12$: $J/\Delta_0=0.02,\eta=1.003$ (dashed line histogram); $J/\Delta_0=0.48,\eta=0.049$ (full line histogram). Full curves show $P_P(s)$ and $P_W(s)$; $N_S>2.5\times10^4$, $N_D=100$, $\delta=\Delta_0$.

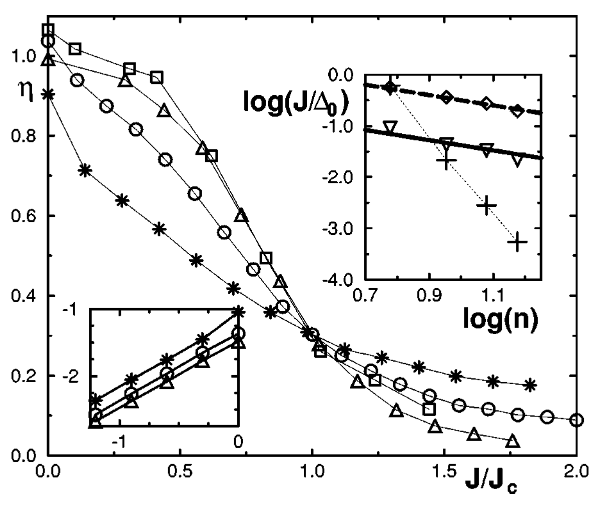

FIG. 2. Dependence of η on the rescaled coupling strength J/J_c for the states in the middle of the energy band for $n=6$ (*), 9 (○), 12 (triangles), 15 (squares); $\delta=\Delta_0$. The upper inset shows $\log_{10}(J_c/\Delta_0)$ (diamonds) and $\log_{10}(J_{cs}/\Delta_0)$ (triangles) versus $\log_{10}(n)$; the variation of the scaled multiqubit spacing Δ_n/Δ_0 with $\log_{10}(n)$ is shown for comparison (+). Dashed line gives the theoretical formula $J_c=C\Delta_0/n$ with $C=3.16$; the solid line is $J_{cs}=0.41\Delta_0/n$. The lower inset shows $\log_{10}(J_{cs}/\Delta_0)$ versus $\log_{10}(\delta/\Delta_0)$ for $n=6$ (*), 9 (○), 12 (triangles); straight lines have slope 1. Logarithms are decimal.

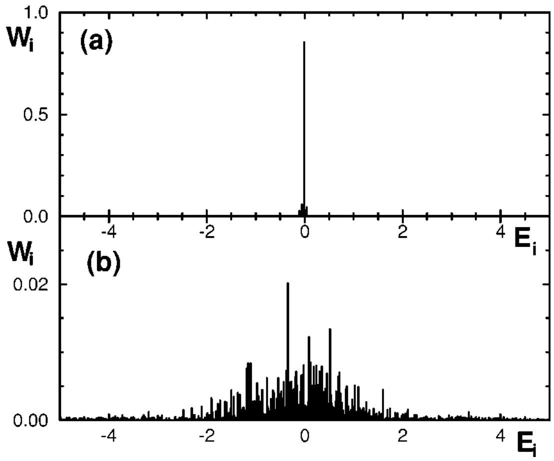

FIG. 3. Two quantum computer eigenstates of model (1) in the basis of noninteracting multiqubit states, i.e., $W_i = |\langle \psi_i | \phi \rangle|^2$ as a function of noninteracting multiqubit energy E_i for $n=12$ and $\delta = \Delta_0$ with $J_c/\Delta_0 = 0.273$ (see text): (a) $J/\Delta_0 = 0.02$; (b) $J/\Delta_0 = 0.48$.

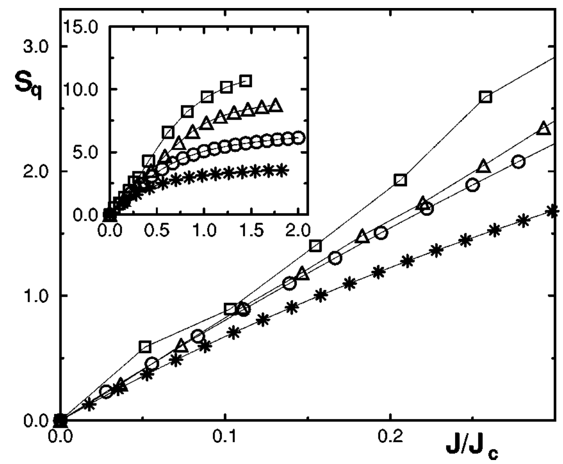

FIG. 4. Dependence of the quantum eigenstate entropy S_q on J/J_c for $\delta = \Delta_0$ and $n=6$ (*), 9 (○), 12 (triangles), 15 (squares); $10^4 < N_S \leqslant 1.6 \times 10^5$. Inset shows the dependence on larger scale.

critical coupling is exponentially larger than the multiqubit level spacing $\Delta_n \sim n 2^{-n} \Delta_0$, as is shown on Fig. 2. For the case $\delta \ll \Delta_0$, the total spectrum at $J=0$ is composed of n bands with interband distance $2\Delta_0$ and a bandwidth of $\sqrt{n}\delta$. Within one band, one multi-qubit state is coupled to about n states in an energy interval of 2δ, so that $J_c \approx \Delta_c \sim \delta/n$. This quantum chaos border is still much bigger than $\Delta_n \sim \sqrt{n}\delta/(N_H/n) \sim n^{3/2} 2^{-n} \delta$.

The transition in the level statistics reflects the drastic change in the multiqubit structure of the eigenstates of Eq. (1). Indeed, Fig. 3 shows that for $J < J_c$ one eigenstate is formed only by one or few noninteracting states $|\psi_i\rangle$, while for $J > J_c$ a huge number of them are required. In the latter case, the computer eigenstates become a random mixture of quantum register states $|\psi_i\rangle$, making it rather difficult to perform computation.

To study this drastic change in the structure of eigenstates, it is convenient to use the quantum eigenstate entropy S_q, defined by: $S_q = -\Sigma_i W_i \log_2 W_i$, where W_i is the quantum probability to find the noninteracting multiqubit state $|\psi_i\rangle$ in the eigenstate $|\phi\rangle$ of Eq. (1) ($W_i = |\langle \psi_i | \phi |^2 \rangle$). In this way $S_q = 0$ if $|\phi\rangle$ is one noninteracting state ($J=0$), $S_q = 1$ if $|\phi\rangle$ is equally composed of two $|\psi_i\rangle$, and the maximal value is $S_q = n$ if all 2^n states contribute equally to $|\phi\rangle$. The variation of the average quantum entropy with J is shown in Fig. 4 for $\delta = \Delta_0$. It shows that S_q grows with J and the transition to ergodic states with large S_q takes place in the vicinity of J_c. In addition these data show that the critical coupling J_{cs} at which $S_q = 1$ is $J_{cs} \approx 0.13 J_c$. The ratio J_{cs}/J_c stays within 15% of the average value when n changes from 6 to 15, while the ratio Δ_n / J_c varies from 1 to 3×10^{-3} (see upper inset of Fig. 2). The dependence of J_{cs} on δ is shown on the lower insert of Fig. 2; it clearly shows the linear decrease of J_{cs} with δ and can be well described by $J_{cs} = 0.4\delta/n$. Naturally, the quantum chaos border drops to zero with δ due to the quasidegeneracy inside the energy bands at $J=0$.

We note that for $n=1000$ and $\delta = \Delta_0 = 1$ K, only two multiqubit states will be mixed at $J_{cs} \approx 0.4 \Delta_0 / n \approx 0.4$ mK. This critical coupling is much larger than the multiqubit level spacing $\Delta_n \sim 10^{-298}$ K. Even if the quantum border J_{cs} corresponds to a relatively low coupling strength it seems reasonable that the residual interaction between qubits can stay below this threshold with current technologies (but not below Δ_n).

The pictorial image of the quantum computer melting under the influence of the interqubit coupling J is shown on Fig. 5. The melting starts in the middle of the spectrum (high energy) and progressively invades low-energy states and the whole computer, destroying its operability. We stress that this destruction takes place in an isolated system without any external decoherence process. Nevertheless the thermalization in this closed system, which appears because of the in-

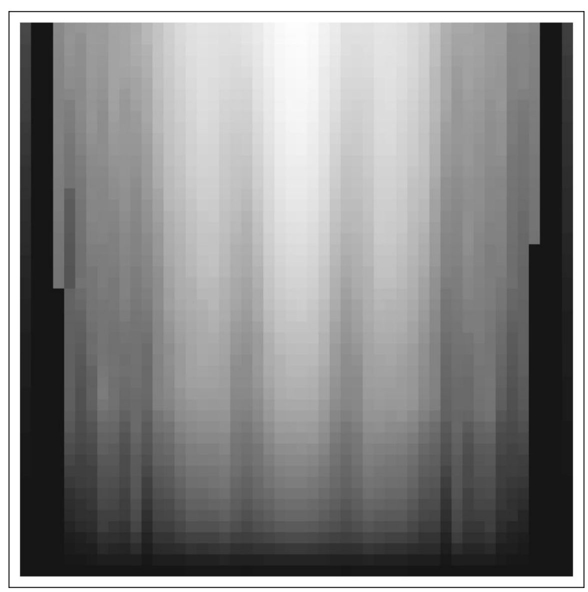

FIG. 5. The quantum computer melting induced by the coupling between qubits. Grayness represents the level of quantum eigenstate entropy S_q, from black ($S_q = 0$) to white ($S_q \approx 11$). Horizontal axis is the energy of the computer eigenstates counted from the ground state to the maximal energy ($\approx 2n\Delta_0$). Vertical axis is the value of J/Δ_0, varying from 0 to 0.5. Here $n=12$, $\delta = \Delta_0$, $J_c/\Delta_0 = 0.273$, and one random realization of Eq. (1) is chosen. A color figure is available on http://xyz.lanl.gov/format/quant-ph/9909074

terqubit coupling, can mimic the effect of a coupling with the external world and external decoherence. Above the quantum chaos border an initial register state $|\psi_i\rangle$ will spread quickly with time [22] over an exponential number of eigenstates of the system with residual interaction, destroying gates operability.

Our studies of a realistic isolated quantum computer hardware show that the mixing of multiqubit states and onset of quantum chaos induced by interqubit coupling leads to its melting and destruction of its operability; however, the quantum chaos border found for this process corresponds to a relatively strong interaction, being exponentially larger than the energy level spacing between multiqubit states. We expect that below this border, error-correcting codes [3,4] will allow us to perform efficient quantum computing with a large number of qubits. Above this border these codes should operate much faster compared to the rate with which chaos sets in [22] to allow to suppress it. Due to that, it is much more efficient to operate the computer below the quantum chaos border. Finally, we note that quantum chaos sets in very easily if the fluctuation amplitude δ of individual qubit spacing drops to zero ($J_c \propto \delta$).

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