Propagating structures in globally coupled systems with time delays

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We consider an ensemble of globally coupled phase oscillators whose interaction is transmitted at finite speed. This introduces time delays, which make the spatial coordinates relevant in spite of the infinite range of the interaction. In the limit of short delays, we show that the ensemble approaches a state of frequency synchronization, where all the oscillators have the same frequency, and can develop a nontrivial distribution of phases over space. Numerical calculations on one-dimensional arrays with periodic boundary conditions reveal that, in such geometry, the phase distribution is a propagating structure.

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I. INTRODUCTION

Standard models for the study of collective complex behavior in natural systems consist of ensembles of interacting dynamical elements [1]. Such models have proved to be extremely versatile both in the analytical and in the numerical description of a wide variety of phenomena within the scope of physics, biology, and other branches of science [2]. According to the range of the interactions involved, these models can be divided into two distinct classes. Local interactions—which are paradigmatically represented in reaction-diffusion systems [1]—give rise to macroscopic evolution in which space variables play a relevant role, such as the appearance of spatial structures and propagation phenomena. On the other hand, with global interactions—where the coupling range is of the order of, or larger than, the system size—space becomes irrelevant and collective behavior is observed to develop in time, typically, in the form of synchronization [3].

A basic, well-known model of globally coupled elements is given by a set of N identical oscillators described, in the so-called phase approximation, by phase variables $\phi_i(t)$ $(i=1,2,\dots,N)$. Their evolution is governed by the equation

$$\dot{\phi}_i = \omega + \frac{\epsilon}{N}\sum_{j=1}^{N} \sin(\phi_j - \phi_i). \tag{1}$$

It is known that, for any value of the coupling intensity ϵ, all the elements converge to a single periodic orbit whose frequency ω coincides with that of an individual oscillator [3]. In this case, ϵ^{-1} measures the time required to reach such a synchronized state.

In this paper we present results of a generalization of the above model when time delays are introduced. The effect of time delays in synchronization phenomena has already been considered for two-oscillator systems, both periodic [4] and chaotic [5]. Ensembles with local interactions [6] and globally interacting inhomogeneous systems have also been studied [7]. None of these contributions, however, makes explicit reference to the relevant case where interactions are global but their propagation occurs at a finite velocity v. This situation, where time delays appear in a quite natural way, provides a realistic description of highly connected systems where the time scales associated with individual evolution and with signal transmission between elements are comparable. Instances of such systems are neural and computer networks [8], and biological populations with relatively slow communication mechanisms—such as sound propagation [9]. Our main result is that, since a finite signal velocity makes spatial variables relevant even when interactions are global, globally coupled ensembles with time delays exhibit typical features of systems driven by local interactions, in particular, structure formation and propagation.

II. MODEL AND ITS SOLUTION FOR SHORT DELAYS

We consider an ensemble of N identical oscillators in the phase approximation, governed by the equation

$$\dot{\phi}_i(t) = \omega + \frac{\epsilon}{N}\sum_{j=1}^{N} \sin[\phi_j(t-\tau_{ij}) - \phi_i(t)], \tag{2}$$

where $\tau_{ij}=d_{ij}/v$ is the time required for the signal to travel from element j to element i at velocity v, and d_{ij} is the distance between i and j. Note that coupling is still global, since its intensity ϵ does not depend on the distance between elements. However, the relative position of the oscillators now becomes relevant through time delays.

The full specification of our system requires us to fix the topology and the metric properties (i.e., the geometry) of the ensemble, by fixing the values d_{ij} for all $i,j=1,\dots,N$. Moreover, initial conditions for ϕ_i must be provided. In the case of delay equations like Eq. (2), it is necessary to specify the evolution of ϕ_i at times prior to $t=0$ [10], namely, for $T_i<t<0$ with $T_i=-\max\{\tau_{ij}\}_j$.

Note that, in contrast with the case without delays, the natural frequency ω cannot be eliminated from Eq. (2) by a homogeneous phase shift $\phi_i \to \phi_i + \omega t$. In fact, in the system with delays this would introduce phase differences $\omega\tau_{ij}$ in the coupling terms. Analogous phase differences have been considered in ensembles of coupled oscillators, in particular, in connection with neural network dynamics [11]. The natural frequency ω can, however, be given any nonzero value by rescaling time, the coupling intensity ϵ, and the delays τ_{ij}. Alternatively, the value of ϵ can be changed by rescaling time, ω, and the delays.

The analytical treatment of time-delay equations is known to be a difficult task. This is especially true for nonlinear multidimensional problems with many delays like Eq. (2), for which essentially no mathematical results of practical application are available [10]. In general, numerical approaches are necessary to deal with such problems. Our model (2), however, admits an approximate analytical solution for arbitrary geometry in the limit of short delays. In fact, when the delays τ_{ij} are much smaller than the typical evolution times of the system, the solution to Eq. (2) is expected to be similar to the case without delays, where all the oscillators become synchronized in both phase and frequency. In our model, characteristic time scales related to the dynamics of a single oscillator and to the relaxation due to the global interaction are given, respectively, by the natural frequency ω and the coupling intensity ϵ. The short-delay limit is therefore defined by the condition $\tau_{ij} \ll \omega^{-1}, \epsilon^{-1}$ for all i,j. In this limit, we assume that $|\phi_j(t-\tau_{ij}) - \phi_i(t)| \ll 1$ for all i,j and t. Equation (2) becomes

$$\dot{\phi}_i(t) = \omega - \epsilon \phi_i(t) + \frac{\epsilon}{N} \sum_{j=1}^{N} \phi_j(t-\tau_{ij}), \tag{3}$$

i.e., a linear equation with delays.

Let us first consider the evolution of the frequencies $\Omega_i = \dot{\phi}_i$. Taking $\Omega_i(t-\tau_{ij}) \approx \Omega_i(t) - \tau_{ij}\dot{\Omega}_i(t)$, we find

$$\dot{\Omega}_i = -\epsilon \Omega_i + \frac{\epsilon}{N} \sum_{j=1}^{N} (\Omega_j - \tau_{ij}\dot{\Omega}_j), \tag{4}$$

where all the functions are now evaluated at the same time t. It is convenient to rewrite this equation in matrix notation, as

$$\dot{\mathbf{\Omega}} = \left(\mathcal{I} + \frac{\epsilon}{N}\mathcal{T} \right)^{-1} \mathcal{M}\mathbf{\Omega} \approx \left(\mathcal{I} - \frac{\epsilon}{N}\mathcal{T} \right) \mathcal{M}\mathbf{\Omega}, \tag{5}$$

where $\mathbf{\Omega} = (\Omega_1, \Omega_2, \ldots, \Omega_N)$, $\mathcal{I} = \{\delta_{ij}\}$ is the identity matrix, $\mathcal{T} = \{\tau_{ij}\}$ is the delay matrix, and $\mathcal{M} = \{-\epsilon\delta_{ij} + \epsilon/N\}$.

Equation (5) can be seen as a perturbation of the problem without delays, $\dot{\mathbf{\Omega}}^0 = \mathcal{M}\mathbf{\Omega}^0$, whose solution is

$$\mathbf{\Omega}^0(t) = \langle \Omega^0 \rangle [1 - \exp(-\epsilon t)]\mathbf{e} + \exp(-\epsilon t)\mathbf{\Omega}^0(0). \tag{6}$$

Here, $\mathbf{e} = (1,1,\ldots,1)$, and $\langle \Omega^0 \rangle = N^{-1}\Sigma_j \Omega_j^0(t) = N^{-1}\mathbf{e}\cdot\mathbf{\Omega}^0$ is a constant of motion associated with the invariance of the unperturbed problem upon a homogeneous shift in the frequencies. As expected, this solution describes the asymptotic approach of all the frequencies to the same value $\langle \Omega^0 \rangle$.

The solution to Eq. (5) can now be written as $\mathbf{\Omega} = \mathbf{\Omega}^0 + \mathbf{\Xi}$, where the components of $\mathbf{\Xi}$ should be of the same order as the time delays. Expanding Eq. (5) up to the first order in the components of $\mathcal{T}$, we find

$$\dot{\mathbf{\Xi}} = \mathcal{M}\mathbf{\Xi} - \frac{\epsilon}{N}\mathcal{T}\mathcal{M}\mathbf{\Omega}^0 = \mathcal{M}\mathbf{\Xi} - \frac{\epsilon^2}{N}\exp(-\epsilon t)\mathbf{\Delta}, \tag{7}$$

with $\mathbf{\Delta} = \mathcal{T}[\langle \Omega^0 \rangle \mathbf{e} - \mathbf{\Omega}^0(0)]$. If the unperturbed problem is solved with the initial conditions for the perturbed equations, $\mathbf{\Omega}^0(0) = \mathbf{\Omega}(0)$, Eq. (7) is to be solved with $\mathbf{\Xi}(0) = \mathbf{0}$. The solution reads

$$\mathbf{\Xi}(t) = -\frac{\epsilon^2}{N} t \exp(-\epsilon t)\mathbf{\Delta}, \tag{8}$$

so that the full solution for the perturbed problem is

$$\mathbf{\Omega}(t) = \Omega[1 - \exp(-\epsilon t)]\mathbf{e} + \exp(-\epsilon t)\mathbf{\Omega}(0) - \frac{\epsilon^2}{N} t \exp(-\epsilon t)\mathbf{\Delta}, \tag{9}$$

with $\Omega = N^{-1}\mathbf{e}\cdot\mathbf{\Omega}(0)$. The phases are then given by the time integral of the respective frequencies $\phi_i(t) = \phi_i(0) + \int_0^t \Omega_i(t')dt'$.

Thus, in the limit of short delays, all the frequencies in the ensemble approach asymptotically the same value, $\Omega_i(t) \to \Omega$ for all i, and the oscillators become synchronized in frequency. On the other hand, it can be seen from the linearized equations (3) that their phases do not synchronize. This is a direct by-product of the presence of time delays. For asymptotically large times, in fact, we can write $\phi_i(t) = \Omega t + \psi_i$. Replacing in Eq. (3), we get

$$\psi_i = \Psi - \Omega\langle \tau_i \rangle, \tag{10}$$

where $\langle \tau_i \rangle = N^{-1}\Sigma_j \tau_{ij}$ is the average delay associated with the ith oscillator. The value of the constant Ψ can be arbitrarily chosen, as a consequence of the symmetry of our system upon a homogeneous phase shift. In the generic situation where the geometrical properties of the ensemble are inhomogeneous, such that the sites occupied by the oscillators are not equivalent with respect to their relative positions, the average delays $\langle \tau_i \rangle$ are expected to differ from site to site. In such a case, the oscillators will generally have different phases. According to Eq. (10), oscillators with smaller average delays have larger phases, and are therefore relatively ahead in the evolution.

If, on the other hand, the geometrical properties are homogeneous, the average delay is the same for all oscillators, and the system becomes synchronized both in frequency and in phase. This is the case of the one-dimensional regular array with periodic boundary conditions considered in the next section. In such a geometry, indeed, we find that full synchronization is the asymptotic state observed for short time delays. However, as we show in the following, other kinds of phase distributions can develop when the delays become larger.

III. ONE-DIMENSIONAL ARRAYS

We now focus attention on a specific geometry, and consider ensembles of N oscillators in a regular one-dimensional array with periodic boundary conditions, i.e., forming a ring. The distance d_{ij} between two sites i and j is in principle not well defined, since the difference between their positions can be measured in both directions around the ring. To solve this ambiguity we choose $d_{ij} = N^{-1}L\min\{|i-j|, N-|i-j|\}$, where L is the linear size of the array. This choice satisfies all the formal requirements that define a well-behaved metric. The time delays associated with these distances are

$$\tau_{ij} = \frac{T}{N}\min\{|i-j|, N-|i-j|\}, \tag{11}$$

where $T=L/v$ is the time needed by the signal to travel around the whole ring at velocity v.

Extensive numerical calculations on one-dimensional periodic arrays, which are presented in detail in the next section, strongly suggest that the asymptotic large-time evolution of model (2) with time delays (11) corresponds to a state of frequency synchronization. That is, all our numerical realizations have shown that the ensemble approaches a state where all the oscillators have the same frequency. We recall that this asymptotic state was derived in Sec. II for short delays in arbitrary geometries. In the numerical calculations, frequency synchronization has been found far beyond the range where the short-delay approximation holds. In the following, we study the asymptotic solutions to Eq. (2) assuming that the ensemble becomes synchronized in frequency, and postpone the presentation of the evidence supporting such an assumption to the next section.

If at large times all the oscillators have the same frequency Ω, their phases can be written as $\phi_i(t)=\Omega t+\psi_i$ (cf. Sec. II). Replacing this ansatz in Eq. (2), we find

$$\Omega=\omega-\frac{\epsilon}{N}\sum_{j=1}^{N}\sin(\Omega\tau_{ij}+\psi_i-\psi_j). \tag{12}$$

Note that the sums $S_i=\Sigma_j\sin(\Omega\tau_{ij}+\psi_i-\psi_j)$ are in general different for each i. However, their numerical values must coincide if the synchronization frequency is to be well defined. For a given value of Ω, this constraint provides $N-1$ independent equations for the phases ψ_i:

$$S_1=S_2\cdots=S_N. \tag{13}$$

Since phases are defined up to an additive constant, we can, for instance, fix the value of ψ_1 and solve these equations for $\psi_2,\ldots,\psi_N$.

To find a solution to Eq. (13), it is convenient to analyze the explicit expression for S_i for the one-dimensional periodic array:

$$S_i=\sum_{j=1}^{N}\sin\left(\frac{\Omega T}{N}\min\{|i-j|,N-|i-j|\}+\psi_i-\psi_j\right). \tag{14}$$

Since the delay τ_{ij} depends on the labels i and j through the difference $i-j$ only, we note that S_i can be made independent of i if the phase difference $\psi_i-\psi_j$ is also a function of $i-j$. Under such a condition, i acts in fact as an irrelevant origin in the sum over j and can be eliminated by redefining the summation variable as $j\to(j+i)\bmod N$. The condition is met for all i and j only if the phase ψ_i depends linearly on i, $\psi_i=Ai+B$ with A and B constants. Requiring the phase to be single valued under the transformation $i\to i+N$ (up to the addition of an integer multiple of 2π), and choosing $\psi_1=0$, we get

$$\psi_i=2\pi\frac{m}{N}(i-1), \tag{15}$$

where m is any integer in the interval $[-N+1,N-1]$.

The solution (15) represents a state where the oscillator phases vary linearly along the array. A total phase difference $\Delta\phi=2\pi m$ accumulates in a whole turn around the ring. The

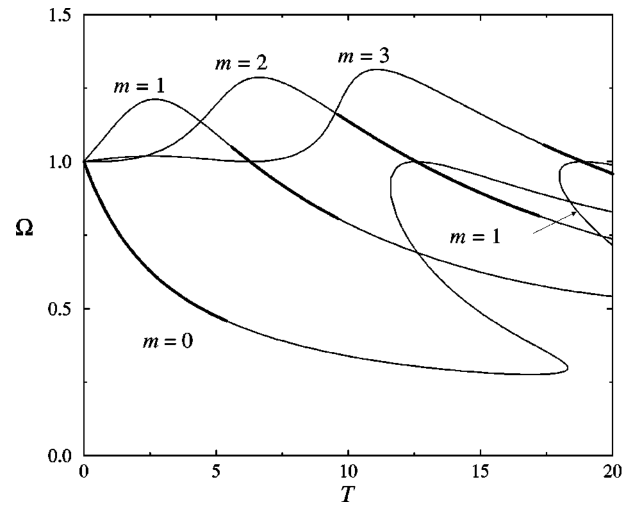

FIG. 1. Synchronization frequency Ω of the asymptotic modes $m=0,\ldots,3$ in a one-dimensional array of $N=100$ globally coupled oscillators with periodic boundary conditions, as a function of the parameter T that defines delays in Eq. (11). Frequencies are measured in units of the natural frequency ω. Bold lines indicate the range where each mode attracts the random-phase initial conditions defined in the text.

simplest mode, $m=0$, corresponds to the state of full synchronization. Taking into account the time evolution of phases,

$$\phi_i(t)=\Omega t+2\pi\frac{m}{N}(i-1), \tag{16}$$

we realize that each mode m represents a propagating structure, whose velocity is $V=L\Omega/2\pi m$, and whose shape is preserved. The corresponding synchronization frequency is given by

$$\Omega=\omega-\frac{\epsilon}{N}\sum_{j=1}^{N}\sin\left(\frac{\Omega T}{N}\min\{j-1,N-j+1\}-2\pi\frac{m}{N}(j-1)\right). \tag{17}$$

The solutions to this equation can be found numerically, for instance, using a built-in function of a program of algebraic manipulation [12]. Figure 1 displays the results for the first few modes ($m\geqslant 0$) for $N=100$. Replacing $j\to N-j$ in Eq. (17), it is immediately shown that the synchronization frequency is independent of the sign of m.

In principle, the stability of these propagating modes can be analyzed by means of a linearization of Eq. (2). Taking $\phi_i(t)=\Omega t+\psi_i+\delta\phi_i(t)$, expanding to the first order in $\delta\phi_i$, and proposing a solution of the form $\delta\phi_i(t)=A_i\exp(\lambda t)$, we end with the eigenvaluelike problem

$$\lambda A_i=\frac{\epsilon}{N}\sum_{j=1}^{N}\cos(\Omega\tau_{ij}+\psi_i-\psi_j)[A_j\exp(\lambda\tau_{ij})-A_i]. \tag{18}$$

It has nontrivial solutions for the amplitudes A_i if the determinant of a matrix with elements

$$H_{ij}=\frac{\epsilon}{N}\cos(\Omega\tau_{ij}+\psi_i-\psi_j)\exp(\lambda\tau_{ij})-\delta_{ij}\left(\frac{\epsilon}{N}\sum_{k=1}^{N}\cos(\Omega\tau_{ik}+\psi_i-\psi_k)+\lambda\right)\quad(19)$$

vanishes. This determinant has the form of a polynomial in λ and in the $N(N-1)/2$ variables $\exp(\lambda\tau_{ij})$. The eigenvalue λ then satisfies a transcendental equation, to be solved in the complex plane. Typically, it has infinitely many solutions, corresponding to exponential growth and decay, and to oscillatory behavior, as well as to their possible combinations. For moderate or large values of N, the resolution of such an equation is unfortunately impracticable, even with the help of numerical algorithms.

The stability of the propagating modes can, however, be decided when the delays vanish, $T=0$. In this limit, the above linearization leads to a standard eigenvalue problem for the matrix $\mathcal{K}$ with elements

$$K_{ij}=\frac{\epsilon}{N}\cos(\psi_i-\psi_j)-\delta_{ij}\frac{\epsilon}{N}\sum_{k=1}^{N}\cos(\psi_i-\psi_k).\quad(20)$$

For the full-synchronization mode $m=0$, $\mathcal{K}$ reduces to the matrix $\mathcal{M}$ of Eq. (5), and the problem is trivially solved. As a consequence of the invariance of the system under a homogeneous phase shift, there is a vanishing eigenvalue $\lambda_1=0$. The remaining $N-1$ eigenvalues are identical and negative, $\lambda_2=\cdots=\lambda_N=-\epsilon$. Thus, as is well known in the ordinary situation without delays, full synchronization is stable. On the other hand, all the other modes are unstable for $T=0$. This conclusion is drawn by noting that, for those modes, the trace of $\mathcal{K}$ is positive, $\mathrm{tr}\,\mathcal{K}=\Sigma_j K_{ii}=\epsilon$. Since the trace of a matrix equals the sum of all its eigenvalues, it follows that at least one of them must be positive. A closer analysis reveals that, in fact, $N-2$ eigenvalues are zero, and the remaining two equal 1/2.

In summary, we have found that, under the assumption of frequency synchronization at large times, a set of globally coupled oscillators with time delays in a one-dimensional regular array with periodic boundary conditions admits a class of stationary states that correspond to propagating structures where the phases vary linearly along the array. For arbitrary delays, the stability of these structures cannot be studied analytically. We find, on the other hand, that for vanishing delays the only stable mode is that of full synchronization, where all the phases coincide. In the next section we present numerical results justifying the assumption of frequency synchronization. Moreover, these results show that the propagating structures derived above can be stable for larger delays and, as a matter of fact, correspond to the asymptotic states of our system for a wide class of initial conditions.

IV. NUMERICAL RESULTS

In our numerical analysis of globally coupled oscillators with time delays, we have solved Eq. (2) by means of a fourth-order Runge-Kutta scheme [13]. The time increment Δt is chosen such that the time delays τ_{ij} are all integer multiples of Δt. In this way, the times $t-\tau_{ij}$ at which the phases ϕ_j contribute to the evolution of $\phi_i(t)$ coincide with nodes of the time discretization, and no interpolations are needed to calculate the values $\phi_j(t-\tau_{ij})$. These values are stored in a matrix which is updated at each time step, and can then be retrieved directly from the matrix. In our calculations we fix $\omega=\epsilon=1$, so that the typical time scales are of the order of unity. For these scales, a time increment $\Delta t\sim10^{-2}$ yields reasonably precise results. In fact, comparison with test calculations using much shorter increments reveals maximal relative differences of the order of 1% along the whole time domain.

As for the initial conditions, we have assumed that for $t<0$ the oscillators evolve independently from each other with their natural frequency ω and with random relative phases. That is, for $t<0$ we have taken $\phi_i(t)=\omega t+\phi_i(0)$, where $\phi_i(0)$ is drawn at random from a uniform distribution in $(-\pi,\pi)$. At $t=0$ coupling is switched on, so that we formally have a time-dependent coupling intensity $\epsilon(t)=\epsilon\theta(t)$, where θ is the Heaviside step function.

Our numerical calculations for one-dimensional regular arrays with periodic boundary conditions have been performed for systems of $N=10$ to 10^4 elements. If, as N grows, time delays are rescaled in such a way that the time T needed for the signal to travel along the whole array is held constant—which, for a given signal velocity, corresponds to preserving the linear size L of the system while the density of oscillators increases (cf. Sec. III)—the results become essentially independent of N at relatively small values of N. We therefore choose to present the results for $N=100$, which are fully representative of those for larger systems.

We have performed extensive numerical realizations for values of T of the form $T=k\Delta T$, with $\Delta T=0.1$ and $k=1,2,\ldots,200$, i.e., up to $T=20$. For the present choice $\omega=\epsilon=1$, most of these values of T are well beyond the range where the approximation considered in Sec. II holds ($T\lesssim1$). The calculations were run from the above described random-phase initial conditions up to $t=10^3$. In *all* cases, we found that the system approaches an asymptotic state where the frequencies of all the oscillators converge to the same value, so that the ensemble becomes synchronized in frequency. To illustrate this fact, Fig. 2 displays the time evolution of the mean dispersion of frequencies $\sigma_\Omega=\sqrt{\langle(\Omega_i-\langle\Omega\rangle)^2\rangle}$, for some selected values of T up to $t=200$. It is seen that, after a certain transient, σ_Ω decreases exponentially with time, $\sigma_\Omega\sim\exp(\Lambda t)$ with $\Lambda<0$. Note that the slope Λ of this exponential decay is a direct measure of the (real part of the) largest eigenvalue of the linearized problem around the state of frequency synchronization. For small values of T the slope is well approximated by the value expected for short delays, $\Lambda=-\epsilon=-1$ (cf. Sec. II), as shown by the dashed line in Fig. 2. For larger values of T, $|\Lambda|$ is smaller. We have measured the slope as a function of T for values of the form $T=k\Delta T$, where now $\Delta T=1$ and $k=1,2,\ldots,20$, by least-squares fitting of an exponential to the large-time evolution of σ_Ω. The results are shown as full dots in Fig. 3.

Figure 2 shows, moreover, that transients become longer as T grows. The duration of transients is given by the time needed to lose information on the initial conditions. In our time-delay system, in fact, the initial conditions are specified

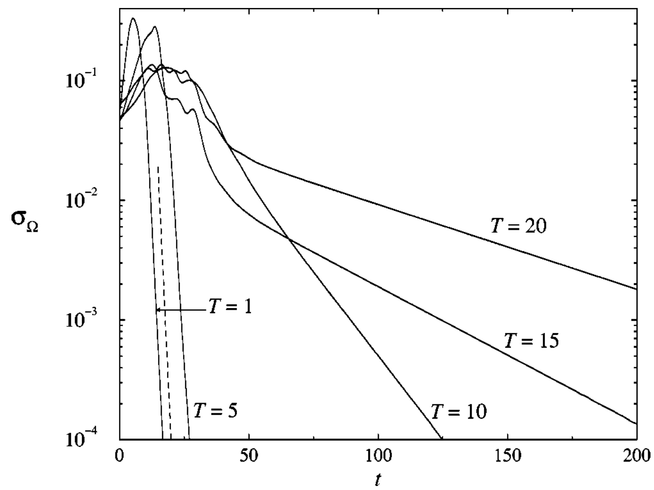

FIG. 2. Evolution of the mean dispersion of frequencies σ_Ω, measured in units of the natural frequency ω, in a one-dimensional array of $N=100$ globally coupled oscillators with periodic boundary conditions, for various values of T. The dashed line stands for the analytical slope obtained in Sec. II for short delays.

over time intervals whose maximum length is, precisely, T.

The analysis of the distribution of phases at large times, i.e., when the state of frequency synchronization has been reached, shows that the ensemble converges to the propagating structures studied in Sec. III. It turns out, however, that the order m of the asymptotic solution approached from random-phase initial conditions depends on T. For small values of T $(T \lesssim 5.6)$, the system approaches the state of full synchronization, $m=0$. As T grows, we find well-defined ranges where the asymptotic states are propagating structures with increasing values of $|m|$. For the first few modes, the boundaries are at $T \approx 5.6$, 9.8, and 17.4. Different realizations of the same class of initial conditions lead to both signs of m with equal probability, as expected. The ranges where the ensemble tends toward each mode m are also shown in Fig. 1, with bold lines.

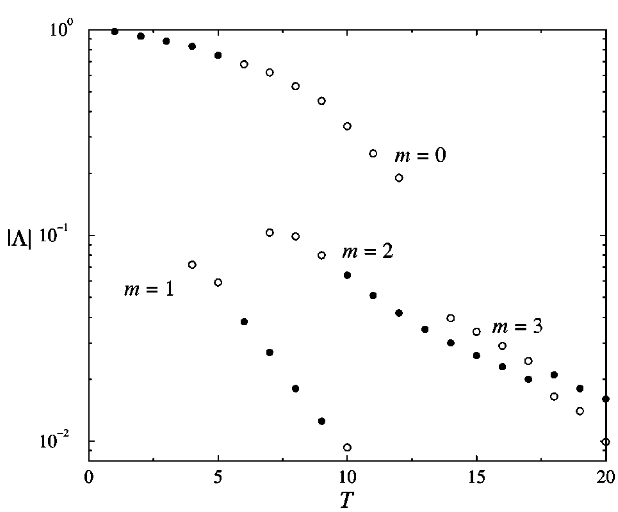

FIG. 3. Slope Λ of the exponential decay of the mean frequency dispersion (see Fig. 2) as a function of T. For each set of data, the label m indicates the mode that describes the observed asymptotic state. Full dots stand for the results of numerical calculations with random-phase initial conditions, whereas empty dots correspond to initial conditions prepared as perturbations of the propagating modes, as explained in the text.

The existence of a well-defined interval of T where each mode is observed in the numerical calculations can be alternatively ascribed to two facts. Either each mode is stable within the corresponding interval and unstable elsewhere or, at a given value of T, several modes are stable but only one is selected by the random-phase initial conditions. In this latter case, the selection of a mode should be related to the relative size of the respective attraction basin. Since, from a probabilistic viewpoint, most initial conditions are of the random-phase type, the mode that attracts such initial states for a given T should have the largest basin.

To decide between the two alternatives, we have performed a series of calculations with different initial conditions. We have taken the evolution of phases for $t<0$ to be given by a small perturbation of a given mode, namely, $\phi_i(t)=\Omega t+2\pi(i-1)m/N+2\pi\rho_i$, where ρ_i is a random number drawn from a uniform distribution in the interval $(0,r)$. The corresponding value of Ω has been calculated from Eq. (12), as in Fig. 1. Using such initial conditions with sufficiently small perturbations (typically, $r \sim 10^{-2}$ to 10^{-1}) we have found that each mode is approached asymptotically, and is therefore stable, not only for the values of T quoted above but in a wider interval. Thus, in general, our system is multistable, as different initial conditions can lead to different asymptotic states for the same set of time delays. In Fig. 3, empty dots stand for measurements of the slope $|\Lambda|$ for different modes in these extended intervals.

It is observed, however, that the range where each mode is stable is not infinite. For sufficiently low and high values of T, even very small perturbations ($r \lesssim 10^{-6}$ in our calculations) drive the system away from the mode prepared as initial condition, and a different mode is asymptotically approached. This indicates that, for each value of m, there is a bounded interval where all the corresponding eigenvalues are negative or have negative real parts, such that the associated mode is stable there. This picture is partially supported by the fact that for $T=0$ the only stable mode is the fully synchronized state $m=0$, as shown in Sec. III.

In summary, extensive numerical calculations for one-dimensional arrays with periodic boundary conditions show quite convincingly that, in such geometry, our ensemble of globally coupled phase oscillators approaches, for a wide range of time delays, an asymptotic state where all the oscillators have the same frequency. For short delays, they have also the same phase and are fully synchronized. For larger delays we obtain the propagating structures studied in Sec. III, where the phases vary linearly around the array. These structures coexist for any set of time delays. The ranges where they are stable have, however, finite length. The system is multistable, since the stability ranges are partially overlapping.

V. SUMMARY AND CONCLUSION

We have considered an ensemble of globally coupled identical phase oscillators with finite interaction velocity. This finite velocity introduces time delays in the evolution equations, and makes the spatial coordinates of oscillators relevant in spite of the infinite range of their interaction. As a matter of fact, we have found that the system develops spatial structures, much like ensembles of locally interacting dynamical elements. This feature, which is reminiscent of the behavior of reaction-diffusion systems, points out sharp differences from the collective motion of coupled oscillators without time delays.

In the limit of short delays, we have been able to derive an approximate analytical solution for arbitrary geometries. This solution describes the asymptotic approach toward a state where all the oscillators have the same frequency. When the geometry is inhomogeneous—for instance, in the presence of boundaries—a stationary spatial distribution of phases appears. Numerical realizations for one-dimensional regular arrays with periodic boundary conditions reveal that the state of frequency synchronization is also attained for larger delays. In this homogeneous geometry, a class of propagating spatial structures, where the oscillator phases vary linearly around the system, is observed. The existence of such structures can be derived analytically, and a numerical study shows that, for given parameter sets, they can be simultaneously stable. Thus, as for many other instances of globally coupled ensembles, our system exhibits multistability [6].

Our results with short delays suggest that it would be interesting to consider systems with larger delays in geometries different from the one-dimensional arrays specifically studied here. We have run a series of preliminary realizations for several geometries, such as one-dimensional arrays with free boundaries, tree (ultrametric) structures, and two-dimensional lattices with periodic and free boundaries. These results, which will be presented elsewhere [14], provide strong evidence that the state of frequency synchronization presented here is a quite generic form of asymptotic collective evolution for globally coupled systems with time delays, in both homogeneous and inhomogeneous geometries. Unfortunately, an analytical approach to our system in these more general situations appears to be impossible in practice even at the level of deciding the existence of stationary solutions, and most conclusions rely at the moment on numerical evidence.

Of course, the possible variations of the model with respect to the individual dynamics of each coupled element are not exhausted with phase oscillators as in Eq. (2). The next step will be to study the effects of the present kind of time delays in ensembles formed by chaotic oscillators, where coupling competes as a stabilizing mechanism against the inherently unstable dynamics of individual elements.

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