Anderson localization as a parametric instability of the linear kicked oscillator

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We rigorously analyze the correspondence between the one-dimensional standard Anderson model and a related classical system, the ''kicked oscillator'' with noisy frequency. We show that the Anderson localization corresponds to a parametric instability of the oscillator, the localization length being related to the rate of exponential growth of the energy of the oscillator. Analytical expression for a weak disorder is obtained, which is valid both inside the energy band and at the band edge.

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I. INTRODUCTION

Recently it was shown that quantum one-dimensional tight-binding models with any diagonal site potential can be formally represented in terms of a two-dimensional (2D) Hamiltonian map [1]. On the other hand, this classical map is associated with a linear oscillator subjected to a linear force given in the form of time-dependent δ kicks. In this picture, both the frequency of the unperturbed oscillator and the period of the kicks are determined by the energy of an eigenstate, and the amplitudes of the kicks are defined by the site potential in the original quantum model. It was shown that by exploring the dynamics of this classical system, one can obtain global characteristics of the eigenstates, such as the localization length defined by the Lyapunov exponent of classical trajectories.

In particular, analytical estimates have been obtained in [1] for a specific diagonal site-potential potential with short-range correlations (the so-called dimer model, see [2]). Other applications to the case of general correlated diagonal [3] and off-diagonal [4] disorder have revealed very important peculiarities. One of the most interesting results has been obtained in [3] where it was shown how to construct random potentials with specific two-point correlators which result in the emergence of the mobility edge in one-dimensional geometry. Based on these predictions, very recently experimental realization of this effect has been done in one-mode microwave guides [5].

In this paper, we perform an analytical study of the standard Anderson model with diagonal uncorrelated disorder, paying main attention to the problem of the mathematical correspondence between the quantum model and its classical representation in the form of a linear kicked oscillator. More specifically, we are interested in the connection between the Anderson localization and the parametric instability of the corresponding classical system. Although the results obtained for the localization length in the weak disorder limit are well known from other studies, the method we use here is a new one and it may explain the mechanism of the Anderson transition in new terms. Moreover, this approach may be very useful for 2D and 3D cases, for which analytical results for global properties of eigenstates are very restricted.

II. DEFINITION OF THE MODEL

In this paper we study the relation existing between the standard 1D Anderson model and a related physical system, a linear oscillator with noisy frequency. The quantum model is defined by the stationary Schrödinger equation [6]

$$\psi_{n+1}+\psi_{n-1}+\epsilon_n\psi_n=E\psi_n, \tag{1}$$

where ψ_n represents the electron wave-function at the nth lattice site, and the site energies ϵ_n are independent random variables with a common distribution $p(\epsilon)$. In the standard Anderson model the probability $p(\epsilon)$ has the form of a box distribution,

$$p(\epsilon)=\frac{1}{W}\,\theta\left(\frac{W}{2}-|\epsilon|\right), \tag{2}$$

whose width W sets the strength of the disorder. In the following, however, we will not restrict our considerations to the specific form (2) of the probability distribution, but simply assume that the variables ϵ_n have zero mean ($\langle\epsilon_n\rangle=0$) and a finite variance $\langle\epsilon_n^2\rangle$.

The kicked oscillator is a harmonic oscillator that undergoes periodic and instantaneous variations of the momentum (the kicks). Such a system is defined by the Hamiltonian

$$H=\omega\left(\frac{p^2}{2}+\frac{x^2}{2}\right)+\frac{x^2}{2}\,\xi(t), \tag{3}$$

where

$$\xi(t)=\sum_{n=-\infty}^{+\infty}A_n\delta(t-nT). \tag{4}$$

The random coefficients A_n that appear in the definition of the noise (4) represent the intensity of the kicks, i.e., they are proportional to the sudden momentum changes experienced by the oscillator at times $t=nT$. In other words, the system (3) represents a harmonic oscillator with a mean frequency ω perturbed by the noise term $\xi(t)$. Using the definition (4), one can easily reconduct the statistical properties of the noise $\xi(t)$ to the corresponding properties of the variables A_n; in particular, the mean and the variance of $\xi(t)$ can be expressed as

$$\langle \xi(t)\rangle = \lim_{\tau\to\infty}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\xi(t)\,dt$$

$$= \lim_{N\to\infty}\frac{1}{NT}\sum_{n=-N/2}^{N/2}A_n = \frac{\langle A_n\rangle}{T}$$

and

$$\langle \xi(t)\xi(t+s)\rangle = \lim_{\tau\to\infty}\frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\xi(t)\xi(t+s)\,dt$$

$$= \frac{1}{T}\delta(s)\lim_{N\to\infty}\frac{1}{N}\sum_{n=-N/2}^{N/2}A_n^2 = \frac{\langle A_n^2\rangle}{T}\delta(s).$$

The equivalence of the models (1) and (3) has been discussed in [1]. There it was shown how the two-dimensional map

$$\begin{aligned} x_{n+1} &= x_n\cos(\omega T) + (p_n - A_n x_n)\sin(\omega T),\\ p_{n+1} &= -x_n\sin(\omega T) + (p_n - A_n x_n)\cos(\omega T), \end{aligned} \tag{5}$$

can be derived by integrating over a period T the Hamiltonian equations of motion of the kicked oscillator (3). Note that in the map (5) x_n and p_n stand for the coordinate and momentum of the oscillator immediately before the nth kick. Eliminating the momentum from Eqs. (5), one eventually obtains the relation

$$x_{n+1} + x_{n-1} + A_n\sin(\omega T)x_n = 2x_n\cos(\omega T),$$

which coincides with the Anderson equation (1) if one identifies the site amplitude ψ_n with the coordinate x_n of the oscillator and if the parameters of the models (1) and (3) are related to each other by the equalities

$$\epsilon_n = A_n\sin(\omega T);\quad E = 2\cos(\omega T). \tag{6}$$

In Ref. [1] the classical map (5) was used as a tool to investigate the properties of the model (1); here we focus instead on the direct analysis of the Hamiltonian model (3).

III. THE OSCILLATOR WITH NOISY FREQUENCY

The dynamics of the kicked oscillator (3) is determined by the Hamiltonian equations of motion:

$$\dot p = -(\omega + \xi(t))x,\quad \dot x = \omega p. \tag{7}$$

In order to study the behavior of the kicked oscillator, it is convenient to substitute the couple of differential equations (7) with the system of stochastic Itô equations,

$$\begin{aligned} dp &= -\omega x\,dt - x\sqrt{\langle A_n^2\rangle/T}\,dW(t),\\ dx &= \omega p\,dt, \end{aligned} \tag{8}$$

where $W(t)$ is a Wiener process with $\langle dW(t)\rangle = 0$ and $\langle dW(t)^2\rangle = dt$.

The systems (7) and (8) can be considered equivalent inasmuch as the shot noise $\xi(t)$ is adequately represented by a Wiener process $W(t)$. That is the case if the strength of the single kicks is weak, i.e., if the condition

$$\langle A_n^2\rangle \ll 1 \tag{9}$$

is fulfilled. That can be understood by considering that the present situation is analogous to the one that occurs in the Brownian motion of a heavy particle suspended in a fluid of light molecules. The instantaneous impacts of the fluid molecules on the massive particle can be successfully described by a continuous Wiener process, provided that each single collision does not produce a significant displacement of the heavy particle. When the mass of the suspended particle is not much bigger than the one of the impinging molecules, the nature of the motion changes and the effect of the molecular collisions can no longer be depicted by a Wiener process.

A similar analogy can be drawn between the present case and the random-walk problem. To be exact, let us consider a one-dimensional random walk made by someone that takes steps of length l at times nT (with n integral). At each step the walker is supposed to go to the right or to the left with equal probability. In this model the walker's position changes with each step much in the way the momentum of the kicked oscillator does under the action of a kick: in both cases the relevant physical quantity is varied in a sudden and random way at regular time intervals. This analogy makes interesting to notice that, by going to the limit

$$l\to 0,\quad T\to 0,$$

while holding fixed the ratio

$$D = \frac{l^2}{T}$$

the discrete time random walk evolves in a Brownian motion with diffusion constant D (see, e.g., Ref. [7]). In other words, a Wiener process can be regarded as a limit case of random walk in the limit of very small and fast-spaced steps. In a similar way, the ''jump process'' $\xi(t)$ can be described by a ''diffusion process'' $W(t)$ when the condition (9) is satisfied, with the ratio

$$k = \frac{\langle A_n^2\rangle}{\omega T} \tag{10}$$

playing a role analogous to that of the diffusion constant D.

This conclusion, although substantially correct, must be made more precise. The analogy between the kicked oscillator (3) and the free Brownian particle or the random walker, although of much use, cannot be complete because the kicked oscillator, unlike the two latter systems, is endowed with an autonomous dynamics dictated by the elastic force and independent from the noise. As a consequence, in assessing the equivalence of the shot noise $\xi(t)$ with the continuous process $W(t)$, one must also take into account the possibility that the interplay of the discrete noise (4) with the unperturbed motion of the oscillator might produce different effects than those induced by the addition of the continuous process $W(t)$ to the deterministic dynamics of the noiseless oscillator. More specifically, one can guess that possible ''resonance'' effects due to the conmensurability of the frequency $1/T$ of the kicks with the frequency $\omega/2\pi$ of the unperturbed oscillator might occur. This is actually the case at the band center, when the two frequencies stand in the ratio $\omega T/2\pi = 1/4$, as we will discuss in Sec. V. Apart from this exceptional case, however, the dynamical features of the oscillators (3) and (8) do not differ, as will appear in the following analysis.

In conclusion, the kicked oscillator (7) and the stochastic oscillator (8) are equivalent when the individual kicks of the first model are weak. On the other hand, since the kicked oscillator (7) is equivalent to the Anderson model (1), one can conclude that the stochastic oscillator (8) provides a correct description of the Anderson model for the weak disorder case. This allows one to analyze the solutions of Eq. (1) in terms of phase-space ''trajectories'' of the stochastic oscillator (8). In this picture, localized states for the Anderson model correspond to unbounded trajectories of the oscillator in the phase space, while extended states translate into bounded trajectories.

In the following we will restrict our considerations to the case defined by the weak disorder condition (9) and thus ensure the equivalence between the Anderson model (1) and the stochastic oscillator (8).

IV. LYAPUNOV EXPONENT

Once we have established the correspondence of the Anderson model with the stochastic oscillator (8), we can proceed to redefine essential features of the first model in the dynamical language of the second. In particular, we are interested in deriving a formula for the Lyapunov exponent, which gives the inverse localization length for the eigenstates of Eq. (1). Since these eigenstates correspond to trajectories of the stochastic oscillator (8), the Lyapunov exponent is naturally redefined as the exponential divergence rate of neighboring trajectories, i.e., through the limit

$$\lambda = \lim_{T\to\infty} \lim_{\delta\to 0} \frac{1}{T}\int_0^T dt \frac{1}{\delta} \ln \frac{x(t+\delta)}{x(t)}, \qquad (11)$$

which corresponds to the standard expression

$$\lambda = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \ln \frac{\psi_{n+1}}{\psi_n}$$

for the Anderson model (1). By taking the limit $\delta \to 0$ first, the expression (11) can be put in the simpler form

$$\lambda = \langle z \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, z(t), \qquad (12)$$

where the Ricatti variable $z = \dot{x}/x$ has been introduced and the symbol $\langle z \rangle$ stands for the (time) average of z.

To compute the Lyapunov exponent, as defined by Eq. (12), it is necessary to analyze the dynamics of the variable $z = \omega p/x$. The time evolution of this quantity is determined by the Itô stochastic equation

$$dz = -(\omega^2 + z^2)dt - \omega\sqrt{\langle A_n^2\rangle/T}\, dW(t), \qquad (13)$$

which can be easily derived using Eq. (8) and the standard rules of the Itô calculus.

Notice that, while the position and momentum of the oscillator (8) do not evolve independently from each other, the dynamics of their ratio $z = \omega p/x$ is totally autonomous from that of any other variable. As a consequence, one deals with the single differential equation (13) instead of having to cope with a set of coupled equations like Eq. (8). Thus, the introduction of the variable z, which is suggested by the definition (12) of the Lyapunov exponent, turns out to be beneficial also for the study of the stochastic oscillator (8).

As is known, the Itô stochastic differential equation (13) is equivalent to the Fokker-Planck equation [7,8]

$$\frac{\partial p}{\partial t}(z,t) = \frac{\partial}{\partial z}[(\omega^2 + z^2)p(z,t)] + \frac{\omega^2\langle A_n^2\rangle}{2T}\frac{\partial^2 p}{\partial z^2}(z,t), \qquad (14)$$

which gives the time evolution of the probability density $p(z,t)$ of the stochastic variable z. In other words, the evolution of $z(t)$ dictated by Eq. (13) is a diffusion process with a deterministic drift coefficient $(\omega^2 + z^2)$ and a noise-induced diffusion coefficient $\omega^2\langle A_n^2\rangle/T$.

The stationary solution of Eq. (14) is

$$p(z) = \left[C_1 + C_2\int_{-\infty}^{z} dx \exp\{\Phi(x/\omega)\}\right]\exp\{-\Phi(z/\omega)\},$$

where C_1 and C_2 are integration constants and the function $\Phi(x)$ is given by the relation

$$\Phi(x) = \frac{2}{k}\left(x + \frac{x^3}{3}\right), \qquad (15)$$

which contains the parameter k defined by Eq. (10). Since $p(z)$ is a probability distribution, it must be integrable and therefore the constant C_1 must vanish. The residual constant C_2 is determined by the normalization condition $\int_{-\infty}^{\infty} p(z)dz = 1$. The resulting distribution is

$$p(z) = \frac{1}{N\omega^2}\int_{-\infty}^{z} dx \exp\{\Phi(x/\omega) - \Phi(z/\omega)\} \qquad (16)$$

with

$$N = \sqrt{\frac{\pi k}{2}}\int_0^{\infty} dx \frac{1}{\sqrt{x}} \exp\left[-\frac{2}{k}\left(x + \frac{x^3}{12}\right)\right]. \qquad (17)$$

Once the steady-state probability distribution (16) is known, one can use it to compute the average of z that defines the Lyapunov exponent (12)

$$\lambda = \langle z \rangle = \int_{-\infty}^{\infty} dz\, z\, p(z). \qquad (18)$$

By this way one obtains

$$\lambda = \frac{\omega}{2N}\int_0^{\infty} dx\, \sqrt{x} \exp\left[-\frac{2}{k}\left(x + \frac{x^3}{12}\right)\right]. \qquad (19)$$

Formula (19) is the central result of this paper. It gives an expression for the inverse localization length in the Anderson model that turns out to be valid both inside the energy band and at the band edge (although it fails to reproduce the anomaly of the Lyapunov exponent at the band center). The extended validity range of the expression (19) is a remarkable feature, because the behavior of the localization length at the band edge is known to be anomalous [9,10] and has been previously derived with methods well distinct (and more complicated) than the ones used to study the localization length inside the band. In the next two sections we will show how expression (19) reproduces the known formulas for the localization length inside the energy band as well as in a neighborhood of the band edge.

Before proceeding along this line, however, it is opportune to complete this section with a couple of further remarks. In the first place, it is interesting to notice that expressions very similar to those of Eqs. (16) and (19) have been obtained for a different but related model: that of a particle in a one-dimensional random potential (see, e.g., [11] and references therein). The problem is defined by the continuous Schrödinger equation

$$\psi''(x)+[E-U(x)]\psi(x)=0, \tag{20}$$

where $U(x)$ represents white noise, i.e., a δ correlated random potential with zero mean:

$$\langle U(x)\rangle=0;\quad \langle U(x)U(x')\rangle=2D\,\delta(x-x').$$

This correspondence is not surprising on two grounds: first, Eq. (1) is the discrete counterpart of the continuous Schrödinger equation (20), and second, the stationary equation (20) is the formal analog of the dynamical equation for the kicked oscillator

$$\ddot{x}(t)+(\omega^2+\omega\xi(t))x(t)=0,$$

which represents an equivalent form of the system (7).

As a second observation, we note that in the present section we have described the dynamics of the random oscillator in terms of its Cartesian phase-space coordinates (p,x). In a equivalent way, we could have used the polar coordinates

$$r=\sqrt{x^2+p^2},$$

$$\theta=\arctan(x/p),$$

which represent the action-angle variables of the oscillator. The dynamics of the oscillator is then dictated by the couple of Itô equations

$$dr=\frac{\langle A_n^2\rangle}{2T}r\sin^4\theta dt-\sqrt{\frac{\langle A_n^2\rangle}{T}}r\sin\theta\cos\theta dW(t),$$

$$d\theta=\left(\omega+\frac{\langle A_n^2\rangle}{T}\cos\theta\sin^3\theta\right)dt+\sqrt{\frac{\langle A_n^2\rangle}{T}}\sin^2\theta\,dW(t),$$

which can be easily derived from Eqs. (8) using the standard rules of the Itô calculus. A glance at the stochastic equation for the angular variable reveals that the latter evolves independently of the radius r; one can then associate to the Itô equation for θ a Fokker-Planck equation whose stationary solution gives the invariant measure $\rho(\theta)$ which is the counterpart of the distribution (16) for the Ricatti variable z. Actually, the two distributions are closely related as a consequence of the simple relation $z=\omega\cot\theta$ which links the two variables z and θ. One could, therefore, have evaluated the Lyapunov exponent (19) in a alternative way by determining the invariant measure $\rho(\theta)$ first and then by using it to compute the average $\lambda=\omega\langle\cot\theta\rangle=\int d\theta\rho(\theta)\cot\theta$. The result that is obtained in this way obviously coincides with the one expressed by formula (19).

V. WEAK DISORDER EXPANSION

The weak disorder case is defined by the condition (9). This condition implies that, except that at the band edge (i.e., for $\omega T\to 0$), the parameter (10) must satisfy the requirement $|k|\ll 1$. In this section we analyze, therefore, the expansion of the Lyapunov exponent (19) in the limit $k\to 0$. This corresponds to studying the behavior of the localization length inside the energy band for the Anderson model (1) with a weak disorder.

Making use of expressions (19) and (17), it is easy to verify that for $k\to 0$ the Lyapunov exponent can be written in the form

$$\lambda=\frac{\langle A_n^2\rangle}{4T}\frac{\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(3n+3/2)}{n!}\left(\frac{k^2}{48}\right)^n}{\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(3n+1/2)}{n!}\left(\frac{k^2}{48}\right)^n}. \tag{21}$$

To the lowest order in k this expression reduces to

$$\lambda=\frac{\langle A_n^2\rangle}{8T}, \tag{22}$$

which represents the basic approximation for the inverse localization length in the weak disorder case.

Taking into account the relations (6) between the parameters of the Anderson model (1) and those of the stochastic oscillators (3) and (8), the variance $\langle A_n^2\rangle$ that appears in formula (22) can be expressed as

$$\langle A_n^2\rangle=\frac{\langle\epsilon_n^2\rangle}{1-E^2/4}.$$

When the distribution for the random site energies ϵ_n is the box distribution (2), one can further write $\langle\epsilon_n^2\rangle=W^2/12$; as a consequence, expression (22) takes the form

$$\lambda=\frac{1}{T}\frac{W^2}{96(1-E^2/4)}, \tag{23}$$

which, for $T=1$, coincides with the well-known standard formula for the inverse localization length in the Anderson model [12].

The extra factor $1/T$ stems from definition (12) of the Lyapunov exponent, which implies that $\lambda=\langle\dot{x}/x\rangle$ has the dimension of a inverse time. In order to have the correct physical dimension, therefore, λ must be inversely proportional to a time parameter which, on the other hand, must be a specific feature of the noise (4), since that is the physical origin of the orbit instability. This requirement singles out the period T between two kicks as the only parameter which can endow λ with the proper physical dimension; the proportionality $\lambda \propto 1/T$ is thus fully justified.

The expression (23) corresponds to the result derived by Thouless using standard perturbation methods [12]. As such, formula (23) fails to reproduce the correct behavior of the Lyapunov exponent at the band center, where the second-order perturbation theory of Thouless breaks down and an anomaly appears which was first explained in Ref. [13]. This deviation of the inverse localization length from the behavior predicted by formula (23) is a resonance phenomenon, which can be conveniently understood by considering the dynamics of the kicked oscillator (3) [10]. In fact, the band center corresponds to the case $\omega T = \pi/2$ and this equality can be interpreted as the condition that the frequency $1/T$ of the kicks be exactly four times the frequency $\omega/2\pi$ of the unperturbed oscillator. This generates a resonance effect that manifests itself in a small but clear increase of the localization length with respect to the value predicted by formula (23) for $E=0$. Once the origin of the anomaly at the band center is explained in these terms, it is not surprising that the model (8) fails to reproduce this feature, because it is obvious that the Wiener noise $W(t)$ cannot conveniently mimic the regularly time-spaced character of the shot noise (4). One might then worry that the model (8) provides an inadequate description of the Anderson model whenever the period of the unperturbed oscillator and that of the kicks stand in any rational ratio. This is not the case, however, because the resonance effect at the band center is the only one that affects the localization length at the second order of the perturbation theory ([10]) and is thus of interest for the present work. For all the other ''rational'' values of the energy $E = 2\cos \pi\alpha$ with α rational, the effect of the resonance on the Lyapunov exponent can be seen only by going beyond the second-order approximation in the weak disorder expansion (see details in Ref. [10]).

VI. THE NEIGHBORHOOD OF THE BAND EDGE

Besides the $k \to 0$ limit considered in the previous section, one can also study the behavior of the Lyapunov exponent (19) in the complementary case $k \to \infty$. Physically speaking, the limit $k \to \infty$ can be interpreted in different ways, depending on the reference model. If one bears in mind the kicked oscillator (3), then taking the limit $k \to \infty$ is tantamount to studying the case of a very strong noise. More precisely, the condition $k \gg 1$ implies that the kicks play a predominant role in the oscillator dynamics with respect to the elastic force. Notice that this is not in contrast with the requirement that each single kick be weak, as established by the condition (9). In fact, regardless of how weak the individual kicks are, their collective effect can be arbitrarily enhanced by making the interval T between two successive kicks sufficiently shorter than the fraction $\langle A_n^2 \rangle / \omega$ of the period of the unperturbed oscillator.

From the point of view of the Anderson model (1), the analysis of the case $k \gg 1$ is equivalent to the study of the localization length in a neighborhood of the band edge. To understand this point, one should consider that in the weak disorder case, as defined by the relation (9), the only way to fulfil the condition $k = \langle A_n^2 \rangle / (\omega T) \to \infty$ is to have $\omega T \to 0$. Correspondingly, the energy $E = 2\cos(\omega T)$ must approach the limit $E \to 2^-$, i.e., the edge of the band. Using relations (6), one can also express the condition $k \gg 1$ in the significant form

$$2 - E \ll \langle \epsilon_n^2 \rangle^{2/3},$$

which shows that the investigation of the case $k \gg 1$ corresponds to studying the behavior of the inverse localization length for energy values which are close to the band edge on a distance scale set by the fluctuations of the random site potential.

With the physical meaning of the limit $k \to \infty$ clear in mind, we can proceed to verify that Eq. (19) reproduces the correct behavior of the Lyapunov exponent in a neighborhood of the band edge. For this, it suffices to notice that the substitution

$$k = \frac{\langle \epsilon_n^2 \rangle}{\omega T \sin^2(\omega T)} \to k' = \frac{\langle \epsilon_n^2 \rangle}{(\omega T)^3} \tag{24}$$

transforms formula (19) in the expression originally obtained by Derrida and Gardner for the Lyapunov exponent at the band edge [9]. This implies that Derrida and Gardner's expression coincides with our own for $\omega T \to 0$, since in this limit the difference between parameters k and k' vanishes. The limit $\omega T \to 0$, on the other hand, identifies the band-edge case: this proves that formula (19) is correct not only inside the energy band (except that for $E=0$), but also for $E \to 2$. The extended validity range of expression (19) is a relevant feature; indeed, to the best of our knowledge, no other formula encompassing the *whole* energy band has been previously found for the Lyapunov exponent in the Anderson model.

To conclude our discussion of the $k \to \infty$ limit, we observe that in this case it may be appropriate to expand the integrals that appear in expressions (19) and (17) in the series of the inverse powers of k. One thus obtains

$$\lambda = \left(\frac{3}{4k^2} \right)^{1/3} \frac{\langle A_n^2 \rangle}{T} \frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-2\sqrt[3]{6})^n}{n!} \Gamma\left(\frac{2n+3}{6} \right) (k^{-1})^{2n/3}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-2\sqrt[3]{6})^n}{n!} \Gamma\left(\frac{2n+1}{6} \right) (k^{-1})^{2n/3}},$$

which is the counterpart of the expansion (21) of the preceding section. To the lowest order in k^{-1} this expression reduces to

$$\lambda = \frac{1}{T} \frac{\sqrt[3]{6}}{2} \frac{\sqrt{\pi}}{\Gamma(1/6)} \left(\frac{\omega T}{\sin(\omega T)} \right)^{2/3} \langle \epsilon_n^2 \rangle^{1/3},$$

so that for $E=2$ (i.e., for $\omega T = 0$), the Lyapunov exponent turns out to be

$$\lambda = \frac{1}{T} \frac{\sqrt[3]{6}}{2} \frac{\sqrt{\pi}}{\Gamma(1/6)} \langle \epsilon_n^2 \rangle^{1/3},$$

in perfect agreement with the result originally found in [9] (see also [10]).

VII. CONCLUDING REMARKS

In this work we have analyzed in a thorough way the correspondence that exists between the 1D Anderson model (1) on one hand and the random oscillators (3) and (8) on the other. We have shown how the exponential divergence of nearby trajectories of the oscillator(s) is equivalent to the localization of electronic eigenstates in the Anderson model. This equivalence manifests itself in the fact that the Lyapunov exponent for the random oscillator (8) coincides with the inverse localization length for the Anderson model (1). This equality holds across the *whole* energy band of the latter system, with the single exception of the band center, where the Lyapunov exponent for the stochastic oscillator (8) does not exhibit the anomaly which characterizes the inverse localization length of the Anderson model (1). This discrepancy can be simply explained with the impossibility for a continuous Wiener process to reproduce the ''resonant'' effects which are specifically due to the discrete nature of the kicks; this is actually the only limitation that prevents the analogy between models (8) and (1) from being complete.

Once the extent of the equivalence between the 1D Anderson model and the random oscillator has been clarified, it becomes possible to envision an extension of this correspondence to more complicated systems. One can, for instance, consider the 1D Anderson model with weak and correlated disorder and interpret it in terms of a linear oscillator with a frequency perturbed by a colored noise. The exploration of the correspondence between these two systems could lead to a better understanding of both, since it might allow one to transfer results and techniques from one model to the other. Very recently it was shown in [3] how to determine the localization length for the Anderson model with *any* correlated potential; the result, however, is valid only to the second order of perturbation theory. On the other hand, for the stochastic oscillator, a well-established cumulant expansion theory exists (see [14,8]) which allows one to go beyond the second order of the perturbation theory but can be applied only if the correlation time of the noise is short enough. It would therefore be extremely interesting to extend the application of these two complementary techniques from the system they were originally conceived for to the different but corresponding model. As a further and more ambitious goal one can think to extend the current approach to the 2D and 3D Anderson model. Relating this quantum model to a classical system of kicked oscillators could open the way to a better understanding of the mechanism of the Anderson localization in 2D and 3D disordered lattices.

Finally, we would like to point out that the classical model (3) of a kicked linear oscillator may find interesting applications in different physical problems. One example is the motion of a charged particle in modern accelerators. In this application, the unperturbed part of Eq. (3) describes transverse one-dimensional oscillations (''betatron oscillations'') of a particle moving in accelerator rings (see, for example, [15]). These oscillations are stable; however, the presence of a large number of thin magnetic lenses located along the ring originates an external perturbation that may lead to instability of the transverse oscillatory motion. It is clear that such a perturbation has the form of a succession of δ kicks since the particle moves at high speed around the ring and the lenses are thin, so that the passage of the particle through each lens can be considered to be instantaneous. The amplitudes of the kicks are different for different lenses; moreover, there are small time-dependent variations of the magnetic field due to experimental imperfections. As a result, this perturbation, albeit weak, can affect the long-time dynamics of the particles moving in the accelerator and produce a significant increase of their transverse energy, thus provoking a loss of beam particles (transverse dimensions of accelerators are typically small). It is quite amazing that these very different effects, quantum localization in disordered solids and the loss of particles in accelerators due to the influence of thin magnetic lenses, have much in common.

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