Reply to Comments on ''Generation of focused, nonspherically decaying pulses of electromagnetic radiation''

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The criticism made by Hannay [preceding Comment, Phys. Rev. E **62**, 3008 (2000)] is unfounded since the steps, familiar from the subluminal regime, that are taken in his argument are not mathematically permissible when the distribution pattern of the source is moving and has volume elements that approach the observer with the speed of light and zero acceleration along the radiation direction. In the superluminal regime, the retarded time is a multivalued function of the observation time and so the retarded potential for the radiation from a localized source cannot be represented, as Hannay assumes, by an integral over all space whose integrand entails a differentiable retarded distribution of the source density. Contrary to what is claimed by Hewish [Comment in this issue, Phys. Rev. E **62**, 3007 (2000)], moreover, there is no discrepancy between conventional antenna theory and the analysis that appears in Phys. Rev. E **58**, 6659 (1998). The characteristics of the new type of emission predicted by this analysis, and received from pulsars, differ from those of the radiation that is produced by known leaky waveguides because there are at present no antennas in which the emitting electric current is both volume-distributed and has the time dependence of a traveling wave with an accelerated superluminal motion.

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I. INTRODUCTION

The preceding two Comments [1,2] are concerned with two entirely different issues: the regularization of the divergent integrals that result from differentiating the retarded potential under the integral sign [1], and the connection of the analysis in [3] with antenna theory and radiation by leaky waveguides [2]. That the divergent integrals in question cannot be regularized by the canonical method (as claimed in [1]) and need to be handled by means of Hadamard's technique (as in [3]) will be discussed in Sec. II (see also [4]). That there is no discrepancy between the results reported in [3] and the existing data on pulsars and on fast traveling-wave antennas will be discussed in Secs. III and IV (see also [5]).

II. DIFFERENTIATION OF THE RETARDED POTENTIAL IN THE SUPERLUMINAL REGIME

Hannay's argument in Comment I [1] is not based on the analysis of any concrete example but on the assumption that all extended source distributions, including those with superluminally moving distribution patterns, would in general have the following three properties: (a) their volume would be finite, (b) the retarded values of both their density and the gradient of their density would be bounded and smooth as functions of the spatial coordinates, for an observation point that lies outside the source, and (c) their density could always be represented in such a way that the integration volume in the classical expression for the retarded potential [Eq. (2) of [1]] would consist of the entire space.

Property (c), which enables Hannay to differentiate the retarded potential under the integral sign without reference to the contribution from the limits of integration, is not compatible with the differentiability requirement (b) when the motion of the source is superluminal. Here we shall demonstrate this by adopting a specific source distribution that is both bounded and smooth, in its own rest frame, and explicitly showing that (i) Eqs. (4) and (5) of Comment I [1] do not follow from its Eq. (3) if there are any source elements that approach the observer with the speed of light and zero acceleration, and (ii) the density of the source in question has a retarded distribution in its globally valid representation that is not a differentiable function of the spatial coordinates [6]. (See also Appendix B of [3] and the references in [4].)

Let us consider a spherical source with the radius a whose center moves on a circle of radius r_0 with the constant angular frequency ω and whose density smoothly reduces from a maximum ρ_0 at its center to zero at its boundary, e.g., it has the form

$$\rho(r,\hat{\varphi},z)=\begin{cases}\rho_0\cos^2[\pi R_0/(2a)] & \text{if } R_0\leqslant a\\ 0 & \text{otherwise,}\end{cases} \tag{1}$$

where

$$R_0\equiv(z^2+r^2+r_0^2-2rr_0\cos\hat{\varphi})^{1/2}$$

is the distance of a point $(r,\hat{\varphi},z)\equiv(r,\varphi-\omega t,z)$ that is stationary in the rotating frame from the center $(r=r_0,\ \hat{\varphi}=0,\ z=0)$ of the sphere. The circle in broken lines in Fig. 1 shows the intersection, with the plane $z=0$, of the boundary of the above source in the $(r,\hat{\varphi},z)$ space for $r_0=\frac{3}{2}c/\omega$ and $a=\frac{1}{2}c/\omega$, where c is the speed of light *in vacuo*. (The axes in this figure are marked in units of c/ω and the larger dotted circles designate the light cylinder $r=c/\omega$ and the orbit $r=\frac{3}{2}c/\omega$ of the center of the source, respectively.)

Once the quantities $|\mathbf{x}-\mathbf{x}_P|$ and $d^3\mathbf{x}$ in Eq. (22) of [3] [or Eq. (2) of [1]] are expressed in terms of cylindrical coordinates and the above expression for the source density is inserted in the resulting form of the retarded potential, one arrives at an integral over the (r,φ,z) space,

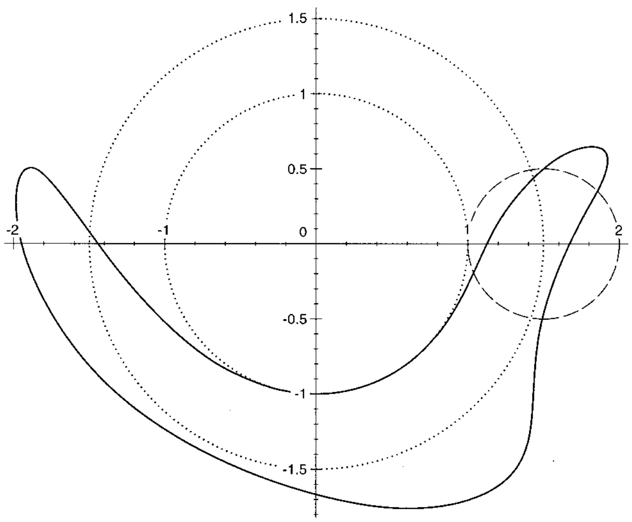

FIG. 1. The retarded shape of the source boundary in the (r,φ,z) space (the solid curve) compared to its original shape in the $(r,\hat{\varphi},z)$ space (the smallest circle in broken lines).

$$A_0(r_P,\hat{\varphi}_P,z_P)=\int r\,dr\,d\varphi\,dz\,\rho(r,z,\hat{\varphi}|_{t=t_P-R/c})/R$$

$$=\rho_0\int_{R_0|_{t=t_P-R/c}\leqslant a} r\,dr\,d\varphi\,dz \times\cos^2\left(\frac{\pi}{2a}R_0|_{t=t_P-R/c}\right)\Big/ R, \qquad (2)$$

with

$$R\equiv[(z-z_P)^2+r^2+r_P^2-2rr_P\cos(\varphi-\varphi_P)]^{1/2},$$

for which the domain of integration is automatically bounded.

Note that not only do we need to replace $\hat{\varphi}$ in the above expression for ρ by its retarded value

$$\hat{\varphi}|_{t=t_P-R/c}\equiv\hat{\varphi}_{\mathrm{ret}}=\varphi-\omega t_P+R(r,\varphi,z;r_P,\varphi_P,z_P)\omega/c \qquad (3)$$

when substituting Eq. (1) in the first member of Eq. (2), but in addition we need to delineate the domain of integration in Eq. (2) by mapping the source boundary $R_0=a$ from the $(r,\hat{\varphi},z)$ space onto the (r,φ,z) space. The image of the source boundary under the mapping $\hat{\varphi}\rightarrow\varphi$ expressed in Eq. (3) is a surface whose shape is different for different observers, or at different observation times, and bears no direct relationship with the sphere $R_0=a$ that appears in Eq. (1).

To specify the boundary of the domain of integration in Eq. (2), we need to solve the transcendental equation (3) for φ at every point $(r,\hat{\varphi},z)$ of the sphere $R_0=a$. In the case of the source depicted in Fig. 1, and of an observer that is located at $(r_P,\varphi_P,z_P)=(\frac{5}{2}c/\omega,0,0)$ at the observation time $t_P=(2\pi-\arccos\frac{2}{5}+\sqrt{21}/2)\omega^{-1}$, the intersection of this domain of integration with the plane $z=0$ has the shape shown by the solid curve in Fig. 1.

The boundary of the irregular volume occupied by the source in the (r,φ,z) space intersects a circle $r=$const, $z=$const (with $1<r\omega/c<2$ and $-\frac{1}{2}<z\omega/c<\frac{1}{2}$) at either two, four, or six values of φ. If we let $(\varphi_l^{(n)},\varphi_u^{(n)})$, with $n=1,2,\ldots$, denote the various intervals in φ that are occupied by the source at any given $(r,z;r_P,\varphi_P,z_P,t_P)$, then the volume integrals in Eq. (2) may be written as a triple integral over the variables φ, z, and r, respectively, in which the functions $\varphi_l^{(n)}(r,z;r_P,\varphi_P,z_P,t_P)$ and $\varphi_u^{(n)}(r,z;r_P,\varphi_P,z_P,t_P)$ constitute the various limits of integration with respect to φ.

Differentiation of the integral in question entails the differentiation of these limits of integration, limits that are given by the solutions φ of Eq. (3) for a point $(r,\hat{\varphi},z)$ on the boundary of the source distribution. Differentiating Eq. (3) with respect to $\mathbf{x}_P$ while holding $(r,\hat{\varphi},z)$ and the observation time $\hat{\varphi}_P$ constant, we find that the gradient of any of the $\varphi_l^{(n)}$ or $\varphi_u^{(n)}$ is given by an expression whose denominator both vanishes and has a vanishing derivative at the boundary points that approach the observer with the speed of light and zero acceleration [see Eq. (B3) of [3]].

In Eqs. (3) and (4) of [1], Hannay uses Leibniz's formula for the differentiation of a definite integral assuming that there are no contributions from the limits of integration. Leibniz's formula, on the other hand, is not applicable if there are any points at which the limits of integration are not differentiable [7]. In the case considered here, where the derivatives of the limits of integration are singular, the gradient of the integral in question does not consist solely of the integral of the gradient of its kernel, as claimed by Hannay. There is an additional contribution to the gradient of the potential: that which arises from the singularities of the gradients of the limits of integration in Eq. (2), and which comprises the boundary contribution to the Hadamard finite part of the gradient of the integral in Eq. (24b) of [3].

The singularities of the gradients of the limits of integration in Eq. (2) are the images, under the mapping $\hat{\varphi}\rightarrow\varphi$, of the singularities of the integrand of the gradient of Eq. (24b) in [3]. By overlooking the contribution from the limits of integration in his Eqs. (3) and (4), Hannay has discarded the boundary term in Hadamard's finite part of the divergent integral that results from the differentiation of the alternative form of the retarded potential given in Eq. (24b) of [3]. (For detailed discussions of this point, see the references in [4].)

Let us now consider an alternative representation of the same source density which complies with Hannay's requirement (c), i.e., which allows us to extend the domain of integration in Eq. (2) to the entire (r,φ,z) space. By introducing a step function that incorporates the vanishing of the density of the source outside its boundary $(R_0=a)$ into the expression for ρ, we can rewrite Eq. (1) in a form that is globally valid:

$$\rho(r,\hat{\varphi},z)=\rho_0\cos^2[\pi R_0/(2a)]\theta(a-R_0), \qquad (4)$$

where $\theta(x)$ is 1 when $x>0$ and zero when $x<0$. The expression that is obtained by inserting Eq. (4) into Eq. (2) may be differentiated under the integral sign without giving consideration to boundary contributions.

However, differentiation of the step function in Eq. (4) results in an additional contribution to the derivative of the integrand of Eq. (2) that entails the Dirac δ function: The gradient of the potential with respect to the coordinates $\mathbf{x}_P$ of the observation point is given by

$$\nabla_P A_0 = -\rho_0 \int r\,dr\,d\varphi\,dz \Big\{ R^{-1}\cos^2\!\left(\frac{\pi}{2a}R_{0\mathrm{ret}}\right)\nabla_P R_{0\mathrm{ret}} \times \delta(a-R_{0\mathrm{ret}}) - \nabla_P\!\left[R^{-1}\cos^2\!\left(\frac{\pi}{2a}R_{0\mathrm{ret}}\right)\right] \times \theta(a-R_{0\mathrm{ret}})\Big\}, \tag{5}$$

in which $R_{0\mathrm{ret}} \equiv R_0|_{\hat\varphi=\hat\varphi_{\mathrm{ret}}}$. Note that, contrary to the statement made by Hannay [1], the vector $\nabla_P R_{0\mathrm{ret}}$ that appears in Eq. (5) has nothing to do with the normal, $\nabla R_{0\mathrm{ret}}$, to the boundary $R_{0\mathrm{ret}}=a$ of the retarded distribution of the source: $R_{0\mathrm{ret}}$ depends on (r_P,φ_P,z_P) and on (r,φ,z) in radically different ways.

That the term entailing the δ function in the integrand of Eq. (5) is ill-defined even as a generalized function may be seen by performing the integration with respect to φ:

$$\nabla_P A_0 = -\rho_0 \int r\,dr\,dz \sum_{\varphi_l^{(n)},\varphi_u^{(n)}} R^{-1}\cos^2\!\left(\frac{\pi}{2a}R_{0\mathrm{ret}}\right)\nabla_P R_{0\mathrm{ret}} \times |(\partial R_0/\partial\hat\varphi)(\partial\hat\varphi_{\mathrm{ret}}/\partial\varphi)|^{-1} + \cdots, \tag{6}$$

where the dots stand for the second term in Eq. (5). The zeros $\varphi=\varphi_l^{(n)}$ and $\varphi=\varphi_u^{(n)}$ of $R_{0\mathrm{ret}}-a=0$, at which the expression under the summation sign must be evaluated, are the same values of φ as those which earlier appeared as limits of integration. The factor $|\partial\hat\varphi_{\mathrm{ret}}/\partial\varphi|^{-1}$, thus implicit in the definition of the above δ function, diverges algebraically at those points of the boundary that approach the observer with the speed of light and zero acceleration [see Eqs. (5) and (7) of [3]].

The product of this divergent factor with the vanishing value of the cosine-squared term on the source boundary constitutes a contribution toward the derivative of the retarded potential that is neither infinite nor zero. This product is an *indeterminate* quantity, contravening the differentiability requirement (b), that would have to be evaluated by means of a physically meaningful procedure [4]. The contribution arising from the inserted step function cannot be any different from that which arises from the limits of integration in Eq. (2). If the product in question were zero, as claimed in [1], then it would follow that Leibniz's rule could be applied to an integral with undifferentiable limits whose integrand vanishes at the boundaries of its integration domain, a conclusion that would manifestly contradict the statement of Leibniz's theorem [7].

Far from side-stepping the singularity encountered earlier that needs to be handled by Hadamard's technique [4], therefore, the adoption of an infinite domain of integration would merely replace the indifferentiability of the limits of the integral in question by a closely related indifferentiability of its integrand. Note that the localized source adopted here does not have a sharp edge; the gradient of its density vanishes at its boundary. Because the retarded time is a multivalued function of the observation time in the superluminal regime, the retarded distribution of the density of a moving source (such as that whose contour $\rho=0$ is depicted in Fig. 1) can lack differentiability even when its original distribution is smooth. The anomalous field decay encountered in the present case [3] is fundamentally different from that which can occur when the density, or the gradient of the density, of the source distribution itself is not smooth. Both the above argument and the analysis in [3] are independent of the order with which the gradient of the source density vanishes at the source boundary.

The contribution from the limits of integration to the right-hand side of Eq. (3) in [1] is zero, as assumed by Hannay, only in the familiar subluminal regime where the derivatives of these limits are singularity-free. In the case of a superluminally moving accelerated source, this contribution is nonvanishing and has a value that may be calculated by means of Hadamard's method [3,4]. The upper bound derived by Hannay applies only to the contribution to the derivative of the retarded potential that arises from the derivative of its integrand, i.e., to the contribution that is retained by Hannay, not to the contribution from the limits of integration that is overlooked by him.

III. CONNECTION WITH ANTENNA THEORY AND RADIATION BY LEAKY WAVEGUIDES

Charge and current distributions moving faster than light do occur inside waveguides and traveling-wave antennas as pointed out by Hewish [2]. However, the charge-current distributions occurring in these systems are invariably limited to either one or two dimensions. Unless the superluminally moving charges and currents are distributed over a volume, the locus of source points that approach the observer with the speed of light and zero acceleration at the retarded time would not be extended and so the resulting radiation would not have the properties that are predicted by the analysis in [3]. It is possible to design modified versions of the existing leaky waveguides that could generate the predicted radiation [8], but there are at present no known antennas in which the source is both volume-distributed and has an accelerated superluminal motion.

The existing versions of fast wave antennas normally entail a source whose distribution moves with a constant phase speed, $u>c$, and whose radiation is beamed at an angle arccos(c/u) to the electric current. As a manifestation of the Čerenkov effect in vacuum, the radiative process in such antennas is fully consistent with, and its observed characteristics confirm, the principles on which the analysis in [3] is based. The radiative process that would come into play when the motion of a superluminal *volume* source is *accelerated*, however, is a different one that has not yet been explored experimentally.

Contrary to what is claimed by Hewish [2], there is no distinction between conventional antenna theory and the type of analysis that appears in [3]. Both are based on the very same solution of Maxwell's equations: the retarded four-potential. The analysis presented in [3] is somewhat more involved than those normally encountered in antenna theory because (i) it does not invoke the far-field approximation, which ignores the curvature of wave fronts and so obliterates the formation of caustics, until the end of the calculation, (ii) it is performed in the time domain, without the use of Fourier decomposition which obscures sharp gradients, and (iii) it is fully three-dimensional. Before he can ''conclude'' that this analysis ''is flawed'' [2], it is clearly essential that Hewish should pinpoint, or at least identify the source of, a possible error in it.

The only objection of this kind raised by Hewish [2] is the one regarding a reference to the Doppler effect in Appendix C of [3], a reference that is made merely to facilitate the physical interpretation of the derived relationship between the emission and reception time intervals of the pulses. This relationship is fundamentally different from that which holds in the case of the Doppler effect: It is a consequence of the fact that, for superluminally moving source elements, the retarded time is a multivalued function of the observation time. There is an analogy with the Doppler effect, therefore, only in the way in which the emitted wave fronts pile up in certain directions, and not in the mechanism by which superluminal sources give rise to the elongation of the emission time interval [5].

Hewish's objection to the estimate of the intensity of the radiation in pulsars is not independent of his contention that the analysis in [3] is flawed [2]. Given the superluminal model of pulsars [5], the large enhancement factor to which he objects is a direct consequence of the violation of the inverse-square law by the intensity of the propagating caustics constituting the pulses that emerges from that analysis.

Finally, the statement ''Ardavan now claims that my analysis can, at best, be suggestive'' [2] stems from a misunderstanding: It is not Hewish's analysis that is regarded as suggestive in the last paragraph of Appendix C [3]. What is referred to as suggestive in that paragraph are the implications of the analysis presented in Appendix C itself (which is based on geometrical optics) in connection with a result (the discontinuity across the bifurcation surface) that is already derived (in Sec. V) from a more exact theory (physical optics).

IV. CONCLUDING REMARK

An experimental device is currently being built at Oxford University that will be capable of producing a superluminally moving polarization current and thereby testing the theoretical results whose validity is questioned in the preceding Comments [1,2].

The apparatus in this experiment consists of an arc of a circular dielectric rod, an array of electrode pairs positioned opposite to each other along the rod, and the means for applying a voltage to the electrodes sequentially at a rate sufficient to induce a polarization current whose distribution pattern moves along the rod with a speed exceeding the speed of light *in vacuo*. A superluminal speed is achieved for a circular rod of diameter $\sim$10 m (and arc length $\sim$1 m) by arranging the frequency of the applied voltage and the phase difference between neighboring electrode pairs such that the distribution pattern of the polarization rotates with the frequency $\sim$10 MHz. If the amplitude of the resulting polarization current is in addition made to fluctuate at $\sim$500 MHz, then the device would generate a nonspherically decaying (coherent) radiation at $\sim$500 MHz, and a spherically decaying (incoherent) radiation with a spectrum that extends as far as $\sim$1 THz [8].

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[5] H. Ardavan, ASP Conf. Ser. **202**, 365 (2000).

[6] Here we are not referring to the familiar breakdown of differentiability in the case of an observation point that coincides with a source point; our discussion is concerned with the potential in the radiation zone.

[7] R. Courant, *Differential and Integral Calculus* (Blackie, New York, 1967), Vol. 2, p. 220.

[8] A. Ardavan and H. Ardavan, International Patent Application No. PCT/GB99/02943 (6 September, 1999).