Comment II on ''Generation of focused, nonspherically decaying pulses of electromagnetic radiation''

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The claim [H. Ardavan, Phys. Rev. E **58**, 6659 (1998)] that a smooth, fast rotating source distribution can radiate with an intensity decaying more slowly than the inverse square distance violates a rigorous upper bound on intensity and is therefore false. The bound had been derived in response to earlier claims and the derivation is repeated here.

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The claim [1] that a smooth, fast rotating source distribution can radiate with an intensity decaying more slowly than the inverse square distance violates a rigorous upper bound [2] on intensity and is therefore false. I noted this bound in response to several earlier papers by Ardavan [3–8], claiming unexpectedly strong waves from such sources. The fact that the new claim violates my bound escapes direct mention in [1], though my paper is cited (p. 6674 of Ref. [1]) with commentary to the effect that I have overlooked the special motion of the source. No mention of this motion (subluminal or superluminal) is necessary—the source distribution variation in my bound is quite general. I repeat the derivation here. The source distribution in question is finite in extent, and the integrals are over all space unless otherwise specified.

The retarded solution of the wave equation $\nabla^2\psi - c^{-2}\partial^2\psi/\partial t^2 = -4\pi s(\mathbf{r},t)$ for any time-dependent source $s(\mathbf{r},t)$ is, at an observation point $\mathbf{r}_p$,

$$\psi(\mathbf{r}_p,0)=\tfrac{1}{2}\int_{-\infty}^{0}dt\int s(\mathbf{r},t)\,\delta((\mathbf{r}-\mathbf{r}_p)^2-c^2t^2)d^3\mathbf{r} \tag{1}$$

$$=\int \frac{s(\mathbf{r},-c^{-1}|\mathbf{r}-\mathbf{r}_p|)}{|\mathbf{r}-\mathbf{r}_p|}d^3\mathbf{r}. \tag{2}$$

The derivative of this is required (since the intensity is its square):

$$\frac{\partial\psi(\mathbf{r}_p,0)}{\partial\mathbf{r}_p}=\frac{\partial}{\partial\mathbf{r}_p}\int \frac{s(\mathbf{r},-c^{-1}|\mathbf{r}-\mathbf{r}_p|)}{|\mathbf{r}-\mathbf{r}_p|}d^3\mathbf{r} \tag{3}$$

$$=\int \frac{\partial}{\partial\mathbf{r}_p}\left[\frac{s(\mathbf{r},-c^{-1}|\mathbf{r}-\mathbf{r}_p|)}{|\mathbf{r}-\mathbf{r}_p|}\right]d^3\mathbf{r} \tag{4}$$

$$=\int \frac{\boldsymbol{\nabla}s(\mathbf{r},-c^{-1}|\mathbf{r}-\mathbf{r}_p|)}{|\mathbf{r}-\mathbf{r}_p|}d^3\mathbf{r}, \tag{5}$$

where $\boldsymbol{\nabla}$ differentiates $s(\mathbf{r},t)$ with respect to the first argument only.

If the source distribution is zero outside some fixed volume, say, a sphere of radius b centered on the origin, and its gradient has a maximum magnitude $\overline{s'}$, then one has the inequality

$$\left|\frac{\partial\psi(\mathbf{r}_p,0)}{\partial\mathbf{r}_p}\right|\leqslant\overline{s'}\int_{r<b}|\mathbf{r}-\mathbf{r}_p|^{-1}d^3\mathbf{r}$$

$$=\frac{\frac{4}{3}\pi b^3\overline{s'}}{r_p}\quad\text{for}\quad r_p\geqslant b \tag{6}$$

$$\left(\text{and}\quad\frac{2}{3}\pi\overline{s'}(3b^2-r_p^2)\quad\text{for}\quad r_p\leqslant b\right). \tag{7}$$

The intensity is the square of the field gradient (7), which therefore decays no more slowly than r_p^{-2}, disproving the claim of [1].

The step from (3) to (4) is justified merely by the smoothness of $s(\mathbf{r}t)$ provided that, as in [1], the observation point lies outside the source region. Otherwise, if the denominator can vanish, the justification (from [2]) is briefly as follows. It was straightforwardly adapted from that for electrostatics by Courant and Hilbert [9] (p. 246). Let $t_a=[(\mathbf{r}-\mathbf{r}_p)^2+a^2]/2ac$ for $|\mathbf{r}-\mathbf{r}_p|\leqslant a$ and $t_a=|\mathbf{r}-\mathbf{r}_p|/c$ otherwise, a being a small positive constant. The value and gradient of t_a is continuous at $|\mathbf{r}-\mathbf{r}_p|=a$. The modified equality with t_a replacing $|\mathbf{r}-\mathbf{r}_p|/c$ in Eqs. (3) and (4),

$$\frac{\partial}{\partial\mathbf{r}_p}\int\left[\frac{s(\mathbf{r},-t_a)}{ct_a}\right]d^3\mathbf{r}=\int\frac{\partial}{\partial\mathbf{r}_p}\left[\frac{s(\mathbf{r},-t_a)}{ct_a}\right]d^3\mathbf{r}, \tag{8}$$

is true by virtue of the ordinary criteria for interchange. But the integral on the left side differs from that in Eq. (3) by $O(a)$, and the integral on the right side differs from that in Eq. (4) by $O(a)$, with the bounds given in [2]. This suffices for the equality of Eqs. (3) and (4).

Refer to the Reply [10] by Ardavan for his response.[1]

[1]Finally the points raised in the Reply [10] to this Comment may be countered as follows. The correct domain of integration for the solution (1) of the wave equation is all space. This is *not* an assumption. It *is* an assumption that the source strength variation $s(\mathbf{r},t)$ in spacetime is sufficiently smooth—a rigid rotation of a smooth spatial distribution, for example. This is the nature of the distributions giving rise to the effects claimed [1–8] (entirely different mechanisms are known to allow anomalous field decay for nonsmooth sources). My bound on the intensity then follows irrespective of the superluminal or subluminal source motion. An example is then given in the Reply of a particular source distribution which is smooth (enough) and rotates fast. The smoothness of the consequent retarded source distribution is called into question. Any such objection must be flawed since the smoothness of $s(\mathbf{r},t)$ implies the smoothness of retarded distribution $s(\mathbf{r},-c^{-1}|\mathbf{r}-\mathbf{r}_p|)$ for any external observation position $\mathbf{r}_p$. Specifically $R^{-1}\partial s/\partial \mathbf{r}_p$, which is the relevant part of the argument of the integral in Eq. (5) of the Reply, is bounded (as follows) so that the ''indeterminate'' coefficient of the δ function must be zero (i.e., the zero of the cosine dominates over the implicit Jacobian of the δ). The bound is

$$|R^{-1}\partial s/\partial \mathbf{r}_p| = R^{-1}|\partial s/\partial t|\,|\partial t/\partial \mathbf{r}_p| < (\mathbf{r}_p-\mathbf{r}_0-a)^{-1} \times(\mathbf{r}_0+a)\omega\rho_0\pi/2ac^{-1} < \infty.$$

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