Percolation in sign-symmetric random fields: Topological aspects and numerical modeling

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The topology of percolation in random scalar fields $\psi(\mathbf{x})$ with sign symmetry [i.e., that the statistical properties of the functions $\psi(\mathbf{x})$ and $-\psi(\mathbf{x})$ are identical] is analyzed. Based on methods of general topology, we show that the zero set $\psi(\mathbf{x})=0$ of the n-dimensional ($n\geqslant 2$) sign-symmetric random field $\psi(\mathbf{x})$ contains a (connected) percolating subset under the condition $|\nabla\psi(\mathbf{x})|\neq 0$ everywhere except in domains of negligible measure. The fractal geometry of percolation is analyzed in more detail in the particular case of the two-dimensional ($n=2$) fields $\psi(\mathbf{x})$. The improved Alexander-Orbach conjecture [Phys. Rev. E **56**, 2437 (1997)] is applied analytically to obtain estimates of the main fractal characteristics of the percolating fractal sets generated by the horizontal "cuts," $\psi(\mathbf{x})=h$, of the field $\psi(\mathbf{x})$. These characteristics are the Hausdorff fractal dimension of the set, D, and the index of connectivity, θ. We advocate an unconventional approach to studying the geometric properties of fractals, which involves methods of *homotopic topology*. It is shown that the index of connectivity, θ, of a fractal set is the topological invariant of this set, i.e., it remains unchanged under the homeomorphic deformations of the fractal. This issue is explicitly used in our study to find the Hausdorff fractal dimension of the single isolevels of the field $\psi(\mathbf{x})$, as well as the related geometric quantities. The results obtained are analyzed numerically in the particular case when the random field $\psi(\mathbf{x})$ is given by a *fractional Brownian surface* whose topological properties recover well the main assumptions of our consideration.

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I. INTRODUCTION

Applications of percolation theory (see, e.g., Refs. [1,2]) have led to remarkable advances in the understanding of many phenomena related to the formation of irregular structures. Topological properties of irregular, random configurations have recently received a good deal of attention in association with the possible universal nature of the geometry of percolation in the vicinity of the critical percolation threshold [3,4].

Indeed, consider an infinite, statistically homogeneous, isotropic random scalar field $\psi(\mathbf{x})$, where $\mathbf{x}\in E^n$ is an n-dimensional Euclidean vector ($n\geqslant 2$). (The symbol E^n denotes n-dimensional Euclidean space; the dimensionality n is assumed to be an integer in what follows.) The introduction of an arbitrary threshold h [to be defined as the level of the "cut" of the field $\psi(\mathbf{x})$] makes it possible to divide the space into two topologically different parts: one composed of all regions where $\psi(\mathbf{x})<h$, marked as being "empty"; and the other composed of the regions where $\psi(\mathbf{x})>h$, marked as being "filled." One of these parts will include a *connected* infinite set, which is said to "percolate." Changing the threshold h, one can find the *critical* threshold, h_c, when the topological phase transition occurs (i.e., the nonpercolating part starts to percolate, or vice versa).

It has been recognized [1,2,4,5] that the geometry of the percolating set *at criticality* (i.e., at the levels $h\to h_c$) is a typical *fractal* for length scales χ, varying between some microscopic ("lattice") distance a and the percolation correlation length $\xi\gg a$. Near the point of the percolation transition, the percolation correlation length ξ is known to diverge, $\xi/a\to\infty$ [1]; it has been clearly established [6] that the divergence of the percolation correlation length ξ results in an anomalous behavior of the macroscopic physical quantities near the critical percolation threshold, h_c.

A. Hausdorff fractal dimension and the index of connectivity

The term "fractal" [7] was originally introduced to quantify the geometric features of a variety of natural objects whose fine-scale structure is *statistically self-similar*. "Statistically self-similar" means that any small part of such an object could be considered (in the statistical sense) as a reduced scale image of the whole. The statistically self-similar geometry appears in the *power-law* behavior of the average "mass" density of the fractal sets. More precisely, this power-law behavior is contained in the factor χ^{D-n}, where χ is the length scale; for the percolating fractal sets, χ ranges between two characteristic lengths; the microscopic distance a and the correlation length $\xi\gg a$. The parameter D in the power exponent, $D-n$, is the so-called *fractal dimension* [7] of the set, commonly referred to as the Hausdorff dimension of the fractal, and n is the dimensionality of the embedding Euclidean space, E^n, which is always not less than D, i.e., $D\leqslant n$. Thus the average density of the fractal objects is *length scale dependent* for $D\neq n$. (In standard Euclidean geometry, the Hausdorff dimension D coincides with the embedding dimensionality n, so that the corresponding average density is constant.)

The fractal dimension D is not, however, the only geometric parameter required for the complete description of the self-similar fractals. The other is the index of connectivity θ [4,6,8,9]; contrary to the fractal dimension D which describes the scaling behavior of the averaged ''mass'' density of a fractal set, the index θ quantifies how the ''elementary'' structural units inside the set (e.g., the filled and empty sites for the problem of the site percolation on lattices) are ''glued'' (connected) together to form the entire fractal object. In Euclidean geometry, when D coincides with the Euclidean dimensionality n, $\theta \equiv 0$; however, in fractal geometries, when D differs from the dimensionality of the embedding space n, the index θ may attain nonzero values.

Roughly speaking, the parameter θ describes the ''shape'' of a fractal object, and may be different for fractals even with equal values of the fractal dimension D. A more precise definition of the index θ could be given by using the concept of the geodesic line, i.e., the ''shortest'' line connecting two ''elementary'' structural units of the fractal. Indeed, the topological arguments of Ref. [10] show that the geodesic line on a self-similar fractal object could be treated as a self-affine fractal curve whose own Hausdorff fractal dimension is equal to $(2+\theta)/2$. This issue was discussed in more detail in Ref. [11]. A rigorous definition of the self-affine fractal curve can be found, e.g., in Ref. [2]. The geometric significance of the index θ for the fractal objects was also discussed on a descriptive level in Ref. [12]. (In the notations of Ref. [12], $\theta \equiv \sigma$.) A detailed consideration of the issue of connectivity for the percolating fractal sets (percolating fractal networks) was given in Ref. [4] where the direct relation between the parameter θ and the topological structure of the fractal network was pointed out.

It is worth mentioning that the index of connectivity θ plays an essential role in many dynamical phenomena on fractals, e.g., transport processes in disordered media [6,8,9,12–15], ''bimolecular'' chemical reactions [16,17], localization of waves [4,11,18,19], etc. The original important promotion of the parameter θ was made in a pioneering paper [6] where the concept of range-dependent diffusion on percolating networks was proposed. By applying scaling theory, it was shown [6] that the diffusion constant on a percolating network, for length scales χ ranging between a and ξ, behaves as a power law $\propto \chi^{-\theta}$.

B. Alexander-Orbach conjecture

The latter insight, along with the realization that solving the problem of the range-dependent diffusion was equivalent to solving the (scalar) elastic vibration problem (for more details, see, e.g., Ref. [4]), led Alexander and Orbach [3] to evaluate the density of states for vibrations of a percolating network at criticality (these vibrations were termed fractons), with the introduction of the so-called fracton, or spectral, dimension $\tilde{d}$. This new quantity was defined as a specific combination of the fractal dimension D and the index of connectivity θ, and has the form $\tilde{d} \equiv 2D/(2+\theta) \leqslant D$. In addition, Alexander and Orbach [3] noted that the spectral dimension $\tilde{d}$ for the percolating networks at criticality was numerically remarkably close to the mean-field value 4/3 (exact in Euclidean dimension $n=6$) for all embedding Euclidean dimensions n greater than one, even though the parameters D and θ were by no means constant as functions of n (below $n=6$). This numerical evidence led them to speculate that the spectral dimension $\tilde{d}$ might be exactly 4/3 for the percolating sets at criticality in all embedding dimensions $n \geqslant 2$. This has come to be known as the Alexander-Orbach (AO) conjecture [3].

The AO conjecture is important because, if true, it might allow one to describe the fractal geometry of percolation by using the unique basic concept of the spectral dimension $\tilde{d} = 4/3$ for such fundamental problems as correlated and uncorrelated percolation on lattices, as well as for the more general continuum percolation problem [2,4]. The great interest in this conjecture results not only from the universal value 4/3 assigned to $\tilde{d}$ in all $n \geqslant 2$, but also from the fact that it establishes a relationship between the index of connectivity, θ, which appears in the description of the dynamical processes on fractals, and the Hausdorff fractal dimension D, yielding the scaling behavior of the density of the fractal substrate.

Much theoretical and numerical effort has been made in the attempt to prove or disprove the AO conjecture (for a comprehensive review, see, e.g., Refs. [4,13]). At present, the situation is as follows. For sufficiently high embedding Euclidean dimensions $n \geqslant 6$, a rigorous analytical proof for the AO conjecture has been obtained within the mean-field theory. The mean-field percolation is well modeled by the percolation on the so-called Cayley trees (Bethe lattices). A Cayley tree is defined as a graph without loops in which each node has the same number of branches; the self-similarity of such graphs is not necessarily manifest in their geometric representation, but is seen in their connectivity [20]. The percolation problem on Cayley trees was solved exactly by Coniglio [21].

For lower embedding dimensions $2 \leqslant n \leqslant 5$, the mean-field theory cannot be directly applied, and an analytical consideration of the topology of percolation in these dimensions meets considerable difficulties [13]. Meanwhile, a large body of studies, both theoretical and numerical, indicates that the true value of the spectral dimension $\tilde{d}$ must be slightly *smaller* than 4/3 for $2 \leqslant n \leqslant 5$ (for a review, see, e.g., Ref. [4]). Thus, an improvement of the original AO value 4/3 for $2 \leqslant n \leqslant 5$ was placed at the center of attention [4].

C. Percolation constant

Recently, Milovanov [10] proposed an unconventional analytical approach to study the fractal geometry of percolation at a critical threshold, which involves methods of differential topologies. The idea of his approach was to supply a fractal object with an additional topological structure of the *fractal manifold* through the introduction of local coordinates at each point of the fractal. Milovanov found that the number of independent coordinates must be formally *fractional* and equal to the spectral fractal dimension, $\tilde{d}$. This approach led him to prove that the value of $\tilde{d}$ at the threshold of percolation satisfies some transcendental algebraic equation; the solution of this equation is some universal topological constant $\mathcal{C}$, i.e., $\tilde{d} = \mathcal{C}$ for all $2 \leqslant n \leqslant 5$. This constant is approximately equal to $\mathcal{C} = 1.327 \pm 0.001$ [10], and is indeed slightly smaller than the original AO result, 4/3. These findings lead to the improvement of the AO value for $2 \leqslant n \leqslant 5$:

$$\tilde{d} \equiv 2D/(2+\theta) = \mathcal{C} \approx 1.327 < 4/3. \tag{1}$$

For the sake of convenience, we refer below to the quantity $\mathcal{C} \approx 1.327$ as the ''percolation constant.''

To illustrate result (1), consider, for instance, the so-called uncorrelated percolation on lattices [2]. From a wealth of numerical studies it has become clearly understood that the Hausdorff fractal dimension D and the index of connectivity θ for the uncorrelated percolating networks on lattices do not depend on the particular kind of the lattice assumed (e.g., triangular, square, etc.), as well as on the specific type of the percolation problem (e.g., ''site'' or ''bond'' [2]). The only parameter that actually affects the numerical values of D and θ, is the dimensionality of the embedding Euclidean space, n $(n<6)$. For example, the uncorrelated percolation on the plane lattices $(n=2)$ is described by the values $D = 91/48 \approx 1.896$ and $\theta \approx 0.86$ (Refs. [2] and [4,6] respectively. This yields the estimate of the spectral fractal dimension, $\tilde{d} \approx 1.326$, which coincides, within the numerical errors, with the percolation constant $\mathcal{C} = 1.327 \pm 0.001$.

Strictly speaking, result (1) is exact for *contractible* fractal sets [22] which do not contain isolated ''voids.'' (Such sets can be continuously deformed into a point.) It can be shown [10], however, that the spectral fractal dimension $\tilde{d}$ for the *noncontractible* percolating fractal sets is at least not larger than the percolation constant $\mathcal{C}$, i.e., $\tilde{d} \leqslant \mathcal{C} \approx 1.327$ in general. This can be supported, e.g., by the numerical results of Normand *et al*. [23], which are among the most accurate. These results were obtained for the plane percolation $(n=2)$, yielding $\tilde{d} \approx 1.321$, which is indeed slightly smaller than $\mathcal{C} \approx 1.327$.

In this paper, we concentrate our attention on the geometry of percolation in random scalar fields $\psi(\mathbf{x})$ having the specific property of *sign symmetry*. (We imply that $\mathbf{x} \in E^n$ and $n \geqslant 2$.) ''Sign symmetry'' means that the statistical characteristics of the random function $\psi(\mathbf{x})$ are identical to those of $-\psi(\mathbf{x})$. Without loss of generality, we require that the function $\psi(\mathbf{x})$ has zero mean, i.e., $\langle \psi(\mathbf{x}) \rangle = 0$, where the average $\langle \cdots \rangle$ is taken over an area of E^n with the characteristic linear size of the order of the percolation correlation length ξ or more. Our particular interest in the random fields $\psi(\mathbf{x})$ with the sign symmetry is motivated by their importance for a number of particular physical applications (see, e.g., Refs. [2,12,24,25], and references therein), where the reflection properties of the field $\psi(\mathbf{x})$ play a role.

Below, we advocate a nontraditional approach to studying the fractal geometry of percolation at the threshold which includes the methods of the homotopic topology. In Sec. II, we discuss some general topological features of the sign-symmetric random scalar fields $\psi(\mathbf{x})$ in Euclidean spaces of arbitrary dimensionality $n \geqslant 2$. It is proven in Sec. II that the zero set of the sign-symmetric random function $\psi(\mathbf{x})$, $n \geqslant 2$, always contains a percolating subset if the condition $|\nabla \psi(\mathbf{x})| \neq 0$ holds almost everywhere. In Sec. III, we analyze in more detail the fractal topology of percolation in two dimensions $(n=2)$. Our particular attention in Sec. III is concentrated on (i) the fractal properties of the single isolevels, $\psi(\mathbf{x}) = h$, $h \to 0$, $h \neq 0$, of the random field $\psi(\mathbf{x})$; and (ii) the fractal characteristics of the percolating subsets of the zero set, $\psi(\mathbf{x}) = 0$, of the function $\psi(\mathbf{x})$. The relevant values of the Hausdorff fractal dimension and the index of connectivity are obtained in Sec. III in terms of the percolation constant $\mathcal{C}$. In Sec. IV, the results of the numerical modeling of the random field $\psi(\mathbf{x})$ on a plane are presented, and a comparison with the basic topological conclusions is given. In Sec. V We summarize the results obtained.

II. PERCOLATION PROPERTIES OF THE ZERO SET

Let E^n be n-dimensional Euclidean space $(n \geqslant 2)$. Consider a random function $\psi(\mathbf{x})$ in E^n, which is statistically identical to the function $-\psi(\mathbf{x})$. (Here ''random'' does not necessarily exclude the possible existence of the long-range correlations [2], and could be applied to problems of the correlated and continuum percolation.)

Assume the condition $|\nabla \psi(\mathbf{x})| \neq 0$ everywhere except, perhaps, in domains of negligible measure. This condition implies that the field $\psi(\mathbf{x})$ is nondegenerated almost for all $\mathbf{x}$. Note that the infinite values of $\nabla \psi(\mathbf{x})$, i.e., when $|\nabla \psi(\mathbf{x})| = +\infty$, are allowed.

Then let $a \ll \xi$ be the microscopic length scale where the function $\psi(\mathbf{x})$ behaves as a continuous nonsingular function. [This, of course, implies that $\psi(\mathbf{x})$ could be treated as ''random'' only at length scales $\chi \gg a$. We also assume below that a cannot be infinitely small just as one could expect for the majority of physical applications.] The important inference to be deduced is that the zero set of the function $\psi(\mathbf{x})$ [i.e., the set of points $\mathbf{x} \in E^n$ where $\psi(\mathbf{x}) = 0$] percolates. [More precisely, the zero set of the function $\psi(\mathbf{x})$ must contain the *percolating subset*; the entire zero set of $\psi(\mathbf{x})$ may be *disconnected* in general.]

This assertion, being quite natural, was formulated in Ref. [2], although we are not aware of any rigorous mathematical proof. Considering such a proof, however, might help one achieve a deeper insight of the topology of percolation in random scalar fields $\psi(\mathbf{x})$ from a more abstract point of view.

Let us now prove the above assertion for arbitrary Euclidean dimensionality $n \geqslant 2$. Assume the contrary: the zero set of the field $\psi(\mathbf{x})$ does not percolate. This leads to the conclusion that all zero isolevels of $\psi(\mathbf{x})$ are bounded closed sets. [We use the term ''zero isolevel'' to denote the boundary of the zero set of the function $\psi(\mathbf{x})$.] In fact, the zero isolevel being an unbounded set would imply that this isolevel stretches to infinity and, therefore, percolates. Also, one can immediately become convinced that the set of all points where $\psi(\mathbf{x}) \neq 0$ is open, and hence, the zero set of $\psi(\mathbf{x})$ is closed. Since the interior of the closed set must be open [26], its boundary, i.e., the zero isolevel, in our case, is also a closed set. (We implicitly take into account that, by assumption, the length scale a cannot be arbitrarily small, enabling one to rely on the concept of the topological space [26].)

The next step is to observe that any bounded closed set in Euclidean space E^n is compact [27]. Thus all zero isolevels of $\psi(\mathbf{x})$ must be compact in E^n. Any of these compact sets divides E^n into two topologically different parts: one is bounded (finite in size), and coincides with the interior of the set; the other is infinite, being its exterior. Let F_i ($1\leqslant i \leqslant\infty$) denote, for a given *i*, the zero isolevel of the function $\psi(\mathbf{x})$ along with its interior. (Since the microscopic distance *a* is assumed to be finite, the set of all F_i is denumerable. Note, also, that all F_i are, by definition, compact sets.) Without loss of generality, below we may consider only those sets F_i which do not intersect with each other, i.e., $F_i\cap F_j=\varnothing$. All these F_i are, therefore, disconnected. Since the Euclidean space E^n is connected [27], we finally conclude that the set $E^n\setminus\cup_i F_i$ is also connected. Moreover, this set is infinite in size, because all F_i are compact. In other words, we must infer that $E^n\setminus\cup_i F_i$ percolates.

We again now make use of the fact that the function $\psi(\mathbf{x})$ varies continuously at length scales shorter than the characteristic microscopic distance *a*. This leads to the conclusion that the function $\psi(\mathbf{x})$ preserves its sign over the entire set $E^n\setminus\cup_i F_i$. Assume that $\psi(\mathbf{x})$ is positive there, i.e., $\psi(\mathbf{x})>0$ for $\mathbf{x}\in E^n\setminus\cup_i F_i$. But the topology of the set $E^n\setminus\cup_i F_i$, where $\psi(\mathbf{x})>0$, principally differs from the topology of the sets where $\psi(\mathbf{x})<0$. In fact, whereas $E^n\setminus\cup_i F_i$ is infinite and connected, the sets where $\psi(\mathbf{x})$ is negative are localized inside F_i and, therefore, are finite in size and disconnected. This topological difference, however, is in contradiction with the assumed statistical equivalence of the functions $\psi(\mathbf{x})$ and $-\psi(\mathbf{x})$. Consequently, the zero set of $\psi(\mathbf{x})$ percolates, Q.E.D.

We stress, however, that the percolation nature of the zero set, $\psi(\mathbf{x})=0$, of the random field $\psi(\mathbf{x})$ does not necessarily imply that this set is a *fractal*. Indeed, our previous consideration was based on the most general topological concepts such as continuity and connectedness, and did not take into account some more specific, structural characteristics of the function $\psi(\mathbf{x})$. Let us now assume that the random field $\psi(\mathbf{x})$ exhibits additional properties of *isotropy* and *statistical homogeneity*. Then, for such a field $\psi(\mathbf{x})$, there *always* exists a *critical* percolation threshold h_c, in whose small vicinity $h\to h_c$ the geometry of the ''cut'' $\psi(\mathbf{x})=h$ is a *self-similar fractal* for length scales χ ranging between the microscopic distance *a* and the percolation correlation length $\xi\gg a$ (see Refs. [1,2,4,5]). The case $h_c=0$ is realized, e.g., for a *two-dimensional* sign-symmetric random field $\psi(\mathbf{x})$ [2], so that the zero set, $\psi(\mathbf{x})=0$, of the random, sign-symmetric, isotropic, statistically homogeneous function $\psi(\mathbf{x})$ on a plane ($\mathbf{x}\in E^2$) has the geometry of the percolating fractal object in the range of scales $a\leqslant\chi\leqslant\xi$.

III. TOPOLOGY OF PERCOLATION IN TWO DIMENSIONS

We now discuss in some more detail the topology of percolation in random scalar fields $\psi(\mathbf{x})$ in two dimensions, i.e., in a more particular case when $\mathbf{x}\in E^2$. We assume that (i) the random function $\psi(\mathbf{x})$ has the property of the sign symmetry, so that the fields $\psi(\mathbf{x})$ and $-\psi(\mathbf{x})$ are statistically identical; (ii) the random field $\psi(\mathbf{x})$ is isotropic and statistically homogeneous [this implies the existence of a *critical* percolation threshold h_c for which the ''cut'' $\psi(\mathbf{x})=h$, $h\to h_c$ exhibits a *self-similar fractal geometry* for $a\leqslant\chi\leqslant\xi$]; and (iii) $|\nabla\psi(\mathbf{x})|\neq 0$ everywhere except, perhaps, domains of negligible measure.

First of all, we note that the critical percolation threshold h_c is *unique* in two dimensions. From the statistical equivalence of the functions $\psi(\mathbf{x})$ and $-\psi(\mathbf{x})$, one then immediately concludes that this unique (critical) threshold is zero, i.e., $h_c=0$ for $n=2$. [Conversely, in higher dimensions $n\geqslant 3$, there could exist more than one critical percolation threshold h_c. This, in turn, leads to the possibility of the *simultaneous* percolation through both ''empty'' and ''filled'' regions, since the topology of the space E^n may admit, for $n\geqslant 3$, the nonintersecting, statistically isotropic paths to infinity [2]. For instance, for the three-dimensional embedding Euclidean space E^3, one could introduce two critical percolation thresholds, h_{c1} and h_{c2}, say, so that the simultaneous percolation through the three-dimensional ''empty'' and ''filled'' regions takes place for all values of *h* lying between h_{c1} and h_{c2}. For the sign-symmetric distribution of $\psi(\mathbf{x})$ in three dimensions, one then has $h_{c1}=-h_{c2}\neq 0$ [28].]

In two dimensions, simultaneous percolation ceases to exist as soon as the *critical* percolation threshold $h_c=0$ is the only one. Consequently, all the isolevels of the sign-symmetric function $\psi(\mathbf{x})$ corresponding to some *nonzero* threshold $h\neq h_c=0$ are bounded closed plane curves (the so-called plane ''loops''). [The introduction of the term ''loop'' assumes that $|\nabla\psi(\mathbf{x})|\neq 0$ almost for all $\mathbf{x}$.] Each of these loops is defined by the equation $\psi(\mathbf{x})=h$ where the parameter $h\neq h_c$.

A. Fractal geometry of single loops at $h\to h_c$

It is clear that the equation $\psi(\mathbf{x})=h, h\neq h_c=0$, could define, in general, a number of the *mutually nonintersecting* loops (to be referred to as *single* loops hereafter), each being a *connected* subset of the set of points $\psi(\mathbf{x})=h$, $\mathbf{x}\in E^2$. Denote the *single* loop (i.e., the *single* isolevel) of the field $\psi(\mathbf{x})$ at the threshold $h\neq h_c$ by Φ_h. Then the set of all the single loops Φ_h represents, for given $h\neq h_c$, the entire (disconnected) set of roots of the equation $\psi(\mathbf{x})=h$.

From the topological point of view, each single loop Φ_h is *homeomorphic* to the standard topological circle S^1 [22]. We formalize this result by writing $\Phi_h\sim S^1$. ''Homeomorphic'' means that the loop Φ_h can be *continuously deformed* into the circle S^1, i.e., there exists a *one-to-one mutually continuous* mapping of the points of Φ_h on the points of S^1. (Such a mapping is usually termed ''homeomorphism'' [22]. A homeomorphism between two given topological objects might be treated geometrically as a deformation of one of these objects into another, which is performed both without ''gaps'' and ''gluings.'') We also note that the above homeomorphism, $\Phi_h\sim S^1$, might be violated on a set of points where $|\nabla\psi(\mathbf{x})|=0$; we assume, however, that the measure of these points is negligible. For more details, Refs. [22,29] might be quoted.

The next step is to make use of the properties of *isotropy* and *statistical homogeneity* of the field $\psi(\mathbf{x})$. As already mentioned above, these properties appear in the statistically self-similar, fractal geometry of the ''cut'' $\psi(\mathbf{x})=h$ for the values of *h* sufficiently close to the critical percolation threshold h_c, i.e., $h\to h_c$. (Here the fractal geometry approximation implies the range of scales $a\leqslant\chi\leqslant\xi$. Note, also that the condition $h\to h_c$ is equivalent to $h\to 0$ for the percolation in two dimensions.) Assuming the condition *h*

$\to h_c$, one could now apply the concept of the *fractal geometry* to the description of the isolevels, $\psi(\mathbf{x})=h$, of the random, isotropic, statistically homogeneous field $\psi(\mathbf{x})$ in the range of scales between a and ξ. In fact, one could consider *single* isolevels (i.e., the *single* loops Φ_h for the percolation on the Euclidean plane E^2) of the field $\psi(\mathbf{x})$ as self-similar fractal objects having some Hausdorff fractal dimension d_h and some index of connectivity θ_h. (Note that the formal introduction of the parameters d_h and θ_h does not require the condition $n=2$, and might be also done for percolation in higher embedding dimensions $n\geqslant 3$.)

On the other hand, the condition $h\to h_c$ would mean that the topology of the isolevels, $\psi(\mathbf{x})=h$, of the field $\psi(\mathbf{x})$ approaches the *critical* topology at $h=h_c$; for this critical topology, the Hausdorff fractal dimension d_h and the index of connectivity θ_h can be related to each other through the percolation constant $\mathcal{C}$ [see Eq. (1)]. We have, consequently,

$$2d_h/(2+\theta_h)\to\mathcal{C}\approx 1.327, \quad h\to h_c. \tag{2}$$

We now argue that the index θ_h is equal to zero in the case of the two-dimensional random field $\psi(\mathbf{x})$, i.e.,

$$\theta_h=0, \quad h\to h_c=0. \tag{3}$$

To prove relation (3), we need the following auxiliary assertion: *the index of connectivity of a fractal set is a topological invariant of this set*. A topological invariant is a quantity which remains unchanged under the homeomorphic deformations [22,29]. Thus our assertion is equivalent to the following one: the indexes of connectivity of the homeomorphic fractal sets must coincide.

Indeed, the index of connectivity of a fractal set quantifies, by definition, how different points of the set are ‘‘glued’’ (connected) to each other in space. Because two homeomorphic sets can be deformed one into another without any ‘‘gaps’’ and ‘‘gluings’’ [22,29], the homeomorphic deformation cannot violate the ‘‘rule’’ of the ‘‘gluing’’ of the points of the set into the whole topological object. Hence, a homeomorphic deformation preserves the index of connectivity, Q.E.D.

Contrary to the index of connectivity of the fractal set, the Hausdorff dimension of the fractal cannot be treated as a topological invariant. An example might be the construction of the Koch curve from the unit interval $\overline{I}$ [5,13]. Such a construction provides a homeomorphism between the Koch curve and the interval $\overline{I}\equiv[0,1]$; however, the Koch curve is the fractal object of the Hausdorff dimension $\log 4/\log 3 \approx 1.26\cdots>1$ [5,13], whereas $\overline{I}$ is a segment of a smooth curve whose Hausdorff dimension is equal to one. In the meanwhile, one concludes that the index of connectivity of the Koch curve is equal to that of $\overline{I}$ (see below).

We now make use of the homeomorphism $\Phi_h\sim S^1$ between the single isolevels (i.e., single loops Φ_h) of the random sign-symmetric field $\psi(\mathbf{x})$ and the standard topological circle S^1. In view of the above, the homeomorphism $\Phi_h \sim S^1$ means that the index of connectivity of Φ_h (i.e., the parameter θ_h) is equal to the index of connectivity of S^1. (Meanwhile, the Hausdorff fractal dimension d_h of the set Φ_h differs, in general, from the Hausdorff dimension of the circle S^1, which is defined to be unity [5]).

One finally notes that the standard circle S^1 is homeomorphic to the closed unit interval, $\overline{I}\equiv[0,1]$, with the identified end points, 0 and 1 [22]. Let us now remove one point from the circle S^1. This is equivalent to splitting the two identified end points, so that one directly transforms the circle S^1 into the interval $\overline{I}$. Removing only one point from the circle S^1 cannot, however, violate the index of connectivity θ_h which describes, by definition, the self-similar, *fractal* geometry of the set $\Phi_h\sim S^1$. (The index of connectivity θ_h might be sensible to a ‘‘self-similar’’ removing of an infinite number of points from the circle S^1.) Hence, we conclude that the parameter θ_h is equal to that of the interval $\overline{I}$.

In a similar way, the index of connectivity θ_h remains unchanged if the two end points 0 and 1 are removed from the closed interval $\overline{I}\equiv[0,1]$. Hence, both the closed interval $\overline{I}\equiv[0,1]$ and the open interval $I\equiv(0,1)$ have exactly the same indexes θ_h. Note, further, that the open interval $I \equiv(0,1)$ is homeomorphic to the one-dimensional Euclidean space E^1, i.e., $I\sim E^1$. The homeomorphism $I\sim E^1$ shows that the indexes of connectivity of the sets I and E^1 coincide. Because the index of connectivity of any Euclidean space $E^n, n\geqslant 1$, is defined to be zero [6], one immediately obtains $\theta_h=0$, which proves Eq. (3).

From Eqs. (2) and (3), one finds

$$d_h\to\mathcal{C}\approx 1.327, \quad h\to h_c=0. \tag{4}$$

Expression (4) shows that the fractal dimension d_h of the single isolevels Φ_h of the two-dimensional sign-symmetric random field $\psi(\mathbf{x})$ is equal to the percolation constant $\mathcal{C} \approx 1.327$ for the values of the threshold h sufficiently close to the critical percolation threshold, $h_c=0$.

It is relevant to remark that each single isolevel Φ_h in two dimensions surrounds some plane area, Ω_h, being the *interior* of the (single) loop Φ_h. From the topological point of view, the loop Φ_h might be treated as the *outer boundary* of the set Ω_h. Since the loop Φ_h is homeomorphic to the standard topological circle, S^1, i.e., $\Phi_h\sim S^1$, the set of points Ω_h is homeomorphic to the interior of the circle S^1, which is defined [22] as the standard two-dimensional open disk D^2, i.e., $\Omega_h\sim D^2$ [22]. (For the sake of simplicity, here we ignore the possible appearance of the isolated ‘‘voids’’ inside Ω_h, whose presence may violate the homeomorphism $\Omega_h\sim D^2$.)

The arguments of Ref. [30] suggest that the set of points Ω_h, which is surrounded by the single *fractal* loop Φ_h on a plane, could be considered as a *fractal* object having its own Hausdorff fractal dimension $1\leqslant D_h\leqslant 2$. It is intuitively clear that the larger the fractal dimension d_h of the *outer boundary* of the set Ω_h, the smaller the fractal dimension D_h of the surrounded area, Ω_h, would be. This might be more rigorously quantified by saying that the sum D_h+d_h of the fractal dimensions D_h and d_h is a *topological invariant*, i.e., the quantity D_h+d_h remains unchanged under the homeomorphic deformations of the entire set $\Omega_h\cup\Phi_h$. Because, on the other hand, the set Ω_h is homeomorphic to the standard two-dimensional disk D^2, i.e., $\Omega_h\sim D^2$, and the outer boundary, Φ_h, of the set Ω_h is homeomorphic to the standard one-dimensional circle S^1, i.e., $\Phi_h\sim S^1$, one immediately obtains the homeomorphism $\Omega_h\cup\Phi_h\sim D^2\cup S^1$. Consequently, the sum D_h+d_h of the Hausdorff fractal dimensions of the sets

Ω_h and Φ_h must be equal to the sum of the (Euclidean) dimensionalities of the sets D^2 and S^1, which is easily seen to be $2+1=3$. Hence $D_h+d_h=3$. This expression for the parameters D_h and d_h was earlier proposed in Ref. [30], where the relationship between the fractal properties of the sets Ω_h and Φ_h was discussed on a descriptive level.

Consequently, the fractal dimension D_h could be expressed as $D_h=3-d_h$. Making use of Eq. (4), one then obtains

$$D_h \to 3-\mathcal{C} \approx 1.673, \quad h \to h_c = 0. \tag{5}$$

The value of $D_h \to 3-\mathcal{C}$ yields the Hausdorff fractal dimension of the *interior* of a plane (single) fractal loop Φ_h for the values of the threshold h sufficiently close to the critical percolation threshold h_c, i.e., $h \to h_c = 0$. An estimate of D_h, which is numerically close to the value of $3-\mathcal{C}$, was also obtained in Refs. [12,24], where some fractal properties of the percolating networks were discussed.

Meanwhile, the index of connectivity of the set Ω_h is easily seen to be zero. This follows immediately from the topological invariance of the index of connectivity and the homeomorphism $\Omega_h \sim D^2$. Indeed, taking into account [22] that the open disk D^2 is homeomorphic to the Euclidean space E^2, i.e., $D^2 \sim E^2$ [22], one obtains $\Omega_h \sim E^2$. Hence the index of connectivity of the set Ω_h is the Euclidean one. We mention, however, that the homeomorphism $\Omega_h \sim E^2$ is violated if the isolated fractal ''voids'' appear inside Ω_h; in this case, the index of connectivity of the set Ω_h would be positive.

It is theoretically important to note that the fractal dimension d_h might be subject to change between the above value of $d_h \to \mathcal{C} \approx 1.327$ and the alternative value of $d_h = 7/4$, depending on a moderate change in the definition of the notion of the single isolevel [2,31]. Roughly speaking, the alternative value of $d_h = 7/4$ describes the single isolevels of the random field on a plane, Φ_h, whose ''gulfs'' might have arbitrarily small widths (see, e.g., Ref. [2]); this formally corresponds to the case when the characteristic ''microscopic'' length scale $a \to 0$. Conversely, the consideration presented above clearly assumes that the parameter a cannot be arbitrary small, so that the homeomorphism $\Phi_h \sim S^1$ could be established. This actually leads to some ''smoothing'' of the isolevels when one implicitly dams all the ''gulfs'' whose widths might be less than $\sim a$. In Ref. [31], the fractal dimension of such isolevels was found to be $d_h \approx 4/3$, which is practically very close to $\mathcal{C} \approx 1.327$.

We also remark that the alternative value of $d_h = 7/4$ has been recognized in modeling the uncorrelated percolation on the plane *lattices*, when the geometry of the percolating sets is *discrete* and the issue of *connectedness* is more delicate (see, e.g., Refs. [2,4]). In view of relation (2), the value of $d_h = 7/4$ might be associated with the nonzero value of the index of connectivity, $\theta_h \approx (7-4\mathcal{C})/2\mathcal{C} \approx 0.64 > 0$. The fact that $\theta_h > 0$ would then mean that the corresponding single isolevels of the field $\psi(\mathbf{x})$ are not homeomorphic to the standard topological circle S^1; rather, these isolevels would contain an infinite number of branching points whose presence violates the above homeomorphism $\Phi_h \sim S^1$. The importance of the branching points for the connectivity properties of the percolating networks at criticality was analyzed in more detail in Ref. [4].

B. Fractal geometry of percolating subsets at $h = h_c$

We now determine the fractal characteristics of the *percolating subsets* of the critical ''cut,'' $\psi(\mathbf{x}) = h_c$, of the two-dimensional ($\mathbf{x} \in E^2$) sign-symmetric random field $\psi(\mathbf{x})$. For two-dimensional fields, there exists only one critical cut at $h_c = 0$; hence the critical cut $\psi(\mathbf{x}) = h_c$ coincides with the zero set $\psi(\mathbf{x}) = 0$ of the field $\psi(\mathbf{x})$ when $\mathbf{x} \in E^2$. We are reminded that the entire zero set $\psi(\mathbf{x}) = 0$ of the random field $\psi(\mathbf{x})$ is *disconnected*; the existence of a (connected) *percolating subset* of the zero set $\psi(\mathbf{x}) = 0$ was formally proven in Sec. II.

Under the condition $|\nabla \psi(\mathbf{x})| \neq 0$, the entire zero set $\psi(\mathbf{x}) = 0$ of the field $\psi(\mathbf{x})$ can be treated as a weblike structure ''composed'' of all the single isolevels Φ_h which correspond to a given threshold $h \to 0$. According to the results of Sec. II, this structure contains a (connected) percolating fractal subset.

Denote d_* and θ_* as the Hausdorff fractal dimension and the index of connectivity of the percolating fractal subset of the zero set $\psi(\mathbf{x}) = 0$. The quantities d_* and θ_* should be distinguished from the parameters d_h and θ_h describing the fractal geometry of the single isolevels Φ_h of the field $\psi(\mathbf{x})$ at $h \to 0$. A calculation of the parameters d_* and θ_* is given in what follows.

The index of connectivity θ_* could be obtained in terms of the Hausdorff fractal dimension $d_h, h \to 0$. The relationship between the parameters θ_* and d_h is a manifestation of the fact that the ''shape'' of the connected subsets of the zero set, $\psi(\mathbf{x}) = 0$, of the random field $\psi(\mathbf{x})$ in two dimensions is determined by the fractal properties of the single isolevels of the field, $\psi(\mathbf{x}) = h$, for $h \to 0$.

An expression for quantity θ_* could be easily obtained from a comparison of the two identical representations of the critical diffusion coefficient in a two-dimensional random field $\psi(\mathbf{x})$, one in terms of the fractal dimension d_h [see Eqs. (2.13) and (4.144) in Ref. [2]], and the other in terms of the index of connectivity θ_* [see Eq. (2) in Ref. [15]], yielding

$$\theta_* = 2(d_h - 1)/d_h, \quad h = h_c = 0. \tag{6}$$

For a regular (nonrandom) sign-symmetric function $\psi(\mathbf{x})$, all the isolevels $\psi(\mathbf{x}) = h$ would be smooth rectifiable curves ($d_h = 1$), so that Eq. (6) yields, evidently, $\theta_* = 0$. Another important particular case is the uncorrelated percolation on lattices for which the result $d_h = 7/4$ could be obtained [2,31]. Then the corresponding value of the index of connectivity θ_* would be $\theta_* = 6/7 \approx 0.86$, in good agreement with Refs. [4,6].

For the percolation problem in a two-dimensional sign-symmetric random field $\psi(\mathbf{x})$, the fractal dimension d_h has been expressed in terms of the percolation constant $\mathcal{C}$, i.e., $d_h \approx \mathcal{C} \approx 1.327$ [see Eq. (4)]. Hence the index of connectivity θ_* becomes

$$\theta_* \approx 2(\mathcal{C}-1)/\mathcal{C} \approx 0.49, \quad h = h_c = 0, \tag{7}$$

where Eq. (6) has been used. This estimate coincides with the result obtained in Ref. [32] from a topological model dealing with the issue of the "dual" fractal topologies (see, also, Refs. [12,24]).

The value of the Hausdorff fractal dimension d_*, corresponding to the index of connectivity [Eq. (7)], can be now estimated from the basic equation (1). This equation describes the fractal geometry of the percolating sets at criticality; in particular, it holds for the percolating fractal subsets of the zero set $\psi(\mathbf{x})=0$ of the two-dimensional sign-symmetric random field $\psi(\mathbf{x})$.

Replacing D for d_*, and θ for θ_* in Eq. (1), and making use of Eq. (7), one finds

$$d_* \approx 2\mathcal{C}-1 \approx 1.654, \quad h=h_c=0. \tag{8}$$

Equation (8) yields the Hausdorff fractal dimension of a percolating fractal subset of the zero set, $\psi(\mathbf{x})=h_c=0$, of the field $\psi(\mathbf{x})$. It is interesting to note that this dimension is numerically close, although not exactly equal, to the Hausdorff fractal dimension $D_h \approx 3-\mathcal{C} \approx 1.673$ of the interior of a plane single fractal loop $\Phi_h, h \to h_c=0$ [see Eq. (5)]. The numerical difference between the fractal dimensions d_* and D_h is actually due to the fact that the percolation constant $\mathcal{C} \approx 1.327$ slightly deviates from the AO value, 4/3, in two dimensions (see the discussion in Sec. I).

The basic result $d_h \to \mathcal{C} \approx 1.327$ [see Eq. (4)] might give us the key to find numerically an estimate for the percolation constant $\mathcal{C}$. Indeed, the value of $\mathcal{C}$ could be principally obtained as the Hausdorff fractal dimension d_h of the single isolevels Φ_h of the two-dimensional sign-symmetric random field $\psi(\mathbf{x})$ for the values of h sufficiently close to the critical percolation threshold, $h_c=0$.

In Sec. IV, we evaluate the percolation constant $\mathcal{C}$ from the particular numerical realization of the random field $\psi(\mathbf{x})$ on a plane, commonly referred to as *fractional Brownian surface*. Characteristic properties of these surfaces are discussed in Ref. [2].

IV. RANDOM FIELDS IN TWO DIMENSIONS: NUMERICAL MODELING

We consider the standard numerical representation of the fractional Brownian surface $\psi(\mathbf{x})$ given by (see, e.g., Refs. [2,33,34])

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}} \psi(\mathbf{k}) \exp\{i(\mathbf{k}\cdot\mathbf{x}+\phi_{\mathbf{k}})\}, \tag{9}$$

where $\psi(\mathbf{k})$ is the Fourier amplitude of the mode with wave vector $\mathbf{k}$, and $\phi_{\mathbf{k}}$ are random phases chosen to simulate the random spatial structure in the field $\psi(\mathbf{x})$. Note that because the phases $\phi_{\mathbf{k}}$ are random, the functions $\psi(\mathbf{x})$ and $-\psi(\mathbf{x})$ are automatically statistically equivalent. (The term "automatically," however, might be oversimplified, as soon as it naively ignores the important issue of the long-range correlations customarily present in the systems with the self-affine fractal geometry. In the framework of the present study, nevertheless we leave this issue mostly beyond the scope of our consideration; for a detailed discussion, see Ref. [35].)

The basic periodicity of the Fourier expansion in Eq. (9) is that of the square simulation box of side L, so that $\mathbf{k} \equiv (k_1,k_2) = 2\pi(n_1,n_2)/L$, with integer valued n_1 and n_2. (In the numerical model, the size of the simulation box L plays the role of the percolation correlation length ξ.) The Fourier amplitudes for an isotropic power-law spectrum are assumed in the form

$$\psi(\mathbf{k}) = \frac{2\pi C/L}{(k^2L^2+1)^{(\alpha+1)/4}}, \tag{10}$$

where C is the normalization constant, and $k=|\mathbf{k}| \sim \chi^{-1}$ and lies between $8\pi/L$ and $2\pi/a$ (a long wavelength cutoff at $L/4$ is introduced). The spectrum in Eq. (10) is truncated at $k_{max}=2\pi N/L$, where $N=L/a$, with $a>0$ the smallest wavelength present. (The introduction of the finite $k_{max}=2\pi/a < \infty$ explicitly takes into account that the value of the length scale a has been assumed to be substantially nonzero.) In the numerical model, we used values of the ratio L/a up to 80. With such a choice, the number of independent Fourier modes in Eq. (9) is 10 024.

The parameter α in Eq. (10) is the spectral index of the power-law energy density spectrum, $P(k) \sim k^{-\alpha}$. Indeed, from the definition of the wave vector $\mathbf{k}$ it is clear that $k^2L^2 \geqslant (8\pi)^2 \gg 1$, hence the Fourier amplitude (10) behaves with k as the power law $k^{-(\alpha+1)/2}$, and the square of the amplitude, $\psi^2(\mathbf{k})$, behaves as $k^{-(\alpha+1)}$. Consequently, the energy density spectrum scales as $P(k) \sim 2\pi k \psi^2(\mathbf{k}) \sim k^{-\alpha}$. Note, also, that the inequality $kL \gg 1$ is actually equivalent (through the definition of the numerical parameter L) to $\chi \ll \xi$, the necessary condition for the fractal approach to be valid.

The spectral index α could be related to the Hausdorff fractal dimension δ of the horizontal cross section (horizontal "cut") of the fractional Brownian surface (9). This relation is given by the well-known Berry formula [36]:

$$\alpha = 5-2\delta, \quad 1 \leqslant \delta \leqslant 2. \tag{11}$$

The fractal dimension δ appears in the variance $\langle[\psi(\mathbf{x}) - \psi(\mathbf{x}+\mathbf{x}_0)]^2\rangle \sim |\mathbf{x}_0|^{2(2-\delta)}$, where $\mathbf{x}_0$ is a constant plane vector obeying $a \ll |\mathbf{x}_0| \ll \xi$ (see, e.g., Ref. [2]). This dimension must be distinguished from the parameters d_h and D_h which describe the fractal properties of the single isolevels, Φ_h, $h \to h_c$, of the sign-symmetric random field $\psi(\mathbf{x})$, as well as from the quantity d_* yielding the Hausdorff fractal dimension of a (connected) percolating subset of the critical cut, $\psi(\mathbf{x})=h_c=0$, of the field $\psi(\mathbf{x})$. Conversely, the fractal dimension δ is the global characteristic of the field $\psi(\mathbf{x})$: This quantity defines the Hausdorff fractal dimension of the entire, *disconnected* set of points $\psi(\mathbf{x})=h$, rather than the dimensionalities of the connected subsets of the set $\psi(\mathbf{x})=h$ (also see Sec. II). In other words, the parameter δ yields the Hausdorff fractal dimension of the (disconnected) set composed of all the isolevels of the field $\psi(\mathbf{x})$ at a given threshold h. This issue is discussed in more detail in Refs. [37,38].

The zero horizontal cut $\psi(\mathbf{x})=0$ of the fractional Brownian surface [Eqs. (9) and (10)] for $\alpha=3/2$ is illustrated in Fig. 1. It is clear that the Hausdorff fractal dimension δ of the entire set $\psi(\mathbf{x})=h_c=0$ cannot be smaller than the Hausdorff dimension d_* of its percolating subset, i.e., $\delta \geqslant d_*$.

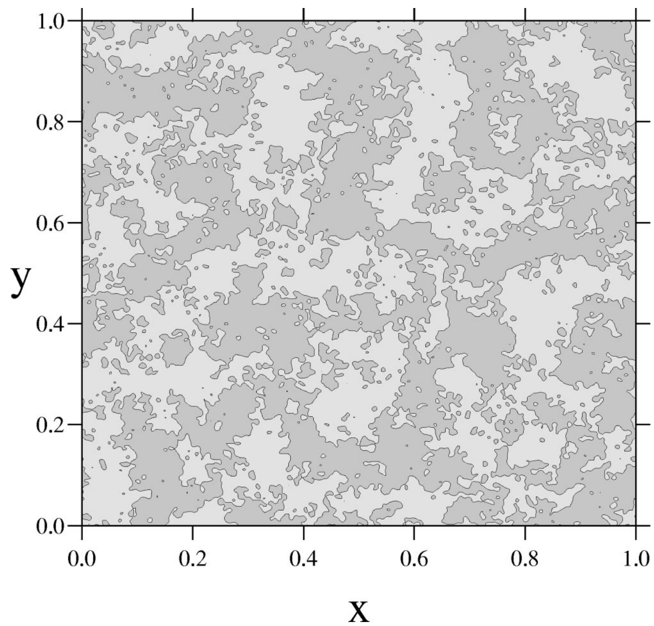

FIG. 1. The zero horizontal ''cut'' of the fractional Brownian surface $\psi(\mathbf{x})$ for $\alpha=3/2$. Dark gray shading corresponds to positive values of $\psi(\mathbf{x})$, and light gray to negative values. One also observes isolated ''spots'' (e.g., dark spots in the light area), where the sign of $\psi(\mathbf{x})$ is reversed. These ''spots'' could be considered as topological ''voids'' whose presence slightly violates the contractibility property. Dimensionless units.

From Eqs. (8) and (11), one concludes that $1\leqslant\alpha\leqslant 7-4\mathcal{C}\approx 1.69$. Hence general topological findings of Secs. II and III could be applied to the fractional Brownian surfaces [Eqs. (9) and (10)] only if their spectral indices α do not exceed the maximum value of $7-4\mathcal{C}$. Physically, this means that the field $\psi(\mathbf{x})$ contains enough energy at large k (i.e., small wavelengths $\sim k^{-1}$) associated with the considerable fine-scale structuring in the cross section $\psi(\mathbf{x})=h_c$. For fractional Brownian surfaces [Eqs. (9) and (10)] having $\alpha\geqslant 7-4\mathcal{C}$, a topological ''smoothing'' of the cut $\psi(\mathbf{x})=h_c$ might be the case as only little energy goes to the small scales.

Since the entire cut $\psi(\mathbf{x})=h$ is *disconnected*, the fractal dimension δ is *independent* of the parameters d_h, D_h, and d_* describing the fractal properties of the *connected* subsets of the set $\psi(\mathbf{x})=h$. From Eq. (11) one then concludes that these parameters are insensitive to the particular value of the spectral index α assumed in the Fourier amplitudes (10). For instance, the result $d_h\approx\mathcal{C}$ [see Eq. (4)] cannot depend on the particular numerical realization of the fractional Brownian surface [Eqs. (9) and (10)], i.e., the fields $\psi(\mathbf{x})$ generated for different values of the spectral index α, must have the same value of the fractal dimension $d_h\approx\mathcal{C}\approx 1.327$. (This might be treated as one of the universal features of the fractal geometry of percolation *at criticality* [4].) Our goal now is to analyze numerically the result $d_h\approx\mathcal{C}\approx 1.327$ for the fractional Brownian surfaces [Eqs. (9) and (10)], assuming different values of the spectral index α $(1\leqslant\alpha\leqslant 7-4\mathcal{C})$. For each realization, we first of all check that the fractal dimension δ of the horizontal ''cut'' of the fractional Brownian surface [Eqs. (9) and (10)] is in accord with the Berry relation (11).

Then, we take into account that the field equations (9) and (10) satisfy the topological conditions (i)–(iii) specified in Sec. III for the random function $\psi(\mathbf{x}),\mathbf{x}\in E^2$. In particular, this enables one to approximate the single isolevels of the field equations (9) and (10) by plane fractal loops, Φ_h.

To estimate the fractal dimension d_h of the loops Φ_h numerically, we exploit the following topological property [29], namely, that each loop Φ_h divides the Euclidean plane E^2 into two distinct parts; one is the *interior* of the loop and is finite in size, the other is its *exterior* and is infinite. (We have already used this property in Sec. III where the fractal dimension, $D_h=3-d_h$, of the internal part, Ω_h, of the fractal loop Φ_h was obtained [see Eq. (5)]).

Let P_h denote the perimeter of the loop Φ_h, and A_h be the area of the internal part, Ω_h, surrounded by Φ_h. A remarkable feature of the fractal loops Φ_h is that the quantities P_h and A_h are related to each other by the scaling law, $P_h\sim A_h^{d_h/2}$ [39]; this law is known as the *area-perimeter relation*. (For the smooth plane loops, one has $d_h=1$, so that the standard relationship $P_h\sim\sqrt{A_h}$ is recovered.) The area-perimeter relation shows that the fractal dimension d_h can be obtained as the slope of the plot $\log P_h^2$ versus $\log A_h$, computed for a series of the loops, $\Phi_h, h\to h_c$, of the random field $\psi(\mathbf{x})$. (We also note that a similar approach based on the application of the area-perimeter relation was recently proposed to studying the coarse-grained texture of the ion distribution functions in Earth's magnetotail [40].)

Given the level $h\neq 0$ of the field $\psi(\mathbf{x})$, we find all the corresponding loops Φ_h inside the simulation box. Let these loops be enumerated by the index j. Then the integration along each loop [which could be performed easily by means of the explicit representation of the field $\psi(\mathbf{x})$ according to Eqs. (9) and (10)] yields their perimeters $P_{h,j}=\oint_{h,j}\sqrt{dx_1^2+dx_2^2}$ where x_1 and x_2 are the components of the plane vector $\mathbf{x}$.

To increase the accuracy of the computation, we introduce the quantity $\mathcal{P}_h=\Sigma_j P_{h,j}$, which is the *total* perimeter of all the loops found for the given value of h [40]. In a similar way, the areas $A_{h,j}$ enclosed by the loops are obtained from Green's formula to give $A_{h,j}=0.5\oint_{h,j}(x_1dx_2-x_2dx_1)$, and the *total* area of all the loops becomes $\mathcal{A}_h=\Sigma_j A_{h,j}$.

Following Ref. [40], we apply the above *area-perimeter relation* to the quantities $\mathcal{P}_h$ and $\mathcal{A}_h$, yielding $\mathcal{P}_h\sim\mathcal{A}_h^{d_h/2}$. [Note that we take into account the properties of *isotropy* and *statistical homogeneity* of the function $\psi(\mathbf{x})$.] It is easy to see that the relationship $\mathcal{P}_h\sim\mathcal{A}_h^{d_h/2}$ implies the condition

$$\mathcal{P}_h^{2/d_h}\sim\sum_j P_{h,j}^{2/d_h}. \tag{12}$$

Equation (12) indicates that the algebraic summation of the areas contained by the *fractal* loops Φ_h cannot be reduced to the (intuitively obvious) summation of *squares* of the perimeters of the loops; rather, these *squares* must be replaced for $2/d_h$, where the power exponent $2/d_h$ is reduced to 2 only in the case of the smooth, non-fractal geometry $(d_h=1)$. Some generalized relations of the form of Eq. (12) are discussed in more detail in, e.g., Refs. [15,41,42].

We evaluated $\mathcal{P}_h$ and $\mathcal{A}_h$ for a statistically reliable number of ''cuts'' $\psi(\mathbf{x})=h$, where h tends toward the critical

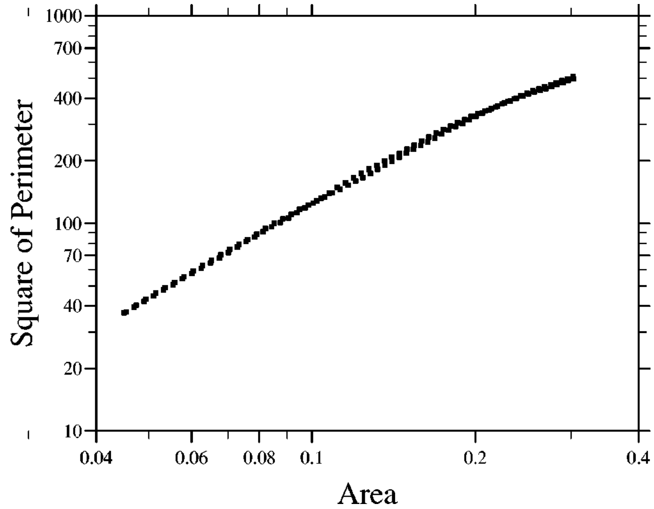

FIG. 2. Log-log plot of $\mathcal{P}_h^2$ vs $\mathcal{A}_h$ (solid squares) for the fractional Brownian surface of Fig. 1. Two sets of points, corresponding either to negative or positive isolevels h, are shown. Dimensionless units.

percolation threshold $h_c=0$, and we plotted $\log \mathcal{P}_h^2$ against $\log \mathcal{A}_h$ to find the slope $d_h \approx 1.32 \pm 0.01$ (see Fig. 2).

According to Eq. (4), the quantity d_h near the critical threshold coincides with the percolation constant, $\mathcal{C}$; hence the slope $d_h \approx 1.32 \pm 0.01$ could be considered as an estimate of the parameter $\mathcal{C}$ from the ''cuts'' of the fractional Brownian surfaces (9).

The result $d_h \approx 1.32 \pm 0.01$ is numerically close to, although somewhat *smaller*, on the average, than the topological value $\mathcal{C} \approx 1.327$ [see Eq. (1)], and is *independent* (within the numerical errors) of the assumed values of the spectral index α ($1 \leqslant \alpha \leqslant 1.69$). The estimate $d_h \approx 1.32 \pm 0.01$ is consistent with the result ≈ 1.321 reported by Normand *et al.*, [23] who used a high-statistics determination of the spectral fractal dimension $\tilde{d}$ at the threshold of percolation.

The fact that the numerical methods applied yield, in both cases, slightly smaller estimates of the percolation constant $\mathcal{C}$ (compared with the topological result $\mathcal{C} \approx 1.327$), leads one to conclude that the percolating fractal sets near the threshold are not exactly *contractible* (as is assumed by the value ≈ 1.327), and that the isolated fractal ''voids'' might play a role (see Ref. [10] and the discussion in Sec. I). [This effect might be recognized from Fig. 1 as evidence of isolated ''spots'' (i.e., topological sign ''holes''), where the sign of $\psi(\mathbf{x})$ is reversed.]

Indeed, it is intuitively clear that the inclusion of the topological ''voids'' would act toward a more efficient percolation, since the convergence of the percolating set to infinity would be ''quicker'' in this case. Hence the percolation threshold could be achieved for a *smaller* value of the spectral fractal dimension $\tilde{d} = 2D/(2+\theta)$ compared with the ''basic'' value of ≈ 1.327 [see Eq. (1)]. (The ''voids'' present actually contribute into the effective index of connectivity θ of the percolating structure. The argument is that the geodesic lines [whose Hausdorff fractal dimension is equal to $(2+\theta)/2$] become ''longer'' as the ''voids'' must be bypassed on all scales. This results in a slight *increase* of the index of connectivity θ and the ensuing *decrease* of the spectral dimension $\tilde{d} \equiv 2D/(2+\theta)$.) In this context, it is worth emphasizing that the topological result $\tilde{d} = \mathcal{C} \approx 1.327$ holds as the limiting case realized for the fractal objects that are *exactly contractible*; the *noncontractibility* effects would always tend to *diminish* the actual value of the parameter $\tilde{d}$ [10].

A comprehensive investigation of the noncontractible fractal sets that include the isolated voids might be a topological problem of outstanding significance. For instance, this might be associated with the problem of the topological classification of the fractal objects from the viewpoint of algebraic ''codes'' when each code identifies the topological type of the fractal set through some classification algorithm. For more details, see Ref. [10], and references therein.

Apart from the noncontractibility effects, the index of connectivity θ might be influenced by the intrinsic correlations (or anticorrelations) operating in the system due to some particular properties assumed. An important example might be self-avoiding random walks (SAW's), characterized by the intrinsic short-range repulsive interaction between different steps of the walk according to their relative orientation [43]. The critical phenomena associated with the SAW's reveal intriguing features; among them is the feasible universality of the critical exponent advocated in Ref. [44]. An extensive Monte Carlo simulation recently performed by Caracciolo *et al.* [43] on the two-dimensional Manhattan lattice shows that the universal properties of the SAW's could be described by the critical exponent 1.3425 ± 0.0003, in agreement with the (theoretical) result 43/32 for the regular lattices. These values are reminiscent of, although a bit *larger* than, the percolation constant $\mathcal{C} \approx 1.327$. The discrepancy between 1.3425 ± 0.0003 [43] and $\mathcal{C} \approx 1.327$ might be the consequence of the short-scale repulsion implied by the SAW's. In fact, such a repulsion would tend to diminish the ''degree of connectedness'' of the SAW structure as the close contacts between the different elements of the SAW trajectories are generally ''discouraged.'' This has a direct effect on the index of connectivity θ, which tends to *decrease*. Hence the corresponding value of the spectral fractal dimension $\tilde{d} \equiv 2D/(2+\theta)$ could be slightly *larger* than $\mathcal{C}$ ≈ 1.327, i.e., $\tilde{d} \geqslant 1.327$, for the SAW's. The effect of the repulsion is, therefore, generally opposite to that of the noncontractibility, and dominates in the case of the SAW's. A topological analysis of the SAW structures associated with the intrinsic interactions present might open perspectives on the critical phenomena research.

V. SUMMARY

In the framework of the present study, we discussed some topological properties of the sign-symmetric random fields $\psi(\mathbf{x}), \mathbf{x} \in E^n, n \geqslant 2$. [''Sign symmetric'' means that the field $\psi(\mathbf{x})$ is statistically equivalent to $-\psi(\mathbf{x})$.]

Applying the concepts of continuity and connectedness, we proved rigorously that the zero set, $\psi(\mathbf{x})=0$, of the field $\psi(\mathbf{x})$ always contains a (connected) percolating subset if the condition $|\nabla \psi(\mathbf{x})| \neq 0$ holds almost everywhere, i.e., except domains of negligible measure. We have shown that this percolating subset could be considered as a *fractal* object if the field $\psi(\mathbf{x})$ observes the additional properties of *isotropy* and *statistical homogeneity*. This fractal geometry approximation holds in the range of spatial scales between some small microscopic distance $a>0$ and the percolation correlation length $\xi\to\infty$.

Our particular attention concentrates on the universal features of the fractal geometry of percolation in the vicinity of the critical percolation threshold. Following Ref. [10], we quantify this universality in terms of Eq. (1); this equation establishes the relation between the two geometric characteristics of a percolating fractal set at criticality: the Hausdorff fractal dimension D and the index of connectivity, θ. Equation (1) is an improvement of the widely known Alexander-Orbach relation [3] for the embedding Euclidean dimensions $2\leqslant n\leqslant 5$; the constant $\mathcal{C}\approx 1.327$, on the right hand side of this equation is termed the *percolation constant*, and is slightly smaller than the original AO value, 4/3 [3].

We applied relation (1) to an analysis of the fractal geometry of percolation in two-dimensional sign-symmetric random fields $\psi(\mathbf{x})$, i.e., when $\mathbf{x}\in E^2$. Our main theoretical findings are the following.

(1) The single isolevels Φ_h of the field $\psi(\mathbf{x})$, i.e., the connected subsets of the set $\psi(\mathbf{x})=h, h\to 0, h\neq 0$, can be treated as fractal loops whose Hausdorff fractal dimension d_h and index of connectivity θ_h are, respectively, $d_h\to\mathcal{C}\approx 1.327$ and $\theta_h=0$. Meanwhile, the area Ω_h surrounded by the fractal loop Φ_h, is a plane fractal set having the Hausdorff fractal dimension $D_h=3-d_h\to 3-\mathcal{C}\approx 1.673$. The index of connectivity of the set Ω_h is equal to zero, provided that no isolated ''voids'' are present inside Ω_h.

(2) The percolating subsets of the zero set, $\psi(\mathbf{x})=0$, are (connected) plane fractal objects characterized by the Hausdorff fractal dimension $d_*\approx 2\mathcal{C}-1\approx 1.654$ and the index of connectivity $\theta_*\approx 2(\mathcal{C}-1)/\mathcal{C}\approx 0.49$.

The fractal dimensions D_h and d_* are numerically close (although not exactly equal) to each other. The small numerical difference between these two fractal dimensions is due to the fact that the percolation constant $\mathcal{C}\approx 1.327$ slightly deviates from the original AO value 4/3.

We found that the *homotopic topology* might be an effective instrument when analyzing the properties of fractal objects. This instrument has been applied in our study to obtain the basic results (1) and (2).

An interesting result formulated in Sec. III is that the index of connectivity of a fractal set is the *topological invariant* of this set. ''Topological invariant'' means that this quantity remains unchanged under the homeomorphic deformations of the set, i.e., under such deformations that are performed without ''gaps'' and ''gluings.'' One thus concludes that *the indexes of connectivity of the homeomorphic fractal sets are equal to each other*.

The result $d_h\approx\mathcal{C}, h\to h_c=0$—i.e., that the Hausdorff fractal dimension d_h of the single isolevels Φ_h of the two-dimensional sign-symmetric random field $\psi(\mathbf{x})$ tends toward the percolation constant $\mathcal{C}$ as the threshold h approaches the critical percolation threshold $h_c=0$—might be the principal way to estimate the value of $\mathcal{C}$ numerically. We performed such an evaluation in the particular case when the field $\psi(\mathbf{x})$ is given by the so-called *fractional Brownian surface*. Our basic idea was to apply the widely known *area-perimeter relation* to the fractal loops Φ_h. The fractal dimension d_h could be then obtained as the slope of the plot $\log P_h^2$ versus $\log A_h$, where P_h is the perimeter of the loop $\Phi_h, h\to 0, h\neq 0$, and A_h is the area of the surrounded domain, Ω_h. This leads to the numerical estimate $d_h\approx 1.32\pm 0.01$ which is reasonably close to, although somewhat smaller, on the average, than the analytical value of the percolation constant $\mathcal{C}\approx 1.327$, exact for the *contractible* percolating sets at criticality [10]. We speculate that the possible deviation of the numerical finding 1.32 ± 0.01 from ≈ 1.327 is due to the slight violation of the contractibility property near the percolation threshold.

We found that the parameters $d_h\to\mathcal{C}$, $\theta_h=0$, $D_h\to 3-\mathcal{C}$, $d_*\approx 2\mathcal{C}-1$, and $\theta_*\approx 2(\mathcal{C}-1)/\mathcal{C}$, which describe the fractal geometry of the *connected* subsets of the ''cut'' $\psi(\mathbf{x})=h, h\to h_c=0$, do not depend on the spectral index α of the fractional Brownian surface $\psi(\mathbf{x})$ [this index defines the spectral energy density of the field $\psi(\mathbf{x})$], nor on the Hausdorff fractal dimension δ of the entire, *disconnected* set of points $\psi(\mathbf{x})=h$. (Meanwhile, the quantities α and δ are related to each other through the Berry formula $\alpha=5-2\delta$ [36].) This might be considered as a possible manifestation of the universal behavior of fractal geometry of percolation at criticality.

Applications of our topological results to the fractional Brownian surfaces [Eqs. (9) and (10)] imply that the spectral index α does not exceed a critical value of $7-4\mathcal{C}\approx 1.69$, i.e., the condition $1\leqslant\alpha\leqslant 7-4\mathcal{C}$ holds. This condition says that the field $\psi(\mathbf{x})$ contains enough energy at smaller scales so that the considerable fine-scale structuring could be recognized in the cross section $\psi(\mathbf{x})=h$.

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