Controlling Hamiltonian chaos by adaptive integrable mode coupling

Ying Zhang,[1] Shigang Chen,[1] and Yugui Yao[2,*]
[1]*LCP, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009(26), Beijing 100088, China*
[2]*State Key Laboratory for Surface Physics, Institute of Physics & Center for Condensed Matter Physics, P.O. Box 603-40, Beijing 100080, China*

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The adaptive integrable mode coupling method is proposed to control two-dimensional Hamiltonian chaos. We demonstrate that this control method can stabilize chaotic motions into regular ones in a model of the standard map. Global stochasticity can be removed from the phase space by the control being switched on and off.

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Chaotic phenomena arise ubiquitously in natural systems and in manmade devices. The pioneering work of Ott, Grebogi, and Yorke (OGY) [1] sparked a great deal of interest in control of chaotic dynamic systems [2–4]. Almost all of the systems studied share one common feature—being dissipative. Dissipative systems exhibit ergodic behavior on the strange attractors, while chaotic Hamiltonian systems have complicated phase space structure, they have no attractor, but interwoven chaotic and regular regions, which have complicated influence on the chaotic dynamics and present a challenge to chaos controlling. There have been some forerunners in this direction [5–9]. Noting that an unstable periodic orbit in Hamiltonian systems often exhibits complex-conjugate eigenvalues at one or more of its orbit points, Lai *et al.* [5] extended the OGY stabilization method to control Hamiltonian chaos by incorporating the notion of stable and unstable directions at each periodic point. Similar to OGY's method, applying this scheme one has to follow the trajectory, and has to calculate the corresponding perturbation parameters on each step by a complicated algorithm. There is another type scheme, whose goal is directing a trajectory to a desired target in the phase space [6–9]. Obviously, it is not a general way to control Hamiltonian chaos. Therefore, some new methods for controlling Hamiltonian chaotic systems ought to be found, which should be both efficient and general.

In this paper, we will propose one method for controlling chaos in two-dimensional Hamiltonian systems, which is called the adaptive integrable mode coupling method. Our goal is to remove the chaotic motions that permeate into the global phase space, and to stabilize them all into regular motions with small perturbation while the final states of the controlled system remain the main features of the original Hamiltonian system. In the following, we will demonstrate this method in a model of the standard map.

The standard map is one of the most frequently occurring models in many different applications [10,11] written in the form

$$J_{n+1}=J_n-\frac{K}{2\pi}\sin(2\pi\theta_n), \quad \mathrm{mod}\,1, \tag{1}$$

$$\theta_{n+1}=\theta_n+J_{n+1}, \quad \mathrm{mod}\,1$$

where it exhibits, in spite of its simplicity, much of the complex and canonical behavior of more complicated models, and this is ideally suited for the study of chaotic dynamics in Hamiltonian systems. For K less than the threshold value [12] $K_c \cong 0.9716\ldots$ motion in J is bounded by the existence of good Kolmogorov, Arnold, and Moser (KAM) [13] surfaces. For $K>K_c$, there is unbounded motion in J, and global chaos sets in.

We are interested in the chaotic orbits, that can reach arbitrary values of J when $K>K_c$. To our knowledge, the greater the nonintegrability of the Hamiltonian system (i.e., the larger K is), the fewer Birkhoff M cycles, the smaller M and N are (rotation numbers of these orbits are N/M, in which M and N are coprime integers) [11]. For the standard map, each of these elliptic M cycles has one fixed point on $\theta=0$, and this is a common feature of all values of K including the integrable case $K=0$. In the case of $K=0$, the solutions corresponding to these elliptic M cycles are integrable modes, which we apply in controlling. Coupling with these integrable modes, the system under control can be described by the following equations:

$$\begin{aligned} J_{n+1}&=J_n-\frac{K}{2\pi}\sin(2\pi\theta_n)+e(y_n-\theta_n), \quad \mathrm{mod}\,1\\ \theta_{n+1}&=\theta_n+J_{n+1}, \quad \mathrm{mod}\,1,\\ X_{n+1}&=X_n[1-\Theta(\theta_n)]+J_n\Theta(\theta_n), \quad \mathrm{mod}\,1\\ y_{n+1}&=y_n[1-\Theta(\theta_n)]+\theta_n\Theta(\theta_n)+X_{n+1}, \quad \mathrm{mod}\,1 \end{aligned} \tag{2}$$

where

$$\Theta(\theta_n)=\begin{cases}1 & \theta_n\leqslant\epsilon, \quad \theta_n\geqslant 1-\epsilon\\ 0 & \epsilon<\theta_n<1-\epsilon.\end{cases} \tag{3}$$

e is a control parameter, and ϵ is set on 0.001. Equation (2) can be considered as a system with two coupling subsystems: $\{J,\theta\}$ and $\{X,y\}$. It is obvious that the subsystem $\{X,y\}$ is integrable in both cases of $\Theta(\theta_n)$, and it drives the other subsystem $\{J,\theta\}$ by a linear coupling of $g=e(y_n-\theta_n)$. Whenever θ_n comes into the vicinity of $[-\epsilon,+\epsilon]$, the integrable subsystem $\{X,y\}$ is reset to $\{X=J, y=\theta+J\}$, and we call this process an *adaptive* exertion.

The Jacobian determinant of Eq. (2) equals

*Author to whom correspondence should be addressed. Email address: ygyao@aphy.iphy.ac.cn

$$\begin{vmatrix} 1 & -K\cos(2\pi\theta_n)-e & 0 & e \\ 1 & -K\cos(2\pi\theta_n)-e+1 & 0 & e \\ \Theta(\theta_n) & (J_n-X_n)\Theta'(\theta_n) & 1-\Theta(\theta_n) & 0 \\ \Theta(\theta_n) & (J_n-X_n-y_n+\theta_n)\Theta'(\theta_n)+\Theta(\theta_n) & 1-\Theta(\theta_n) & 1-\Theta(\theta_n) \end{vmatrix}, \tag{4}$$

where $\Theta'(\theta_n)=\delta(\theta_n-1+\epsilon)-\delta(\theta_n-\epsilon)$. We note that determinant (4) does not always equal ± 1. When $\Theta(\theta_n)=0$, Det. (4) equals 1 and the system of Eq. (2) conserves its measure. When $\Theta(\theta_n)=1$, Det. (4) equals 0, and X is a trivial dimension. If we consider $\{J,\theta,y\}$ only, we obtain

$$\begin{vmatrix} 1 & -K\cos(2\pi\theta_n)-e & e \\ 1 & -K\cos(2\pi\theta_n)-e+1 & e \\ 1 & 1 & 0 \end{vmatrix} = -e, \tag{5}$$

which indicates dissipation. It is notable that for global stochasticity, the probability for the case of $\Theta(\theta_n)=0$ is much bigger than that of $\Theta(\theta_n)=1$ due to $\epsilon=0.001$ and Eq. (3). So, there is a long term of the conservative driving, driving $\{J,\theta\}$ into $\{X,y\}$, and a short term of the dissipative resetting, resetting $\{X,y\}$ to $\{J,\theta\}$. The two terms turn out alternately. Via this interactive course, one can expect the system to turn into the common stable structure of the two subsystems, and that global chaos can be suppressed in this way.

The first question is: Can global chaos under this handling be stabilized into regular motions? The answer is positive, and the control parameter can be so small that the control signal has a mass of order 1% of that of the controlled system. For each K considered, we have determined 4×10^4 to be of random initial values, all of which can be completely controlled. Figure 1 shows an example in which $K=1.9$, $e=0.006$, and the global chaotic motion is controlled by regular motion. After cutting off enough relaxation iterations, all the stable islands maintain their stable regions while becoming smaller, and all the global chaotic orbits have been stabilized into regular motions. Then, when we release the control (i.e., reset $e=0$), the system follows Eq. (1), and iterates in the corresponding stable (again) islands permanently. Figure 2 exhibits the typical orbits in the phase space on different control stages. The global chaotic motions shown in Fig. 2(a) have been controlled by regular ones in the limited regions in Fig. 2(b), and they become regular orbits in the stable islands shown in Fig. 2(c) after releasing the control action. Thus, global chaos in Eq. (1) has been controlled, and global stochasticity has been removed. We have mentioned above that the system under control becomes dissipative, it has local convergent regions. Taking into account our simulation results, these limited regions are just inside the former stable islands region, and all the systems running under control will be trapped in these regions, then turn into regular motions. The control does not change the periodic orbits of 1 and 2. By inserting their corresponding solutions into the matrix corresponding to Eq. (4) and determining their linear stability, we know that the norms of all the eigenvalues for them are no larger than 1, so these special periodic orbits remain stable under control.

Since we only have some limited regions of regular motions in Fig. 2(b) which belong to the former stable islands, the second question is can the stabilized regular orbits under

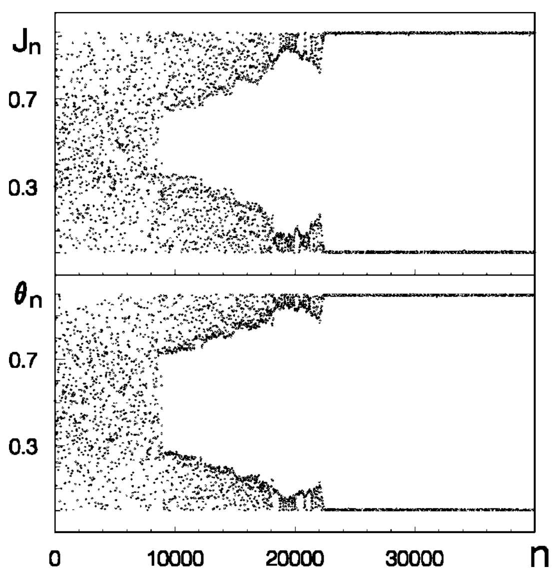

FIG. 1. Stabilized global chaos: (a) J_n vs n; (b) θ_n vs n; $K=1.9$, $e=0.006$, and 10^5 iterations have been cut.

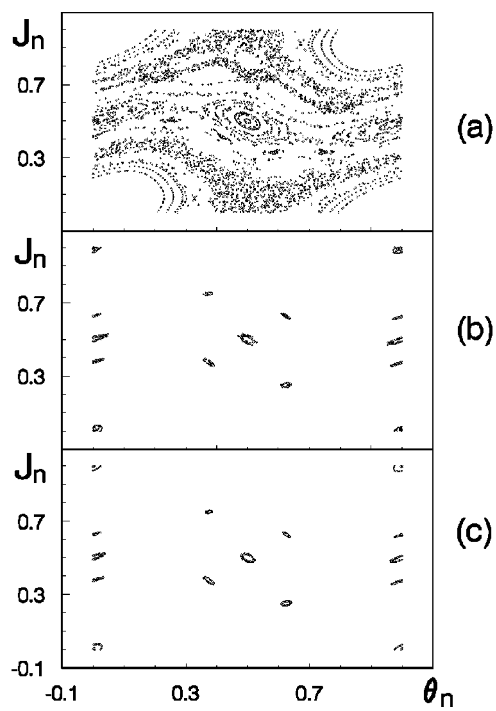

FIG. 2. Typical orbits in the phase space (θ_n, J_n). (a) is from Eq. (1); (b) is from Eq. (2); (c) is from Eq. (2) after the control is switched off, and all the global chaotic motions have been removed from (a). $K=1.1$, $e=0.006$, and 10^7 iterations have been cut.

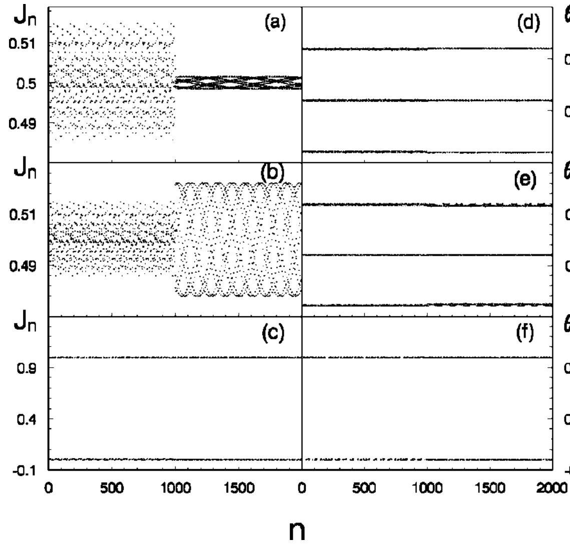

FIG. 3. Typical changes of the system. The control is released from $n=1000$. (a)–(c) are J_n vs n, and (d)–(f) are θ_n vs n corresponding to (a)–(c). $K=1.2$, $e=0.006$, and the only difference among (a), (b), and (c) [(d), (e), and (f)] is the different initial value.

control belong to those of the former stable islands? The answer is negative, and remains negative except for the special periodic orbits mentioned above. This is easy to estimate and to show. After global chaos has been stabilized, we release the control, check the resultant motion, then we have the behaviors in Fig. 3, showing that changes in θ_n are all small, while in J_n there are three kinds of typical changes. They are either shrinking as in (a), swelling as in (b), or changing very little as in (c). Although their regular modes are changed, the resultant motions are still inside the again stable islands, and global chaos has been completely stabilized.

It is worthwhile to note that for different initial values, the relaxation iterations which should be cut are different; a few of them spend 10^6. For example, we control 100×100 initial values with $K=1.9$ and $e=0.006$, and the initial values are distributed uniformly in the phase space of the system without control. We calculate the distribution of the relaxation times in Fig. 4 to quantify the iterations to achieve the final localized regular state. From Fig. 4, we note that the time scale has a peak at about 30 000 iterations in addition to a long time tail. 94.86% of the initial conditions can be controlled in 10^5 iterations, while the other 5.14% need 10^6. This implicitly reflects the third question we are interested in: What are the complex basins of attraction? There are only two families of regular motions under the condition as stated above, most of which correspond to the primary period 1 family, while under the other condition a few correspond to the primary period 2 family, to which the initial values contributed are illustrated in Fig. 5. There is 100×100 points with homogeneous distribution considered in each frame of Fig. 5, where (b) magnifies (a), and (c) magnifies (b). Scattering distribution can be observed in finer and finer scale in Fig. 5. It can be expected that: if one point in the phase space of (θ,J) is controlled in the period 2 family, in its arbitrary small vicinity, there must be another point which will be controlled in the period 1 family. This is obvious in Fig. 6. Figure 6 shows two trajectories of the controlled system

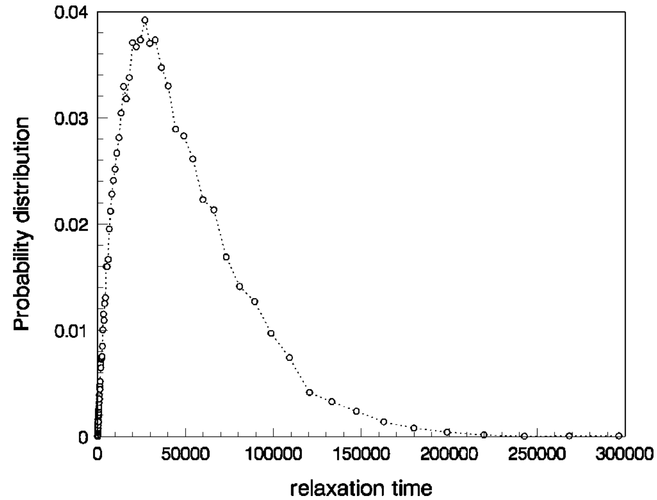

FIG. 4. Distribution of the relaxation times for achieving a stable state. $K=1.9$, $e=0.006$, and 100×100 different initial conditions are considered.

which are stabilized into the period 2 family in (a) and the period 1 family in (b), respectively. The trajectories in both (a) and (b) almost visit the whole phase space of (θ,J). Comparing Figs. 6(a) with 6(b), we can state that the basins of attraction for the two are intermingled with each other.

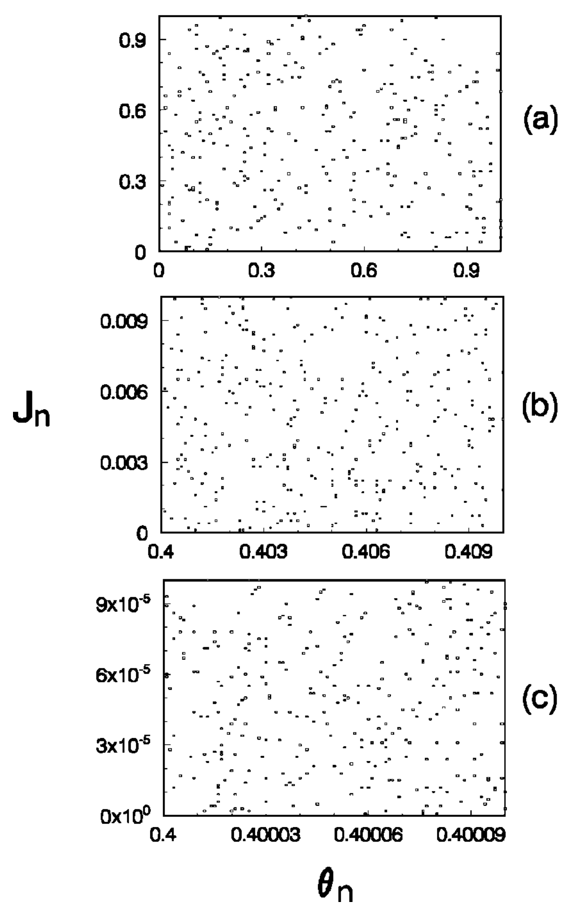

FIG. 5. Initial values are controlled into a period 2 family: (a) full set; (b) enlargement of region defined in $\{(\theta_0,J_0)|0.4\leqslant\theta_0<0.41, 0\leqslant J_0<0.01\}$; (c) enlargement of region defined in $\{(\theta_0,J_0)|0.4\leqslant\theta_0<0.4001, 0\leqslant J_0<0.0001\}$. $K=1.9$, $e=0.006$, and 100×100 points with homogeneous distribution are considered.

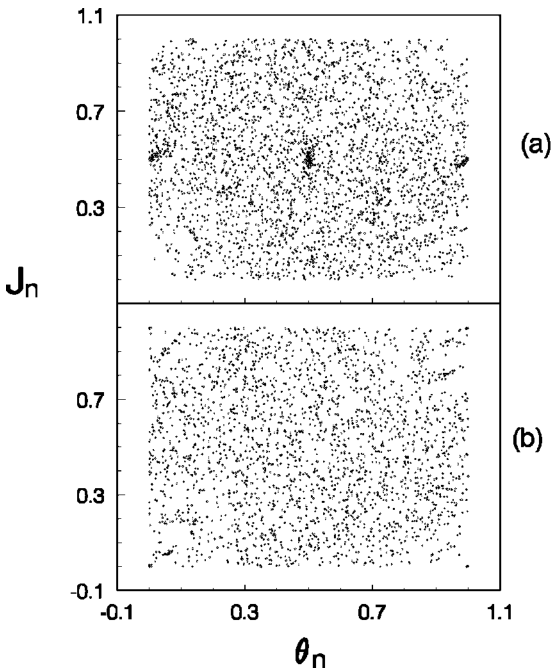

FIG. 6. Typical trajectories of the controlled system with relaxation. The orbit in (a) is controlled into period 2 family, and the orbit in (b) is controlled into period 1 family, and $K=1.2$, $e=0.006$.

Now, the fourth question is: What is the effective range of our control method? It is reasonable to conjecture: if there are stable primary islands in the phase space, all global chaos can be driven into regular motions. The last Birkhoff cycle is a 1 cycle, so the control method might guarantee the upper bounds as $K=4$. Much numerical simulation justifies this conjecture. In fact, even K is larger than 4, for example, of $K=4.3$, we can still stabilize some of the global chaotic motions into regular ones, but we cannot stabilize all of them.

In summary, we proposed an alternative method to control the two-dimensional Hamiltonian chaos in a model of the standard map. And we demonstrated that this method can stabilize the chaotic orbits that permeate throughout the global phase space. All the orbits are controlled into some localized regions similar to the former stable islands but smaller. When the control is switched off, the regularity will be kept. We cannot determine in advance what kinds of regular motions the system under control will finally be stabilized into because the attractive basins for them are complicated being intermingled. The method can be implemented safely in the range of $K<4$, even in the case where K is a little larger than 4, but it still has some effect on global chaotic motions.

The principle behind this method is using the adaptive integrable mode of the system as a dissipative pulse to control Hamiltonian systems. The system under control is conservative in a long term, and dissipative in a short term due to the short dissipative pulse exaction. The width of the dissipative pulse is very short $[\tau(\varepsilon)\propto 10^{-3}]$ and the strength of the pulse is very weak $(e\propto 10^{-2})$ so that the final trajectories of the system remain the main features of the original Hamiltonian system. This method can control all the different kinds of initial values considered, so its efficiency is general. For an experimental system, the integrable mode may be produced either by an integrable real system corresponding to the controlled system or by a signal generator. Since we can have computer in signal's analyzing and controlling, it is not difficult to introduce a weak dissipative pulse to the experimental system, and it is easy to apply this method.

The method studied in this paper is based on the symmetry and continuity between a nonintegrable system and its integrable counterpart. This indicates a possibility of applying it to other two-dimensional Hamiltonian systems, and our further work in this context is in process. One of the most important lessons learned from the kicked rotator model, from which one kind of real physical system of the standard map arises is the classical diffusion excitation taking place above the chaotic threshold, which is quantum mechanically suppressed by interference effects that lead to exponential localization of excitation in momentum space [14]. Furthermore, the formal connection between the rotator problem and the one-dimensional tight-binding model with a time-independent pseudorandom potential was found [15], which led to the recognition that the quantum suppression of the chaotic excitation of the rotator is a sort of dynamical version of Anderson localization. Since then, we can expect this method to apply to controlled dynamical localization phenomenon in a quantum Hamiltonian system.

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