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Stochastic resonance in a bistable system subject to multiplicative and additive noise

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The stochastic resonance (SR) phenomenon in a bistable system under the simultaneous action of multiplicative and additive noise and periodic signal is studied by using the theory of signal-to-noise ratio (SNR) in the adiabatic limit. Two cases have been considered: the case of no correlations between multiplicative and additive noise and the case of correlations between two noises. The expressions of the SNR for both cases are obtained. The effects of intensity of multiplicative and additive noise and the intensity of the correlations between noises on the SNR are discussed for both cases, respectively. It is found that the existence of a maximum in the SNR is the identifying characteristic of the SR phenomenon. In the case of no correlations between multiplicative and additive noise, the SNR is independent of the initial condition of the system. However, the SNR is not only dependent on the intensity of correlations between noises, but also on the initial condition of the system in the presence of correlations between two noises.

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I. INTRODUCTION

Since Benzi *et al.* [1] and Nicolis *et al.* [2] discovered a phenomenon that they termed *stochastic resonance* (SR), a wealth of theoretical and experimental papers has followed, extending the notion of SR and discovering new applications (for extensive reviews see [3–5] and for a collection of papers see [6,7]). Now the SR paradigm has drawn considerable attention in such diverse fields as climatology, chemistry, laser physics, neuroscience (including single-neuron and many-neuron models), biophysics and physiology, particle accelerators, solid-state physics (including superconducting quantum interference devices, bistable magnetic systems, electron paramagnetic resonance, ferroelectrics, ferromagnetics, fluorescence, Ising systems, Josephson junctions, etc.), and even sociology.

There have been many theoretical developments of SR in conventional bistable systems [8–19]. In order to describe SR, McNamara, Wiesenfeld, and Roy [8,9] introduced the signal-to-noise ratio (SNR) to quantify SR, a quantity used in engineering to describe the quality of a signal within a noise background. While both quantities, the response amplitude and the SNR, undergo a resonancelike curve as a function of the noise level, the maxima are located at different values of the noise intensity. Other measures of SR, based on the residence-time distribution of a bistable, periodically driven system, had been introduced to characterize SR. Zhou, Moss, and Jung [14] studied the heights of peaks in the residence-time distribution at odd multiples of the half-period of the driving. They go through maxima as a function of the noise. Gammaitoni, Marchesoni, and Santucci [20] introduced the area under the peak of the residence-time distribution at the half-period of the driving as a measure for SR. They showed that this area goes through a maximum as a function of the noise or the driving frequency and concluded that SR is a bona fide resonance. So far, the majority of the theoretical studies in this area have focused on nonlinear systems with additive white noise. It was concluded that nonlinearity is an essential ingredient of SR since in a linear system the input additive white noise leads to only a trivial decrease in the output SNR.

Recently, it has been shown [21–24] that ''stochastic resonance'' also can be found in a linear system subject to multiplicative rather than to additive noise. Note that the term ''stochastic resonance'' here has been applied to the nonmonotonic behavior of the output signal amplitude rather than to the usually considered SNR mentioned above. It turned out that ''stochastic resonance'' takes place only for multiplicative colored noise (e.g., the dichotomous noise or the O-U noise), but disappears for white noise. Noise multiplicativity and time correlation are the necessary conditions for the ''stochastic resonance'' to occur in a linear system.

The largest amount of work regarding fluctuations has been on the consideration of systems with just one noise source. However, many physical systems require considering various noise sources. Moreover, in certain situations noises may be correlated with each other [25–34]. More recently, considered to be the quadratic-in-field SNR, SR in a linear system subject to multiplicative noise and additive noise has been studied in Ref. [35]. It was shown that, in the linear system, SR is absent for Gaussian white noise, but when the multiplicative noise has the form of an asymmetric dichotomous noise, the SNR becomes a nonmonotonic function of the correlation time and the asymmetry of noise, and the SNR strongly depends on the strength of the cross correlations between multiplicative noise and additive noise. However, it is well known that more realistic models of physical systems are nonlinear. Therefore, it is very important to study the effects of correlations between multiplicative and additive noise on the SR phenomenon of the nonlinear systems. The nonlinear systems with correlations between multiplicative and additive noise have attracted extensive investigations [25–34]. Some of these investigations were concerned with the steady-state statistical properties of nonlinear systems; others were concerned with the transient problems of nonlinear systems.

In this paper, we will use the theory of SNR proposed by

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McNamara and Wiesenfeld [9] to study the SR phenomenon in conventional bistable systems under the simultaneous action of a multiplicative noise and an additive noise and a periodic signal. According to the theory of Ref. [9], the bistable case is reduced to a two-state system, characterized by the occupation probabilities $n_{\pm}=\text{prob}(x=x_{\pm})$ of both stable states $x_{\pm}$. The master equation for these occupation probabilities is

$$\begin{aligned}\dot{n}_{+}=-\dot{n}_{-}&=W_{-}(t)n_{-}-W_{+}(t)n_{+}\\&=W_{-}(t)-[W_{-}(t)+W_{+}(t)]n_{+},\end{aligned}\tag{1}$$

where $W_{\pm}$ is the transition rate out of stable states $x_{\pm}=\pm c$. The general solution of Eq. (1) is

$$n_{+}(t)=g^{-1}(t)\left[n_{+}(t_0)g(t_0)+\int_{t_0}^{t}W_{-}(t')g(t')dt'\right],\tag{2}$$

where $g(t)=\exp\{\int_{t_0}^{t}[W_{+}(t')+W_{-}(t')]dt'\}$. It is assumed that the transition rate $W_{\pm}$ is of the form

$$W_{\pm}(t)=f(\alpha\pm\beta\cos\Omega t).\tag{3}$$

Note that $W_{\pm}(t)$ is time periodic due to the periodic signal. The transition rate can be expanded in the small parameter $\overline{\beta}=\beta\cos\Omega t$:

$$W_{\pm}=\frac{1}{2}(W_0\mp W_1\beta\cos\Omega t+W_2\beta^2\cos^2\Omega t\mp\cdots),\tag{4}$$

where

$$\frac{1}{2}W_0=f(\alpha),\quad \frac{1}{2}W_n=\frac{(-1)^n}{n!}\frac{d^n f(\alpha)}{d\overline{\beta}^n},\tag{5}$$

where $f(\alpha)$ is essentially given by the inverse of the Kramers time. It should be pointed out that the Kramers time is independent of the initial condition $x(t=0)$ in the symmetric bistable system driven by an additive white noise [9]. The power spectrum of the system is given by

$$\begin{aligned}S(\omega)&=S_1(\omega)+S_2(\omega)\\&=\frac{\pi c^2W_1^2\beta^2}{2(W_0^2+\Omega^2)}\delta(\omega-\Omega)+\left[1-\frac{W_1^2\beta^2}{2(W_0^2+\Omega^2)}\right]\frac{2c^2W_0}{W_0^2+\omega^2},\end{aligned}\tag{6}$$

which contains two parts: $S_1(\omega)$ is the signal output which is a δ function at the signal frequency, and $S_2(\omega)$ is the broadband noise output which is a Lorentzian bump centered at $\Omega=0$. Then, the SNR is defined by

$$\Omega_{\text{SNR}}=\frac{P_s}{S_2(\omega=\Omega)},\qquad P_s=\int_0^{\infty}S_1(\omega)d\omega.\tag{7}$$

To obtain the expression of SNR in terms of the output signal power spectrum, the key problem is to calculate the transition rate. It is stressed that the expression for the transition rate would be valid only in the *adiabatic limit*, so the theory of SNR proposed by McNamara and Wiesenfeld [9] is also called the adiabatic approximation. In order to keep our results valid, throughout this paper we will also restrict ourselves in the case of the *adiabatic limit*. The purpose of this paper is twofold. First of all, in Sec. II we will study SR in a conventional bistable system under the simultaneous action of multiplicative and additive noise, and periodic forcing. In this section, the multiplicative noise is independent of the additive noise (i.e., there is no correlation between the two noises). The effects of varying intensity of the multiplicative noise or the additive noise on the SNR will be studied, respectively. Our second goal is to study the effects of correlations between multiplicative and additive noise on the SR in the bistable system in Sec. III. We end with conclusions in Sec. IV.

II. BISTABLE SYSTEM WITH NO CORRELATIONS BETWEEN MULTIPLICATIVE AND ADDITIVE NOISE

We consider the overdamped motion of a Brownian particle in a symmetric bistable potential under the simultaneous action of multiplicative and additive noise and periodic signal (or periodic forcing). The dimensionless form of the Langevin equation for this model reads

$$\dot{x}=ax-bx^3+x\xi(t)+A\cos\Omega t+\eta(t),\tag{8}$$

with

$$\langle\xi(t)\rangle=0,\quad \langle\xi(t)\xi(s)\rangle=2D\delta(t-s),\tag{9}$$

$$\langle\eta(t)\rangle=0,\quad \langle\eta(t)\eta(s)\rangle=2\epsilon\delta(t-s),\tag{10}$$

where D and ϵ describe the intensity of multiplicative and additive noise, respectively, A is the amplitude, and Ω is the frequency of the periodic signal. The deterministic potential of the bistable system

$$U_0(x)=-\frac{a}{2}x^2+\frac{b}{4}x^4\tag{11}$$

has two stable states $x_{-}=-\sqrt{a/b}$, $x_{+}=\sqrt{a/b}$, and an unstable state $x_u=0$. Here we assume that there is no correlation between multiplicative and additive noise:

$$\langle\xi(t)\eta(s)\rangle=\langle\eta(t)\xi(s)\rangle=0.\tag{12}$$

The Fokker-Planck equation corresponding to the Langevin equation (8) with Eqs. (9), (10), and (12) can be written as

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t}=&-\frac{\partial}{\partial x}[ax-bx^3+A\cos\Omega t+Dx]P(x,t)\\&+\frac{\partial^2}{\partial x^2}[Dx^2+\epsilon]P(x,t).\end{aligned}\tag{13}$$

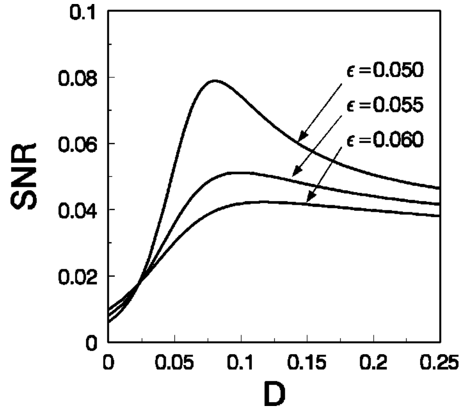

FIG. 1. SNR for the case of no correlations between multiplicative and additive noise, as a function of the multiplicative noise intensity D, for different values of the additive noise intensity ϵ with $A=0.08$ and $\Omega=0.001$.

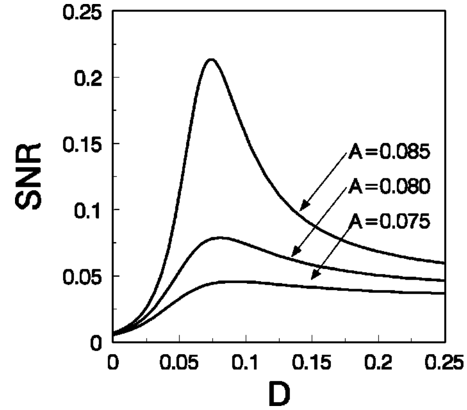

FIG. 2. SNR for the case of no correlations between multiplicative and additive noise, as a function of the multiplicative noise intensity D, for different values of the amplitude of the periodic signal A with $\epsilon=0.05$ and $\Omega=0.001$.

In the absence of a periodic signal ($A=0$), it is well known that the particle will spend most of its time near $x_\pm$, and its steady-state distribution function $P_s(x)$ is

$$P_s(x)=N|Dx^2+\epsilon|^{-1/2}\exp\left[-\frac{\hat{U}(x)}{D}\right], \tag{14}$$

with the modified potential $\hat{U}(x)$,

$$\hat{U}(x)=\frac{b}{2}x^2-\left(\frac{a}{2}+\frac{b\epsilon}{2D}\right)\ln|Dx^2+\epsilon|. \tag{15}$$

One can easily show that the extrema of $\hat{U}(x)$ coincide with those of the deterministic potential $U_0(x)$. On the other hand, under the action of noises, the particle will make occasional transitions over the barrier in the center. In order to calculate the transition rates $W_\pm$ out of the $x_\pm$ states, we can first calculate the mean first passage time (MFPT) $\tau_\pm$ of the process $x(t)$ to reach the state $x_\mp$ with initial condition $x(t=0)=x_\pm$, which is given by the Kramers time [36]

$$\tau_\pm=2\pi|U_0''(x_\pm)U_0''(x_u)|^{-1/2}\exp\left[\frac{\hat{U}(x_u)-\hat{U}(x_\pm)}{D}\right]. \tag{16}$$

Note that the above result is valid only when the intensity of two types of noises, measured by D and ϵ, is small in comparison with the energy barrier height $\Delta\hat{U}=|\hat{U}(x_u)-\hat{U}(x_\pm)|$, that is $\epsilon,D\ll 1$. Thus

$$W_\pm=\tau_\pm^{-1}=\frac{a}{\sqrt{2}\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}+\left(\frac{a}{2}+\frac{b\epsilon}{2D}\right)\ln\left|\frac{aD}{b\epsilon}+1\right|\right]\right\}, \tag{17}$$

where $W_\pm$ is the transition rate out of the $x_\pm$ state. It is shown that the transition rate W_- is equal to the transition rate W_+ when the systemic parameters (a and b) and the intensities of noises (D and ϵ) are given, which means that the transition rate $W_\pm$ is independent of the initial condition $x(t=0)$ of the system in the case of no correlations between multiplicative and additive noise. In other words, the system ''forgets'' its initial position, which is just like that in the bistable system driven by only one additive white noise [9].

In the presence of the periodic signal $A\cos\Omega t$, the potential of the system is modulated by the periodic signal. However, here we assume that the signal amplitude is small enough (i.e., $A\ll 1$) that, in the absence of any noise, it is insufficient to force a particle to move from one well to the other, and it can be considered that $x_\pm=\pm\sqrt{a/b}$ and $x_u=0$ are still the stable states and unstable state of the system. On the other hand, we also assume that the variation of the periodic signal is slow enough (i.e., $\Omega\ll 1$ or the adiabatic limit [9]) that there is enough time to make the system reach local equilibrium in the period of $1/\Omega$. Therefore, the quasi-steady-state distribution function $P_s(x,t)$ corresponding to Eq. (13) can be written as

$$P_s(x,t)=N|Dx^2+\epsilon|^{-1/2}\exp\left[-\frac{\phi(x,t)}{D}\right], \tag{18}$$

with

$$\phi(x,t)=\frac{b}{2}x^2-\left(\frac{a}{2}+\frac{\epsilon b}{2D}\right)\ln|Dx^2+\epsilon| -A\sqrt{\frac{D}{\epsilon}}\arctan(\sqrt{D/\epsilon}\;x)\cos\Omega t. \tag{19}$$

The modified MFPT is given by

$$\tau_\pm=2\pi|U_0''(x_\pm)U_0''(x_u)|^{-1/2}\exp\left[\frac{\phi(x_u,t)-\phi(x_\pm,t)}{D}\right], \tag{20}$$

and the transition rate is

$$W_\pm=\frac{a}{\sqrt{2}\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}+\left(\frac{a}{2}+\frac{b\epsilon}{2D}\right)\ln\left|\frac{aD}{b\epsilon}+1\right| \pm A\sqrt{\frac{D}{\epsilon}}\arctan(\sqrt{aD/b\epsilon})\cos\Omega t\right]\right\}. \tag{21}$$

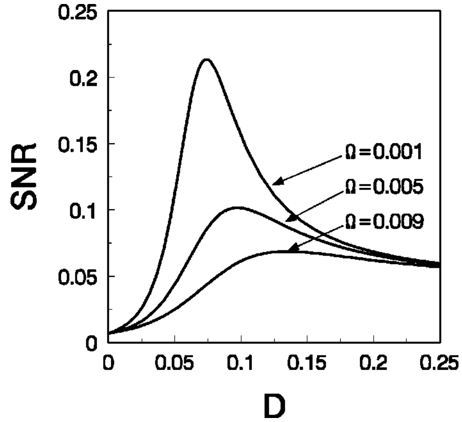

FIG. 3. SNR for the case of no correlations between multiplicative and additive noise, as a function of the multiplicative noise intensity D, for different values of the frequency of the periodic signal Ω with $A=0.085$ and $\epsilon=0.05$.

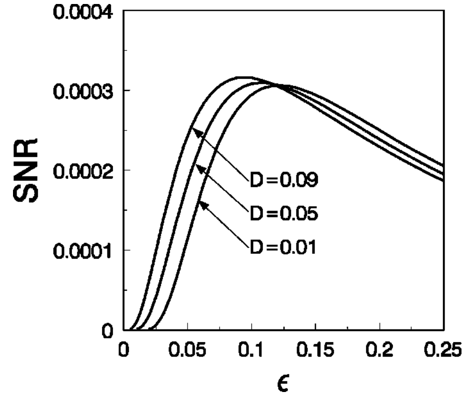

FIG. 4. SNR for the case of no correlations between multiplicative and additive noise, as a function of the additive noise intensity ϵ, for different values of the multiplicative noise intensity D with $A=0.01$ and $\Omega=0.001$.

Within the framework of the theory of SR presented by McNamara and Wiesenfeld [9], the SNR takes the standard form for the bistable system with independent noises in terms of the output signal power spectrum,

$$\Omega_{\mathrm{SNR}}=\frac{\pi A^2 W_0}{4\epsilon D}\left(\arctan\sqrt{\frac{aD}{b\epsilon}}\right)^2 \times\left[1-\frac{W_0^2 A^2[\arctan(\sqrt{aD/b\epsilon})]^2}{2\epsilon D(W_0^2+\Omega^2)}\right]^{-1}, \qquad (22)$$

where

$$W_0=\frac{\sqrt{2}a}{\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}+\left(\frac{a}{2}+\frac{b\epsilon}{2D}\right)\ln\left|\frac{aD}{b\epsilon}+1\right|\right]\right\}. \qquad (23)$$

By virtue of the expression Eq. (22) of SNR, the effects of the multiplicative and additive noise on the SNR can be discussed by numerical computation. For simplicity, we take $a=b=1$ in our computation. In Figs. 1–3 we present the signal-to-noise ratio as a function of the multiplicative noise intensity D, for different values of the additive noise intensity ϵ, the amplitude A, and the frequency Ω of the periodic signal, respectively. The existence of a maximum in these curves is the identifying characteristic of the SR phenomenon. It is shown that the peak is decreased as the additive noise intensity is increased. When the additive noise intensity ϵ is fixed, the maximum of the signal-to-noise ratio is increased as the amplitude of the input signal is increased, but decreased as the frequency of the input signal is increased.

In Figs. 4–6 we present the signal-to-noise ratio as a function of the additive noise intensity ϵ, for different values of the multiplicative noise intensity D and the amplitude A and the frequency Ω of the input signal, respectively. It is shown that the peak of the SNR is increased as the multiplicative noise intensity is increased. When the multiplicative noise intensity D is fixed, as shown in Fig. 5, the interesting point here is that there is only one peak (the first peak) for a small value of the amplitude of the input signal (e.g., the case of $A=0.05$) at a value of ϵ, and the peak is broad and low. However, when the value of the amplitude of the input signal is increased, a second peak appears at a smaller value of ϵ (e.g., the case of $A=0.055$). As the value of the amplitude of input signal continues increasing, the second peak rapidly becomes high and narrow (e.g., the case of $A=0.06$), and the first peak will disappear when A is larger. A similar phenomenon has been shown in Ref. [9], where this phenomenon appears for sufficiently low frequency of the input signal, but for the increasing amplitude of the input signal here. From Fig. 6, we can see that the effect of frequency of the input signal on the signal-to-noise ratio is so little that there is almost no variation for different values of Ω.

III. BISTABLE SYSTEM WITH CORRELATIONS BETWEEN MULTIPLICATIVE AND ADDITIVE NOISE

In this section, consider the bistable system Eq. (8) with correlations between multiplicative and additive noise, and the correlation form between two noises is assumed as follows [26–31]:

$$\langle\xi(t)\eta(s)\rangle=\langle\eta(t)\xi(s)\rangle=2\lambda\sqrt{\epsilon D}\,\delta(t-s), \quad |\lambda|\leqslant 1, \qquad (24)$$

where λ is the cross-correlation intensity. The Fokker-Planck equation corresponding to Eq. (8) with Eqs. (9), (10), and (24) can be written as

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}[ax-bx^3+A\cos\Omega t+Dx+\lambda\sqrt{\epsilon D}]P(x,t) +\frac{\partial^2}{\partial x^2}[Dx^2+2\lambda\sqrt{\epsilon D}\;\;x+\epsilon]P(x,t). \qquad (25)$$

In the absence of a periodic signal ($A=0$), the steady-state distribution function $P_s(x)$ of Eq. (25) is

$$P_s(x)=N|Dx^2+2\lambda\sqrt{\epsilon D}\;\;x+\epsilon|^{-1/2}\exp\left[-\frac{\hat{U}(x,\lambda)}{D}\right] \qquad (26)$$

with the modified potential $\hat{U}(x,\lambda)$,

$$\hat{U}(x,\lambda)=\frac{b}{2}x^2-2b\lambda\sqrt{\frac{\epsilon}{D}}\;x-\left(\frac{a}{2}-\frac{b\epsilon(4\lambda^2-1)}{2D}\right)\times\ln|Dx^2+2\lambda\sqrt{\epsilon D}\;x+\epsilon|+\frac{\lambda}{\sqrt{1-\lambda^2}}\times\left[a+\frac{b\epsilon(3-4\lambda^2)}{D}\right]\arctan\left(\frac{\sqrt{D/\epsilon}\;x+\lambda}{\sqrt{1-\lambda^2}}\right) \tag{27}$$

for $|\lambda|<1$, and

$$\hat{U}(x,\lambda=\pm1)=\frac{b}{2}x^2\mp2b\sqrt{\frac{\epsilon}{D}}\;x+\left(\frac{3b\epsilon}{D}-a\right)\times\ln|\sqrt{D}\;x\pm\sqrt{\epsilon}|\pm\frac{(b\epsilon/D-a)\sqrt{\epsilon}}{\sqrt{D}\;x\pm\sqrt{\epsilon}}. \tag{28}$$

It can also be shown that the extrema of the modified potential $\hat{U}(x,\lambda)$ coincide with those of the deterministic potential $U_0(x)$. The MFPT $\tau_\pm$ of the process $x(t)$ to reach the state $x_\mp$ with initial condition $x(t=0)=x_\pm$ is also given by the Kramers time [30]

$$\tau_\pm=2\pi|U_0''(x_\pm)U_0''(x_u)|^{-1/2}\exp\left[\frac{\hat{U}(x_u,\lambda)-\hat{U}(x_\pm,\lambda)}{D}\right]. \tag{29}$$

Note that the above result is valid only when the intensity of two types of noises, measured by D and ϵ, is small in comparison with the energy barrier height $\Delta\hat{U}=|\hat{U}(x_u,\lambda)-\hat{U}(x_\pm,\lambda)|$. It provides the restriction on the noise intensities, which had been discussed in our previous paper [30]. Then we can obtain the transition rate $W(x(t=0)=x_\pm,\lambda)$ out of the $x_\pm$ states:

$$W(x_\pm,\lambda)=\frac{a}{\sqrt{2}\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}\pm2b\lambda\sqrt{\frac{a\epsilon}{bD}}-\left(\frac{b(4\lambda^2-1)\epsilon}{2D}-\frac{a}{2}\right)\ln\left|\frac{aD}{b\epsilon}\pm2\lambda\sqrt{\frac{aD}{b\epsilon}}+1\right|-\frac{\lambda}{\sqrt{1-\lambda^2}}\left(a+\frac{b(3-4\lambda^2)\epsilon}{D}\right)\times\left(\arctan\frac{\lambda\pm\sqrt{aD/(b\epsilon)}}{\sqrt{1-\lambda^2}}-\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\right]\right\} \tag{30}$$

for $|\lambda|<1$,

$$W(x_\pm,\mp1)=\frac{a}{\sqrt{2}\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}-2b\sqrt{\frac{a\epsilon}{bD}}-\left(\frac{3b\epsilon}{D}-a\right)\ln\left|\sqrt{\frac{aD}{b\epsilon}}-1\right|+\frac{b\epsilon/D-a}{1-\sqrt{b\epsilon/(aD)}}\right]\right\} \tag{31}$$

for $\lambda=\mp1$, and

$$W(x_\pm,\pm1)=\frac{a}{\sqrt{2}\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}+2b\sqrt{\frac{a\epsilon}{bD}}-\left(\frac{3b\epsilon}{D}-a\right)\ln\left|\sqrt{\frac{aD}{b\epsilon}}+1\right|+\frac{b\epsilon/D-a}{1+\sqrt{b\epsilon/(aD)}}\right]\right\} \tag{32}$$

for $\lambda=\pm1$.

It is very important that, when we assume the multiplicative noise and additive noise are correlated with each other, the transition rate out of the x_- state is not equal to the transition rate out of the x_+ state when the systemic parameters (a and b) and the parameters of noises (λ, D, and ϵ) are given. The transition rate is now dependent on the initial condition $x(t=0)$ of the system because of the correlations between multiplicative and additive noise. In other words, the correlations between two noises cause the system to ''remember'' its initial position, which differs from that in the no-correlations case (Sec. II) and that in the only additive white noise case [9]. On the other hand, we find that the transition rate out of the x_- state for $\lambda=+1$ is equal to the transition rate out of the x_+ state for $\lambda=-1$ [see Eq. (31)], and the transition rate out of the x_+ state for $\lambda=+1$ is equal to the transition rate out of the x_- for $\lambda=-1$ [see Eq. (32)].

In the presence of a small periodic signal with very slow frequency (i.e., $A,\Omega\ll1$), the potential of the system is modulated by the periodic signal. However, we assume that the signal amplitude is small enough (i.e., $A\ll1$) that, in the absence of any noise, it is insufficient to force a particle to move from one well to the other, and it can be considered that $x_\pm=\pm\sqrt{a/b}$ and $x_u=0$ are still the stable states and unstable state of the system. Moreover, we also assume that the variation of the periodic signal is slow enough (i.e., $\Omega\ll1$ or the adiabatic limit [9]) that there is enough time to make the system reach local equilibrium in the period of $1/\Omega$. Then, the quasi-steady-state distribution function $P_s(x,t)$ of the system can be written as

$$P_s(x,t)=N|Dx^2+2\lambda\sqrt{\epsilon D}\;x+\epsilon|^{-1/2}\exp\left[-\frac{\phi(x,\lambda,t)}{D}\right], \tag{33}$$

with

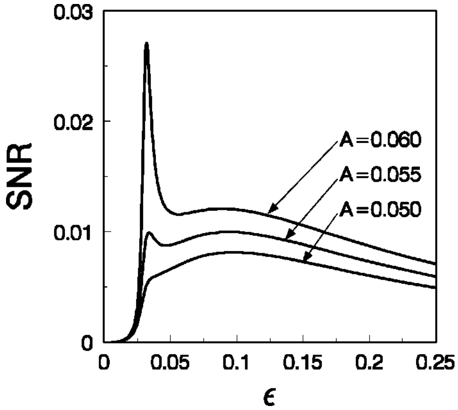

FIG. 5. SNR for the case of no correlations between multiplicative and additive noise, as a function of the additive noise intensity ϵ, for different values of the amplitude of the periodic signal A with $D=0.05$ and $\Omega=0.001$. Note the second peak appears as the value of A increasing.

$$\phi(x,\lambda,t)=\frac{b}{2}x^2-2b\lambda\sqrt{\frac{\epsilon}{D}}\;x+\left(\frac{b\epsilon(4\lambda^2-1)}{2D}-\frac{a}{2}\right)\ln|Dx^2+2\lambda\sqrt{\epsilon D}\;x+\epsilon|+\frac{\lambda}{\sqrt{1-\lambda^2}}\left(a+\frac{b\epsilon(3-4\lambda^2)}{D}-\frac{A}{\lambda}\sqrt{\frac{D}{\epsilon}}\cos\Omega t\right)\arctan\left(\frac{\sqrt{D/\epsilon}\;x+\lambda}{\sqrt{1-\lambda^2}}\right) \qquad (34)$$

for $|\lambda|<1$, and

$$\phi(x,\lambda=\pm1,t)=\frac{b}{2}x^2\mp2b\sqrt{\frac{\epsilon}{D}}\;x+\left(\frac{3b\epsilon}{D}-a\right)\ln|\sqrt{D}\;x\pm\sqrt{\epsilon}|+\left(\frac{b\epsilon}{D}-a\pm\sqrt{\frac{D}{\epsilon}}A\cos\Omega t\right)\times\frac{\sqrt{\epsilon}}{\sqrt{D}\;x\pm\sqrt{\epsilon}}. \qquad (35)$$

The modified MFPT is given by

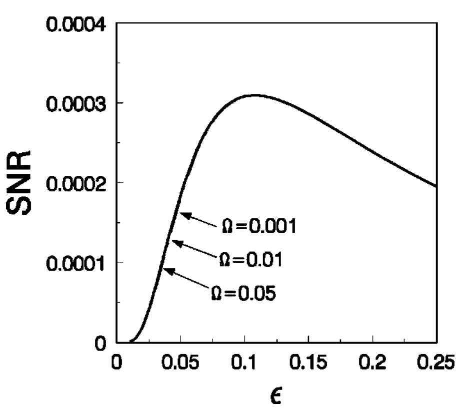

FIG. 6. SNR for the case of no correlations between multiplicative and additive noise, as a function of the additive noise intensity ϵ, for different values of the frequency of the periodic signal Ω with $A=0.01$ and $D=0.05$. Note that there is almost no distinction for different values of Ω.

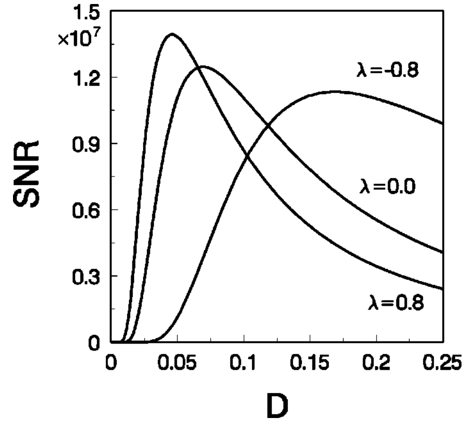

FIG. 7. SNR [Eq. (40)] for the case of correlations between multiplicative and additive noise, as a function of the multiplicative noise intensity D, for different values of the correlated intensity λ with the initial condition $x(t=0)=x_+$. $A=0.0002$, $\Omega=0.0001$, and $\epsilon=1.5D$.

$$\tau_\pm=2\pi|U_0''(x_\pm)U_0''(x_u)|^{-1/2}\exp\left[\frac{\phi(x_u,\lambda,t)-\phi(x_\pm,\lambda,t)}{D}\right]. \qquad (36)$$

Then we can obtain the transition rate $W(x(t=0)=x_\pm,\lambda)$ out of the $x_\pm$ states:

$$W(x_\pm,\lambda)=\frac{a}{\sqrt{2}\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}+2b\lambda\sqrt{\frac{a\epsilon}{bD}}-\left(\frac{b(4\lambda^2-1)\epsilon}{2D}-\frac{a}{2}\right)\ln\left|\frac{aD}{b\epsilon}+2\lambda\sqrt{\frac{aD}{b\epsilon}}+1\right|-\frac{\lambda}{\sqrt{1-\lambda^2}}\left(a+\frac{b(3-4\lambda^2)\epsilon}{D}-\frac{A}{\lambda}\sqrt{\frac{D}{\epsilon}}\;\cos\Omega t\right)\left(\arctan\frac{\lambda+\sqrt{aD/(b\epsilon)}}{\sqrt{1-\lambda^2}}-\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\right]\right\} \qquad (37)$$

for $|\lambda|<1$,

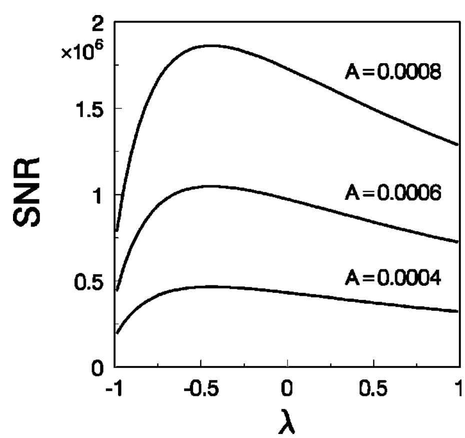

FIG. 8. SNR [Eq. (40)] for the case of correlations between multiplicative and additive noise, as a function of the correlated intensity λ, for different values of the amplitude of the periodic signal A with the initial condition $x(t=0)=x_+$. $D=0.08$, $\Omega=0.0001$, and $\epsilon=0.16$.

$$W(x_\pm,\mp 1)=\frac{a}{\sqrt{2}\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}-2b\sqrt{\frac{a\epsilon}{bD}}-\left(\frac{3b\epsilon}{D}-a\right)\ln\left|\sqrt{\frac{aD}{b\epsilon}}-1\right|+\frac{b\epsilon/D-a-\sqrt{D/\epsilon}\ A\cos\Omega t}{1-\sqrt{b\epsilon/(aD)}}\right]\right\} \quad (38)$$

for $\lambda=\mp 1$, and

$$W(x_\pm,\pm 1)=\frac{a}{\sqrt{2}\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}+2b\sqrt{\frac{a\epsilon}{bD}}-\left(\frac{3b\epsilon}{D}-a\right)\ln\left|\sqrt{\frac{aD}{b\epsilon}}+1\right|+\frac{b\epsilon/D-a+\sqrt{D/\epsilon}\ A\cos\Omega t}{1+\sqrt{b\epsilon/(aD)}}\right]\right\} \quad (39)$$

for $\lambda=\pm 1$.

From Eqs. (37)–(39), it can be seen that, in the case of correlations between multiplicative and additive noise, the transition rate is not only dependent on the intensities (ϵ and D) of noises and the cross-correlation intensity λ of the correlations between multiplicative noise and additive noise, but also on the initial condition $x(t=0)$ of the system. Within the framework of the theory of SR presented by McNamara and Wiesenfeld [9], we can obtain the standard form of the SNR for the bistable system with correlations between multiplicative noise and additive noise in terms of the output signal power spectrum in different cases.

(i) When $|\lambda|<1$,

$$\Omega_{\mathrm{SNR}}(x_\pm,\lambda)=\frac{\pi A^2W_0(x_\pm,\lambda)}{4\epsilon D(1-\lambda^2)}\times\left(\arctan\frac{\lambda\pm\sqrt{aD/(b\epsilon)}}{\sqrt{1-\lambda^2}}-\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)^2\times\left[1-\frac{A^2W_0^2(x_\pm,\lambda)}{2\epsilon D(1-\lambda^2)[W_0^2(x_\pm,\lambda)+\Omega^2]}\times\left(\arctan\frac{\lambda\pm\sqrt{aD/(b\epsilon)}}{\sqrt{1-\lambda^2}}-\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)^2\right]^{-1}, \quad (40)$$

where

$$W_0(x_\pm,\lambda)=\frac{\sqrt{2}a}{\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}\pm 2b\lambda\sqrt{\frac{a\epsilon}{bD}}-\left(\frac{b(4\lambda^2-1)\epsilon}{2D}-\frac{a}{2}\right)\ln\left|\frac{aD}{b\epsilon}\pm 2\lambda\sqrt{\frac{aD}{b\epsilon}}+1\right|-\frac{\lambda}{\sqrt{1-\lambda^2}}\left(a+\frac{b(3-4\lambda^2)\epsilon}{D}\right)\times\left(\arctan\frac{\lambda\pm\sqrt{aD/(b\epsilon)}}{\sqrt{1-\lambda^2}}-\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\right)\right]\right\}. \quad (41)$$

(ii) When $\lambda=\mp 1$ and $x(t=0)=x_\pm$,

$$\Omega_{\mathrm{SNR}}(x_\pm,\mp 1)=\frac{\pi A^2W_0(x_\pm,\mp 1)}{4\epsilon D[1-\sqrt{b\epsilon/(aD)}]^2}\left[1-\frac{A^2W_0^2(x_\pm,\mp 1)}{2\epsilon D[1-\sqrt{b\epsilon/(aD)}]^2[W_0^2(x_\pm,\mp 1)+\Omega^2]}\right]^{-1}, \quad (42)$$

where

$$W_0(x_\pm,\mp 1)=\frac{\sqrt{2}a}{\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}-2b\sqrt{\frac{a\epsilon}{bD}}-\left(\frac{3b\epsilon}{D}-a\right)\ln\left|\sqrt{\frac{aD}{b\epsilon}}-1\right|+\frac{b\epsilon/D-a}{1-\sqrt{b\epsilon/(aD)}}\right]\right\}. \quad (43)$$

(iii) When $\lambda=\pm 1$ and $x(t=0)=x_\pm$,

$$\Omega_{\mathrm{SNR}}(x_\pm,\pm 1)=\frac{\pi A^2W_0(x_\pm,\pm 1)}{4\epsilon D[1+\sqrt{b\epsilon/(aD)}]^2}\left[1-\frac{A^2W_0^2(x_\pm,\pm 1)}{2\epsilon D[1+\sqrt{b\epsilon/(aD)}]^2[W_0^2(x_\pm,\pm 1)+\Omega^2]}\right]^{-1}, \quad (44)$$

where

$$W_0(x_\pm,\pm 1)=\frac{\sqrt{2}a}{\pi}\exp\left\{-\frac{1}{D}\left[-\frac{a}{2}+2b\sqrt{\frac{a\epsilon}{bD}}-\left(\frac{3b\epsilon}{D}-a\right)\ln\left|\sqrt{\frac{aD}{b\epsilon}}+1\right|+\frac{b\epsilon/D-a}{1+\sqrt{b\epsilon/(aD)}}\right]\right\}. \quad (45)$$

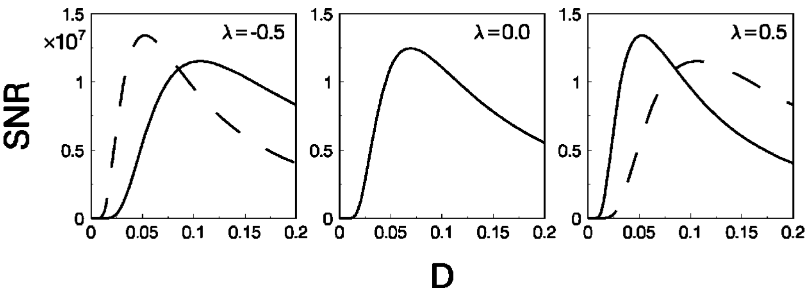

FIG. 9. Comparison of the SNR [Eq. (40)] for the initial condition $x(t=0)=x_+$ (the full line) with that for $x(t=0)=x_-$ (the dashed line). $A=0.0002$, $\Omega=0.0001$, and $\epsilon=1.5D$.

By virtue of the expressions of SNR [Eqs. (40), (42), and (44)] for the different cases, the effects of the correlation between multiplicative and additive noise on the SNR can be presented by the numerical computation. For simplicity, we take $a=b=1$ in our calculations.

When $|\lambda|<1$, the signal-to-noise ratio with the initial condition $x(t=0)=x_+$, as a function of the multiplicative noise intensity D for different values of the correlated intensity λ, is shown in Fig. 7. The maximum of the signal-to-noise ratio for the initial condition x_+ is increased as the correlation intensity λ varies from negative value to positive value. It can also be found that the maximum of the signal-to-noise ratio for the initial condition x_- is decreased as the correlation intensity λ varies from negative to positive. When the intensities of two noises are given, the signal-to-noise ratio with the initial condition x_+, as a function of the correlation intensity λ for different values of the amplitude of the periodic signal A, is shown in Fig. 8. We can see that the height of the peak increases with the increasing of the amplitude of the periodic signal A for both initial conditions. The peak of the signal-to-noise ratio for $x(t=0)=x_+$ is situated in $\lambda<0$. However, because of the symmetry, the peak of the signal-to-noise ratio for $x(t=0)=x_-$ is situated in $\lambda>0$. In Fig. 9, we compare the SNR for the initial condition x_+ with that for the initial condition x_-. It can be seen that there is no distinction between the SNR for the initial condition x_+ and that for the initial condition x_- when the intensity of the correlations between noises is zero, but the distinction of the SNR between $x(t=0)=x_+$ and $x(t=0)=x_-$ is clearly when $\lambda\neq 0$. Moreover, we also find that the value of SNR with $\lambda=-0.5$ for $x(t=0)=x_+$ is equal to that with $\lambda=+0.5$ for $x(t=0)=x_-$. The above results show that the signal-to-noise ratio is not only dependent on the intensities (ϵ and D) of noises and the cross-correlation intensity λ of the correlations between multiplicative noise and additive noise, but also on the initial condition $x(t=0)$ of the system.

When $\lambda=-1$ and $x(t=0)=x_+$ [or $\lambda=1$ and $x(t=0)=x_-$], the signal-to-noise ratio, as a function of the multiplicative noise intensity D for different values of the amplitude of the periodic signal A and for different values of the additive noise intensity ϵ, is shown in Fig. 10 and Fig. 11, respectively. The interesting point here is that the peak is narrow and high, and the value of the signal-to-noise ratio is very large in the order 10^4. Moreover, there is a large variation in the value of the SNR peak over a very narrow range in A (see Fig. 10) and in ϵ (see Fig. 11). However, when $\lambda=1$ and $x(t=0)=x_+$ [or $\lambda=-1$ and $x(t=0)=x_-$], the signal-to-noise ratio, as a function of the multiplicative noise intensity D for different values of the amplitude of the periodic signal A and for different values of the additive noise intensity ϵ, is shown in Fig. 12 and Fig. 13, respectively. It is shown that the peak is broad and the value of the signal-to-noise ratio is very small in the order 10^{-7}.

IV. CONCLUSION

We have studied the SR phenomenon in conventional bistable systems under the simultaneous action of multiplicative and additive noise and periodic forcing by using the theory of SNR proposed by McNamara and Wiesenfeld [9]. Two cases have been considered: one is the case of no correlations between multiplicative and additive noise, and the other is the case of correlations between two noises. We have

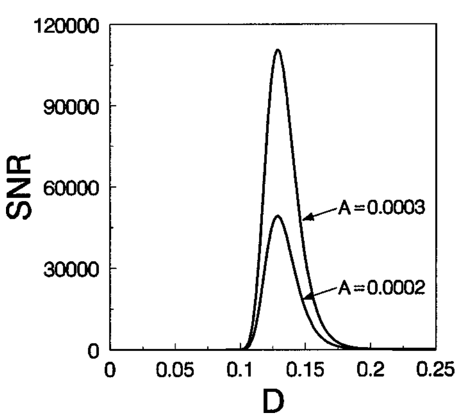

FIG. 10. SNR [Eq. (42)] for the case of $\lambda=-1$ and $x(t=0)=x_+$ [or $\lambda=1$ and $x(t=0)=x_-$], as a function of the multiplicative intensity D, for different values of the amplitude of the periodic signal A. $\epsilon=0.08$ and $\Omega=0.0001$.

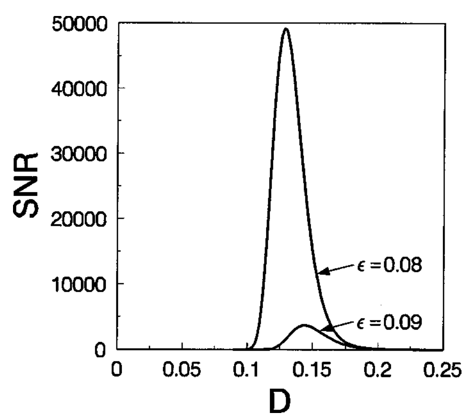

FIG. 11. SNR [Eq. (42)] for the case of $\lambda=-1$ and $x(t=0)=x_+$ [or $\lambda=1$ and $x(t=0)=x_-$], as a function of the multiplicative intensity D, for different values of the additive noise intensity ϵ. $A=0.0002$ and $\Omega=0.0001$.

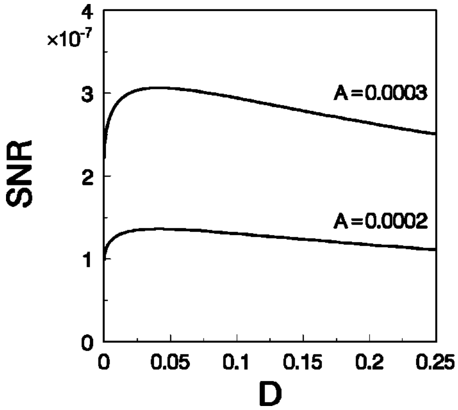

FIG. 12. SNR [Eq. (44)] for the case of $\lambda=1$ and $x(t=0)=x_+$ [or $\lambda=-1$ and $x(t=0)=x_-$], as a function of the multiplicative intensity D, for different values of the amplitude of the periodic signal A. $\epsilon=0.08$ and $\Omega=0.0001$.

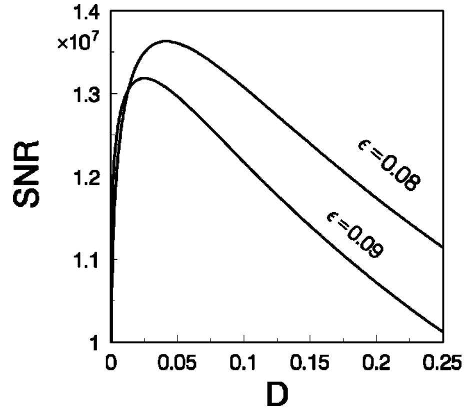

FIG. 13. SNR [Eq. (44)] for the case of $\lambda=1$ and $x(t=0)=x_+$ [or $\lambda=-1$ and $x(t=0)=x_-$], as a function of the multiplicative intensity D, for different values of the additive noise intensity ϵ. $A=0.0002$ and $\Omega=0.0001$.

obtained the expressions of the SNR for both cases. By virtue of the expressions of SNR and through the numerical computation, we have found that the existence of a maximum in the SNR is the identifying characteristic of the SR phenomenon. In the case of no correlations between multiplicative and additive noise, the SNR is independent of the initial condition $x(t=0)$ since the transition rate $W_\pm$ is independent of the initial condition. However, the presence of correlations between two noises changes this picture. The correlations between noises cause the system to ''remember'' its initial position, and the SNR is now dependent on the initial condition since the transition rates depend on the initial condition.

In the case of no correlations between two noises, the effects of varying intensity of the multiplicative D and the additive noise ϵ on the signal-to-noise ratio have been studied, respectively. It has been shown that the effects of D and ϵ on the change of the signal-to noise ratio are opposed to each other. If the SNR is a function of D, there is only one maximum when the amplitude of the periodic signal A is increased, and the value of the maximum in SNR decreases when the frequency of the periodic signal Ω is increased. However, if the SNR is a function of ϵ, a second maximum in the SNR can appear when the amplitude of the periodic signal A is increased, and there is no variation for different values of Ω.

In the case of correlations between two noises, the effects of the intensity λ of correlations between noises have been studied for different cases. When $|\lambda|<1$, the stochastic resonance phenomenon can still appear when the intensity λ of the correlations between two noises is varied from negative to positive, and the appearance of the maximum in the SNR is dependent on the initial condition of the system. When $\lambda=-1$ and $x(t=0)=x_+$ [or $\lambda=1$ and $x(t=0)=x_-$], the value of the SNR is very large. However, when $\lambda=1$ and $x(t=0)=x_+$ [or $\lambda=-1$ and $x(t=0)=x_-$], the value of the SNR is very small. In fact, when $\lambda=-1$ (or $\lambda=1$), the probability distribution of the Brownian particle is fully concentrated at $x(t=0)=x_+$ [or $x(t=0)=x_-$]. On the contrary, when $\lambda=-1$ [or $\lambda=1$], the probability distribution of the Brownian particle is very little at $x(t=0)=x_-$ (or $x(t=0)=x_+$).

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[1] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981); R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, Tellus **34**, 10 (1982).
[2] C. Nicolis and G. Nicolis, Tellus **33**, 225 (1981).
[3] P. Jung, Phys. Rep. **234**, 175 (1993).
[4] F. Moss, *Stochastic Resonance: From Ice Ages to the Monkey's Ear*, in *Contemporary Problems in Statistical Physics*, edited by G.H. Wess (SIAM, Philadephia 1994), pp. 205–253.
[5] K. Wiesenfeld and F. Moss, Nature (London) **373**, 33 (1995).
[6] Proceedings of the NATO RAW on Stochastic Resonance in Physics and Biology, edited by F. Moss, A. Bulsara, and M.F. Shlesinger [J. Stat. Phys. **70**, 1 (1993)].
[7] International Workshop on Fluctuations in Physics and Biology: Stochastic Resonance, Signal Processing and Related Phenomena, edited by A. Bulsara, S. Chillemi, L. Kiss, P.V.E. McClintock, R. Mannella, F. Marchesoni, G. Nicolis, and K. Wiesenfeld [Nuovo Cimento D **17**, 653 (1995)].
[8] B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988).
[9] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).
[10] R.F. Fox, Phys. Rev. A **39**, 4148 (1989).
[11] M.I. Dykman, R. Mannella, P.V.E. McClintock, and N.G. Stocks, Phys. Rev. Lett. **65**, 2606 (1990).
[12] P. Jung and P. Hanggi, Phys. Rev. A **41**, 2977 (1990).
[13] G. Hu, G. Nicolis, and C. Nicolis, Phys. Rev. A **42**, 2030 (1990).
[14] T. Zhou, F. Moss, and P. Jung, Phys. Rev. A **42**, 3161 (1990).

[15] P. Jung and P. Hanggi, Phys. Rev. A **44**, 8032 (1991).
[16] H. Gang, G.R. Qing, D.C. Gong, and X.D. Weng, Phys. Rev. A **44**, 6414 (1991).
[17] G. Hu, H. Haken, and C.Z. Ning, Phys. Lett. A **172**, 21 (1992).
[18] M.I. Dykman, R. Mannella, P.V.E. McClintock, and N.G. Stocks, Phys. Rev. Lett. **68**, 2985 (1992).
[19] M.I. Dykman, H. Haken, Hu Gang, D.G. Luchinsky, R. Mannella, P.V.E. McClintock, C.Z. Ning, N.D. Stein, and N.G. Stock, Phys. Lett. A **180**, 332 (1993).
[20] L. Gammaitoni, F. Marchesoni, and S. Santucci, Phys. Rev. Lett. **74**, 1052 (1995).
[21] A. Fulinski, Phys. Rev. E **52**, 4523 (1995).
[22] V. Berdichevsky and M. Gitterman, Europhys. Lett. **36**, 161 (1996).
[23] A.V. Barzykin and K. Seki, Europhys. Lett. **40**, 117 (1997).
[24] A.V. Barzykin, K. Seki, and F. Shibata, Phys. Rev. E **57**, 6555 (1998).
[25] I.I. Fedchenia, J. Stat. Phys. **50**, 1043 (1988); **52**, 1005 (1988).
[26] A. Fulinski and T. Telejko, Phys. Lett. A **152**, 11 (1991).
[27] L. Cao and D.J. Wu, Phys. Lett. A **185**, 59 (1994).
[28] Y. Jia, L. Cao, and D.J. Wu, Phys. Rev. A **51**, 3169 (1995).
[29] Y. Jia and J.R. Li, Phys. Rev. E **53**, 5764 (1996).
[30] Y. Jia and J.R. Li, Phys. Rev. E **53**, 5786 (1996).
[31] A.J.R. Madureira, P. Hanggi, and H.S. Wio, Phys. Lett. A **217**, 248 (1996).
[32] Y. Jia and J.R. Li, Phys. Rev. Lett. **78**, 994 (1997).
[33] Y. Jia and J.R. Li, Physica A **252**, 417 (1998).
[34] J.H. Li and Z.Q. Huang, Phys. Rev. E **53**, 3315 (1996).
[35] V. Berdichevsky and M. Gitterman, Phys. Rev. E **60**, 1494 (1999).
[36] E. Guardia and M. San Miguel, Phys. Lett. **109A**, 9 (1985).