Distribution of the determinant of a random real-symmetric matrix from the Gaussian orthogonal ensemble

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The Mellin transform of the probability density of the determinant of $N\times N$ random real-symmetric matrices from the Gaussian orthogonal ensemble is calculated. The determinant probability density is given by a single Meijer G function for odd N. The distribution of the potential at the origin, within the Coulomb gas interpretation, is investigated from the Mellin transform of the determinant distribution and is shown to be asymptotically Gaussian.

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I. INTRODUCTION

Random matrix theory finds considerable use in various branches of physics, notably in nuclear physics, quantum chaology, and for investigating Hamiltonians of disordered and strongly interacting quantum systems [1–3], and references therein]. Among the ensembles of $N\times N$ random matrices H_N, three Gaussian ensembles have been studied extensively [1–3] and are still being investigated. Their probability densities are proportional to $\exp[-\mathrm{tr}(H_N^2)]$, where tr means trace. Matrices are real symmetric for the Gaussian orthogonal ensemble (GOE), Hermitian for the Gaussian unitary ensemble (GUE), and quaternion self-dual for the Gaussian symplectic ensemble. Many characteristics, both exact at finite N and asymptotic at large N, of the distributions of eigenvalues of $N\times N$ random matrices of various symmetries are known for the Gaussian ensembles [1–3].

The properties of eigenvalues of random matrix ensembles (RME's) can be interpreted in two dimensions (2D) from the equilibrium characteristics of a gas of N identical point charges on a line [1–10], often referred to as a log gas [9,10], which interact via a logarithmic Coulomb potential and are confined by an external potential. The external potential is harmonic in the case of Gaussian ensembles as can be seen by rewriting Eq. (1) below as a Boltzmann factor at a temperature $1/\beta$ that depends only on the symmetry of the considered ensemble, with $\beta=1$, 2, and 4 for the orthogonal unitary, and symplectic ensembles, respectively [1–3]. The problems of the distribution and associated fluctuations of linear statistics in RME's [1–10], and references therein] are of interest, for instance, in the study of conductance fluctuations in mesoscopic conductors. The linear statistics problem deals with physical quantities F which are given by sums $F=\Sigma_{k=1}^{N}f(\lambda_k)$ over the eigenvalues λ_k of a random matrix, where $f(x)$ may depend nonlinearly on x. General arguments have been used to predict that the distribution of any linear statistic is Gaussian and independent of the confining potential in the scaled asymptotic limit $N\to\infty$ provided its variance is finite [5–10]. The potential fluctuations have been investigated in 2D and 3D classical charged systems [11,12] and more recently for circular ensembles by Baker and Forrester, who further studied the dipole moment statistic [9]. Within the log-gas interpretation, the statistic of the potential at the origin is obtained for $f(x)=-\ln|x|$, that is, for $F=-\ln|D|$. It may thus be derived directly from the distribution of the determinant D of H_N. We notice further that the partition functions of log gases whose confining harmonic potential includes a supplementary logarithmic contribution are by-products of the derivation of the determinant distributions of Gaussian ensembles [see Eqs. (10), (12), (19), and (26) for $\beta=1$]. The RME's associated with the latter log gases belong to the family of generalized Gaussian ensembles [13,14].

The probability densities of the determinant were derived only recently in terms of Meijer G functions by Mehta and Normand [15] for the GUE and for the Gaussian ensemble of complex matrices without further constraints on the entries. The determinant density was expressed with the help of Mellin's inversion integral by Nyquist, Rice, and Riordan [16] in the case of a random matrix ensemble of real matrices with N^2 identically and independently distributed (iid) Gaussian entries while properties of random determinants are discussed by Girko [17], who mentions general applications of random determinants, in particular in solid state physics (Chap. 27). It is worth recalling that Wigner derived the eigenvalue distribution of the GUE because he wanted to obtain an estimate of the value of a determinant of a matrix I_N+H_N where I_N is an $N\times N$ unit matrix and the modulus of each element of the Hermitian matrix H_N is small as compared with 1 (first section of [18]).

Mehta and Normand [15] emphasize that the question of the determinant distribution remains open for the Gaussian orthogonal ensemble and for the Gaussian symplectic ensemble. The present paper solves this problem for the GOE case as it reports the calculation of the Mellin transform of the probability density of the determinant of a $N\times N$ random real-symmetric matrix H_N that belongs to that Gaussian ensemble. The determinant density is shown to be proportional to a single Meijer's G function in the case of odd N. Exact and asymptotic results are established for the potential statistic from the Mellin transforms of the determinant distributions.

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II. DETERMINANT DISTRIBUTIONS

The joint distributions of eigenvalues for the Gaussian ensembles are [1–3]

$$P_{N,\beta}(\lambda_1,\ldots,\lambda_N)=C_{N,\beta}\exp\left[-\frac{1}{2\sigma^2}\left(\sum_{k=1}^{N}\lambda_k^2\right)\right]\times\left(\prod_{1\le j<k\le N}|\lambda_j-\lambda_k|^{\beta}\right), \quad (1)$$

where $\beta=1$, 2, and 4 for the orthogonal, unitary, and symplectic ensembles, respectively, and $C_{N,\beta}$ is the reciprocal of the Mehta integral [2], p. 354. The case $\beta=0$ in Eq. (1), which is of some interest in the present context, corresponds further to an ensemble of diagonal matrices whose eigenvalues λ_k, $k=1,\ldots,N$, are iid Gaussian variables. The elements whose knowledge suffices to construct matrices of the considered Gaussian ensembles, namely, H_{ii} for $\beta=0,1,2$, H_{ij} for $\beta=1$, and $\mathrm{Re}(H_{ij}),\mathrm{Im}(H_{ij})$ for $\beta=2$, with $i,j=1,\ldots,N(j>i)$, are recalled to be independently distributed according to Gauss distributions with zero means and variances $\langle H_{ii}^2\rangle=\sigma^2$ and $\langle H_{ij}^2\rangle=\sigma^2/2$ for $\beta=1$ and $\langle[\mathrm{Re}[H_{ij}]]^2\rangle=\langle[\mathrm{Im}[H_{ij}]]^2\rangle=\sigma^2/2$ for $\beta=2$. The notation $P_N^{(\beta)}(D)$ will be used throughout the text to designate the probability density of the determinant D of $N\times N$ Gaussian random matrices either diagonal, real symmetric, or Hermitian according to the value of β. The distributions $P_N^{(\beta)}(D)$ are symmetric when $N=2p+1$. Indeed, the odd moments calculated from Eq. (1) satisfy the relation

$$\langle(\lambda_1\lambda_2\cdots\lambda_{2p+1})^{2q+1}\rangle=(-1)^{(2p+1)(2q+1)}\times\langle(\lambda_1\lambda_2\cdots\lambda_{2p+1})^{2q+1}\rangle=0$$

for $\beta=1,2$, as can be shown by changing λ_j into $-\lambda_j$ ($j=1,\ldots,2p+1$). The distribution $P_N^{(0)}(D)$ is symmetric for any N, as a consequence of the relation $\langle(\lambda_1\lambda_2\cdots\lambda_N)^{2q+1}\rangle=\Pi_{k=1}^{N}\langle\lambda_k^{2q+1}\rangle=0$. To express the sought-after determinant distributions in terms of Meijer G functions, which are defined as inverse Mellin transforms of ratios of products of gamma functions [19–21], we calculate separately the Mellin transforms of the even and odd parts of the determinant distributions [15]. Defining first

$$P_N^{(\beta)\pm}(D)=\tfrac{1}{2}[P_N^{(\beta)}(D)\pm P_N^{(\beta)}(-D)], \quad (2)$$

the associated Mellin transforms are

$$M_N^{(\beta)\pm}(s)=\int_0^{\infty}D^{s-1}P_N^{(\beta)\pm}(D)dD=\tfrac{1}{2}\int_{R^N}P_{N,\beta}(\lambda_1,\ldots,\lambda_N)\prod_{k=1}^{N}|\lambda_k|^{s-1}\varepsilon^{\pm}(\lambda_k)d\lambda_k \quad (3)$$

with

$$\varepsilon^{+}(x)=1,\quad \varepsilon^{-}(x)=\mathrm{sgn}(x). \quad (4)$$

As described below for $N=2p+1$, the Mellin transform $M_{2p+1}^{(\beta)+}(s)$ is proportional to a product $\Pi_{j=1}^{N}\Gamma(s/2+b_j^{(\beta)})$ defined by N parameters $b_j^{(\beta)}$ ($j=1,\ldots,N$) whose explicit knowledge is necessary to derive the determinant distribution, which is then proportional to a Meijer G function, $G_{0,N}^{N,0}(D^2|b_1^{(\beta)},b_2^{(\beta)},\ldots,b_N^{(\beta)})$ [15,19–21]. These parameters are known for $\beta=2$ [15] but as yet either unnoticed for $\beta=0$ or unknown for $\beta=1$.

A. Diagonal case: $\beta=0$

The Mellin transform of the symmetric distribution of the determinant $P_N^{(0)}(D)$ is easily derived for any N, as the eigenvalues are independent ($\sigma=1/\sqrt{2}$):

$$M_N^{(0)}(s)=\frac{1}{2\pi^{N/2}}\prod_{j=1}^{N}\Gamma\left(\frac{s}{2}\right), \quad (5)$$

that is, $b_j^{(0)}=0$ for any j in the range $1\le j\le N$. The distribution $P_N^{(0)}(D)$ is thus obtained as

$$P_N^{(0)}(D)=\frac{1}{(2\pi\sigma^2)^{N/2}}G_{0,N}^{N,0}(D^2/(2\sigma^2)^N|0,0,\ldots,0) \quad (6)$$

for any value of σ. For $N=2$ [20], p. 128, $P_2^{(0)}(D)=(1/\pi\sigma^2)K_0(D/\sigma^2)$, where $K_0(x)$ is a modified Bessel function. The explicit form of $P_N^{(0)}(D)$ is complicated in general (see Sec. 4.5.2 of [19]).

B. Hermitian case: $\beta=2$

Mehta and Normand [15] have shown that $P_N^{(2)}(D)$ is given by a single Meijer G function whose parameters are $b_j^{(2)}=\lfloor j/2\rfloor$ ($\lfloor x\rfloor$ denotes the largest integer $\le x$) when N is odd, $N=2p+1$:

$$P_N^{(2)}(D)=K_N^{(2)}G_{0,N}^{N,0}(D^2/(2\sigma^2)^N|0,1,1,2,2,\ldots,p,p) \quad (7)$$

and

$$[K_N^{(\beta)}]^{-1}=(2\sigma^2)^{N/2}\prod_{j=1}^{N}\Gamma(\tfrac{1}{2}+b_j^{(\beta)}), \quad (8)$$

while it is a linear combination of two G functions when N is even (see [15]). A sketch of the calculation of the parameters $b_j^{(2)}$ from generalized Hermite polynomials is given in Appendix A.

C. Real-symmetric case: $\beta=1$

The absolute value of the Vandermonde determinant in Eq. (3) renders calculations more difficult in the real-symmetric case. The simplicity of the symmetric distributions $P_{2p+1}^{(\beta)}(D)$ for $\beta=0,2$ suggests, however, that the unknown distributions $P_{2p+1}^{(1)}(D)$ might also be proportional to a single Meijer G function. We prove indeed below that

$$P_{2p+1}^{(1)}(D)=K_{2p+1}^{(1)}G_{0,2p+1}^{2p+1,0}\left(D^2/(2\sigma^2)^N\,\middle|\,0,\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4},\ldots,\frac{2p-1}{4},\frac{2p+1}{4}\right), \quad (9)$$

that is, $b_1^{(1)}=0$ and $b_j^{(1)}=\frac{1}{2}\lfloor (j-1)/2\rfloor+\frac{1}{4}$ $(j\geqslant 2)$, while $K_{2p+1}^{(1)}$ is given by Eq. (8). For $N=1$, all matrices reduce to a single Gaussian element so that $b_1^{(\beta)}=0$ $(\beta=0,1,2)$ is in fact expected from Eq. (6). We define the integral [with $\varepsilon^{\pm}(x)$ given by Eq. (4)]

$$I_{N,\sigma}^{\pm}(s,\beta)=\int_{R^N}\prod_{k=1}^{N}\left[dx_k\varepsilon^{\pm}(x_k)|x_k|^{s-1}\exp\left(-\frac{x_k^2}{2\sigma^2}\right)\right]\times\left|\prod_{1\leqslant i<j\leqslant N}(x_i-x_j)\right|^{\beta}, \tag{10}$$

from which the needed Mellin transforms are directly calculated (Sec. II C 3). We define further the following general integral:

$$I_N^{(V)}=\int_{R^N}\prod_{k=1}^{N}dx_k\exp[-V(x_k)]\prod_{1\leqslant i<j\leqslant N}|x_i-x_j|, \tag{11}$$

where $V(x)$ is an even function of x. We outline in Appendix B the calculation of the latter integral for odd N (see also Sec. 14.3 of [2]), and we obtain

$$I_N^{(V)}=N!\,2^{(N-1)/2}\begin{vmatrix}\langle P_0,P_1\rangle & \langle P_0,P_3\rangle & \cdots & \langle P_0,P_{N-2}\rangle & T(P_0)\\ \langle P_2,P_1\rangle & \langle P_2,P_3\rangle & \cdots & \langle P_2,P_{N-2}\rangle & T(P_2)\\ & & \vdots & & \\ \langle P_{N-1},P_1\rangle & \langle P_{N-1},P_3\rangle & \cdots & \langle P_{N-1},P_{N-2}\rangle & T(P_{N-1})\end{vmatrix}$$

[Eqs. (B14) and (B15)], where the polynomials P_j, $T(\cdot)$ and the inner product $\langle\cdot,\cdot\rangle$ are defined by Eqs. (B1), (B8), and (B11), respectively.

1. Calculation of $I_{N,\sigma}^{+}(s,1)$ for the GOE

In the case of the GOE, $I_N^{(V)}$ is equal to $I_{N,\sigma}^{+}(s,1)$ [Eq. (10)] with a potential

$$V_\sigma(x)=\frac{x^2}{2\sigma^2}-(s-1)\ln|x|. \tag{12}$$

In the following we calculate $I_{N,1}^{+}(s,1)$ as $I_{N,\sigma}^{+}(s,1)=\sigma^{Ns+N(N-1)/2}I_{N,1}^{+}(s,1)$. The monic polynomials $P_m(x)$ [Eqs. (B15) and (B16), Appendix B] are chosen to be $P_m(x)=x^m$. From integral (6.455) of [21], which involves an incomplete gamma function, we deduce the inner product [Eq. (B11)] $\langle x^{2i},x^{2j+1}\rangle$ [Re(S)>0]:

$$\langle x^{2i},x^{2j+1}\rangle=2\int_0^{+\infty}dx\,x^{2j+s}\exp\left(-\frac{x^2}{2}\right)\times\int_0^{x}dy\,y^{2i-1+s}\exp\left(-\frac{y^2}{2}\right)=\frac{\Gamma(s+i+j+\frac{1}{2})}{s+2i}F\left(1,s+i+j+\tfrac{1}{2};\frac{s}{2}+i+1;\tfrac{1}{2}\right), \tag{13}$$

where $F(\cdot)$ is a hypergeometric function. Similarly [Eq. (B8)],

$$T(x^{2i})=\int_{-\infty}^{+\infty}dx\,x^{2i}|x|^{s-1}\exp\left(-\frac{x^2}{2}\right)=2^{i+s/2}\Gamma\left(i+\frac{s}{2}\right). \tag{14}$$

From Eqs. (13) and (14), either using Gauss's recursion functions [21,25] or integrating by parts, we deduce the following relations:

$$(s+2i)T(x^{2i})=T(x^{2i+2}), \tag{15}$$

$$(s+2i)\langle x^{2i},x^{2j+1}\rangle=\langle x^{2i+2},x^{2j+1}\rangle+\Gamma(s+i+j+\tfrac{1}{2}). \tag{16}$$

a. N odd. For N odd, Eqs. (15) and (16) and Eq. (A12) of Mehta and Normand [15], namely,

$$\det[\Gamma(s+i+j)]_{i,j=0,\ldots,N-1}=\prod_{j=0}^{N-1}j!\,\Gamma(s+j), \tag{17}$$

are used to express D_T from the second determinant appearing in Eq. (B15):

$$D_T=2^{(s+N-1)/2}\Gamma\left(\frac{s+N-1}{2}\right)\times\det\left[\frac{\Gamma(s+i+j+\frac{1}{2})}{s+2i}\right]_{i,j=0,\ldots,(N-3)/2}=2^{s/2}\Gamma\left(\frac{s}{2}\right)\prod_{j=0}^{(N-3)/2}j!\,\Gamma(s+j+\tfrac{1}{2}). \tag{18}$$

From Eq. (B14), we obtain finally for $N=2p+1$ [Re(S)>0].

$$I_{2p+1,1}^{+}(s,1)=2^{(N+s-1)/2}\frac{N!}{[(N-1)/2]!}\Gamma\left(\frac{s}{2}\right)\times\left(\prod_{j=1}^{(N-1)/2}j!\,\Gamma(s+j-\tfrac{1}{2})\right). \tag{19}$$

b. N even. For N even, using Eq. (16), Eq. (B16) becomes

$$I^{+}_{N,1}(s,1)=N!2^{N/2}\begin{vmatrix} & \left|\frac{\Gamma(s+i+j+\frac{1}{2})}{s+2i}\right|_{i=0,\dots,(N-4)/2,j=0,\dots,(N-2)/2} \\ & \vdots \\ \langle x^{N-2},x\rangle & \langle x^{N-2},x^3\rangle\cdots\langle x^{N-2},x^{N-1}\rangle \end{vmatrix}. \quad (20)$$

Denoting $n=(N-2)/2$, we replace the last column $C(n)$ by a linear combination of columns $C(j)$ $(0\leqslant j\leqslant n-1)$:

$$C'(n)=C(n)+\sum_{j=0}^{n-1}(-1)^{n-j}\binom{n}{j}\frac{\Gamma(s+n+\frac{1}{2})}{\Gamma(s+j+\frac{1}{2})}C(j). \quad (21)$$

The ith element of $C'(n)$ $(0\leqslant i\leqslant n-1)$ is

$$C'(i,n)=\frac{\Gamma(s+n+\frac{1}{2})}{s+2i}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{\Gamma(s+i+j+\frac{1}{2})}{\Gamma(s+j+\frac{1}{2})}\right), \quad (22)$$

which is zero, as the sum in Eq. (22) is equal to

$$\frac{d^i}{dx^i}\left[\sum_{j=0}^{n}(-1)^j\binom{n}{j}x^{s+j+1/2}\right]_{x=1}=\frac{d^i}{dx^i}[(1-x)^n x^{s+1/2}]_{x=1}. \quad (23)$$

The sole nonzero element, $C'(n,n)$, can be calculated from the Gauss relation [25] $(b-a)F(a,b;c;z)+aF(a+1,b;c;z)-bF(a,b+1;c;z)=0$, and from a relation which can, for instance, be proven by recurrence,

$$n!F(1+n,b;c;z)=\sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}F(1,b+j;c;z)\times\left(\prod_{k=1}^{n}(b+j-k)\right), \quad (24)$$

yielding

$$C'(n,n)=\frac{\Gamma(s+n+\frac{1}{2})}{s+2n}n!F\left(n+1,s+n+\tfrac{1}{2};\frac{s}{2}+n+1;\tfrac{1}{2}\right). \quad (25)$$

From this, integral (20) is finally found to be $(N=2p)$ [Re(S)>0]

$$I^{+}_{2p,1}(s,1)=2^{(N+s-1)/2}\frac{N!}{(N/2)!}\Gamma\left(\frac{s}{2}\right)\left(\prod_{j=1}^{N/2}j!\Gamma(s+j-\tfrac{1}{2})\right)\times\left(\frac{F(s/2,(1-s)/2;(s+N)/2;\frac{1}{2})}{\Gamma((s+N)/2)}\right), \quad (26)$$

a form which differs from the form of the Mellin transform found for odd N [Eq. (19)] because of the last factor.

2. Calculation of $I^{-}_{N,\sigma}(s,1)$ for the GOE (N even)

As $I^{-}_{N,\sigma}(s,1)=\sigma^{Ns+N(N-1)/2}I^{-}_{N,1}(s,1)$, we calculate $I^{-}_{N,1}(s,1)$, which is nonzero only for even N,

$$I^{-}_{N,1}(s,1)=\int_{R^N}\prod_{k=1}^{N}\left[dx_k\,\mathrm{sgn}(x_k)|x_k|^{s-1}\exp\left(-\frac{x_k^2}{2}\right)\right]\times\prod_{1\leqslant i<j\leqslant N}|x_i-x_j|. \quad (27)$$

It suffices to replace $\exp[-V(x)]$ by $\mathrm{sgn}(x)\exp[-V(x)]$ [Eq. (A26) of [15]] in the calculations of Sec. II C 1 to express integral (27). The inner product $\langle x^{2i},x^{2j+1}\rangle$ becomes [Re(S) >0]

$$\begin{aligned}\langle x^{2i},x^{2j+1}\rangle&=-2\int_0^{+\infty}dx\,x^{2j+s}\exp\left(-\frac{x^2}{2}\right)\times\int_x^{+\infty}dy\,y^{2i-1+s}\exp\left(-\frac{y^2}{2}\right)\\&=-\frac{\Gamma(s+i+j+\frac{1}{2})}{s+2j+i}\times F\left(1,s+i+j+\tfrac{1}{2};\frac{s}{2}+j+\tfrac{3}{2};\tfrac{1}{2}\right).\end{aligned} \quad (28)$$

The calculation then proceeds along the same lines as previously, interchanging the role of rows and columns. This yields

$$\det[\langle x^{2i},x^{2j+1}\rangle]_{i,j=0,\dots,(N-2)/2}=(-1)^{N/2}F\left(\frac{N}{2},s+\frac{N-1}{2};\frac{s+N+1}{2};\tfrac{1}{2}\right)\times\prod_{j=0}^{N/2-1}\left(\frac{j!\Gamma(s+j+\frac{1}{2})}{s+2j+1}\right) \quad (29)$$

and finally $(N=2p)$ [Re(S)>0]

$$I^{-}_{2p,1}(s,1)=(-1)^{N/2}2^{(N+s-2)/2}\frac{N!}{(N/2)!}\Gamma\left(\frac{s+1}{2}\right)\times\left(\prod_{j=1}^{N/2}j!\Gamma(s+j-\tfrac{1}{2})\right)\times\left[\frac{F((s+1)/2,1-s/2;(s+N+1)/2;\frac{1}{2})}{\Gamma((s+N+1)/2)}\right]. \quad (30)$$

3. Mellin transforms

To summarize, the Mellin transforms [Eq. (3)] of the determinant distributions are obtained from

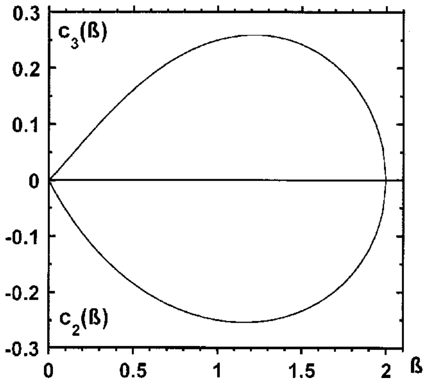

FIG. 1. Parameters $c_2(\beta)=b_2^{(\beta)}-\beta/2$ and $c_3(\beta)=b_3^{(\beta)}-\beta/2$ calculated numerically from $I_{3,1}^{+}(s,\beta)$ [Eq. (10)] and from Eq. (33).

$$M_N^{(1)+}(s)=\sigma^{N(s-1)}\left\{\frac{I_{N,1}^{+}(s,1)}{2I_{N,1}^{+}(1,1)}\right\}, \tag{31}$$

using Eqs. (19) and (26) for odd and even N, respectively, and from

$$M_{2p}^{(1)-}(s)=\sigma^{N(s-1)}\left\{\frac{I_{N,1}^{-}(s,1)}{2I_{N,1}^{+}(1,1)}\right\}, \tag{32}$$

using Eq. (30) for $N=2p$ while $M_N^{(1)-}(s)$ is zero for odd N. For $\beta=0,1,2$ and $N=2p+1$, the integral of Eq. (10) (+ case) can be alternatively written as

$$I_{2p+1,\sigma}^{+}(s,\beta)=2^{Ns/2}\pi^{N/2}\sigma^{[N(s-1)+N_p]}\times\prod_{m=1}^{N}\left(\frac{\Gamma(1+\beta m/2)\Gamma(s/2+b_m^{(\beta)})}{\Gamma(1+\beta/2)\Gamma(\frac{1}{2}+b_m^{(\beta)})}\right) \tag{33}$$

with $N_p=N+\beta N(N-1)/2$ and

$$b_1^{(\beta)}=0,\quad b_m^{(0)}=0,\quad b_m^{(1)}=\frac{1}{2}\left\lfloor\frac{m-1}{2}\right\rfloor+\frac{1}{4},\quad b_m^{(2)}=\left\lfloor\frac{m}{2}\right\rfloor \tag{34}$$

($b_m^{(2)}$ from [15]) ($m\geqslant 2$). Equation (33) reduces as expected to a Mehta integral [Eq. (17.6.7), p. 354 of [2]] for $s=1$. The determinant distribution $P_{2p+1}^{(1)}(D)$ given by Eq. (9) is finally deduced from Eqs. (31) and (33) with parameters $b_m^{(1)}$ as its Mellin transform is proportional to a product of Γ functions whose arguments are linear in s [see below Eq. (4) and also Eq. (3.4) of [15]]. Although the origin of Eq. (33) is understood only for some integer values of β, the question of its validity for any noninteger β ranging between 0 and 2 is naturally raised where the unknown parameters $b_j^{(\beta)}$ ($b_1^{(\beta)}=0$) would be assumed to be explicitly independent of N as in Eq. (34). We display in Fig. 1 the β dependence of the parameters $c_k(\beta)=b_k^{(\beta)}-\beta/2$ ($k=2,3$) which are obtained from a combination of a numerical calculation of $I_{3,1}^{+}(s,\beta)$ and Eq. (33).

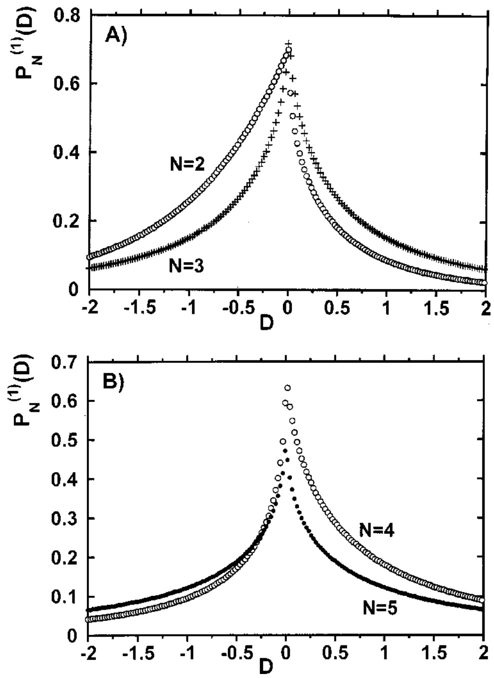

FIG. 2. Determinant distributions $P_N^{(1)}(D)$ from Monte Carlo simulations with 10^8 matrices ($\sigma=1$) for (a) $N=2$ (circles) and 3 (crosses) and (b) $N=4$ (empty circles) and 5 (solid circles).

4. Determinant distribution

A general expression for the determinant distribution has not been found for $P_{2p}^{(1)}(D)$ but the latter is in theory uniquely determined by an inversion of its Mellin transform. It is worth calculating exact distributions for $N=2$ and 3, as they may be of interest in relation to physical properties involving random matrices or random second-rank tensors in 2D or 3D. It is possible to derive an exact density for $N=2$ by different methods. The distribution of the determinant $D=H_{11}H_{22}-H_{12}^2$ of a 2×2 GOB matrix H_{ij} might, for instance, be obtained from a convolution of the distribution $P_2^{(0)}(D)$ given below Eq. (6) and a χ-square distribution. A simpler calculation, sketched in Sec. IV, uses the distribution of the determinant of the associated fixed-trace ensemble [27] to yield

$$P_2^{(1)}(D)=\frac{1}{\sigma^2\sqrt{2}}\exp\left(\frac{D}{\sigma^2}\right),\quad D\leqslant 0,$$

$$P_2^{(1)}(D)=\frac{1}{\sigma^2\sqrt{2}}\exp\left(\frac{D}{\sigma^2}\right)[1-\operatorname{erf}(\sqrt{2D/\sigma^2})],\quad D\geqslant 0, \tag{35}$$

where erf(x) is the usual error function [21]. Its moments are given in Appendix C [Eq. (C6)]. Monte Carlo simulation results [Fig. 2(a)] are in excellent agreement with Eq. (35).

For $N=3$, expressing $G_{0,3}^{3,0}(z|\cdot)$ ([20], p. 98) gives the determinant distribution as

$$P_3^{(1)}(D)=\frac{1}{\sigma^3\pi\sqrt{\pi}}G_{0,3}^{3,0}\left(\frac{D^2}{8\sigma^6}\middle|0,\frac{1}{4},\frac{3}{4}\right)$$
$$=\frac{1}{\pi\sigma^3}\left[(\sqrt{2\pi})_0F_2\left(;\frac{1}{4},\frac{3}{4};-\frac{D^2}{8\sigma^6}\right)-4\Gamma\left(\frac{3}{4}\right)\right.$$
$$\times\left|\frac{D}{2\sigma^3\sqrt{2}}\right|^{1/2}{}_0F_2\left(;\frac{1}{2},\frac{5}{4};-\frac{D^2}{8\sigma^6}\right)+\frac{8}{3}\Gamma\left(\frac{1}{4}\right)$$
$$\left.\times\left|\frac{D}{2\sigma^3\sqrt{2}}\right|^{3/2}{}_0F_2\left(;\frac{3}{2},\frac{7}{4};-\frac{D^2}{8\sigma^6}\right)\right]. \tag{36}$$

For numerical purposes, it is worth mentioning that the determinant distributions considered in the present paper can be calculated using Monte Carlo simulations to generate the $G_{0,N}^{N,0}(z|b_1,\ldots,b_N)$ function, as it is proportional to the distribution of the product of N independent gamma random variables [the jth variable has a density $f_j(x)\propto x^{b_j^{(\beta)}}\exp(-x)$] [19,20]. From the known asymptotic behavior of Meijer's G functions [20], $P_{2p+1}^{(1)}(D)$ varies as $|D|^{(N-1)(N-3)/4N}\times\exp(-N|D|^{2/N}/2\sigma^2)$ for large D while it is a constant whatever N for small values of D, as $b_1^{(\beta)}=0$ [20], p. 145.

Some determinant distributions $P_N^{(1)}(D)$ are shown in Fig. 2. The moments $\langle D^m\rangle_N^{(\beta)}$ of the determinant distributions are simply obtained from the Mellin transforms:

$$\langle D^{2k}\rangle_N^{(\beta)}=2M_N^{(\beta)+}(2k+1), \tag{37}$$

$$\langle D^{2k+1}\rangle_{2p}^{(\beta)}=2M_{2p}^{(\beta)-}(2k+2) \tag{38}$$

$(\langle D^{2k+1}\rangle_{2p+1}^{(\beta)}=0)$. Explicit expressions for the moments of the determinant distribution are given in Appendix C.

III. POTENTIAL STATISTIC

The moment generating function $E_N^{(\beta)}(t)$ of the distribution $g_N^{(\beta)}(V)$ of $V=\ln|D|$, the negative of the potential F at the origin (Sec. I), is related to the Mellin transform of the even part of the determinant distribution [Eq. (3)] by

$$E_N^{(\beta)}(t)=\int_{-\infty}^{+\infty}e^{Vt}g_N^{(\beta)}(V)dV$$
$$=2\int_0^{\infty}D^tP_N^{(\beta)+}(D)dD=2M_N^{(\beta)+}(t+1). \tag{39}$$

[$|t|<1$ in Eqs. (40) and (41)]. The moments are then calculated from $\langle V^k\rangle^{(\beta)}=[d^kE_N^{(\beta)}(t)/dt^k]_{t=0}$, while the cumulants $\kappa_k^{(\beta)}$ of order k [25] are deduced from the derivatives of the cumulant generating function $\ln[E_N^{(\beta)}(t)]$ at $t=0$, $\kappa_k^{(\beta)}=[d^k\ln[E_N^{(\beta)}(t)]/dt^k]_{t=0}$ The central moments of order 2 and 3 are equal to the cumulants of the same order, $\mu_k^{(\beta)}=\langle(V-\langle V\rangle^{(\beta)})^k\rangle=\kappa_k^{(\beta)}$ $(k=2,3)$.

Equation (31) extended to $\beta\neq1$ and Eqs. (33) and (34) yield the following cumulant generating function, which holds whatever N for $\beta=0$ [Eq. (5) and $\beta=2$ [Eq. (2.18) of [15]] and only for N odd, $N=2p+1$, for $\beta=1$:

$$E_N^{(\beta)}(t)=(2\sigma^2)^{Nt/2}\prod_{m=1}^{N}\left(\frac{\Gamma(t/2+\frac{1}{2}+b_m^{(\beta)})}{\Gamma(\frac{1}{2}+b_m^{(\beta)})}\right). \tag{40}$$

For N even, $N=2p$, $E_{2p}^{(1)}(t)$ is obtained from Eq. (26):

$$E_{2p}^{(1)}(t)=2^{t/2}\sigma^{Nt}F\left(\frac{1+t}{2},-\frac{t}{2};\frac{N+1+t}{2};\frac{1}{2}\right)\frac{\Gamma((1+t)/2)\Gamma((N+1)/2)}{\Gamma(\frac{1}{2})\Gamma((N+1+t)/2)}\prod_{m=1}^{p}\left(\frac{\Gamma(t+m+\frac{1}{2})}{\Gamma(m+\frac{1}{2})}\right). \tag{41}$$

In the diagonal case, the mean $\langle V\rangle^{(0)}$ and the variance $\mu_2^{(0)}$ are proportional to N as expected for a sum of iid random variables, being respectively equal to $(N/2)[\psi(\frac{1}{2})+\ln(2\sigma^2)]$, where $\psi(z)$ is the psi function [21,25], and $N\pi^2/8$ as deduced from Eq. (40). In that case, the asymptotic distribution of $|D|$ is a log-normal distribution from the central limit theorem [26].

In the Hermitian and real-symmetric cases, the mean $\langle V\rangle^{(\beta)}$ is

$$\langle V\rangle^{(\beta)}=\frac{N}{2}\ln(2\sigma^2)-\delta_{\beta1}\left(\frac{N-1}{2}\right)\ln2+\Sigma_1(p)+\delta_e$$
$$\times\left[-\frac{1}{2}\psi\left(\frac{N+1}{2}\right)+\delta_{\beta1}F_1(N)\right], \tag{42}$$

where $\Sigma_1(p)=\frac{1}{2}\psi(\frac{1}{2})+\Sigma_{j=1}^{p}\psi(j+\frac{1}{2}),\delta_{\beta1}$ is 1 for $\beta=1$ and 0 otherwise, δ_e is 1 if N is even and 0 otherwise, and

$$F_k(N)=\left[\frac{d^k\ln F((1+t)/2,-t/2;(N+1+t)/2;\frac{1}{2})}{dt^k}\right]_{t=0}.$$

The asymptotic mean $\langle V\rangle_\infty^{(\beta)}=-N(\frac{1}{2}+\ln2)$ is deduced whatever N and β from $\langle\ln|D|\rangle=N\langle\ln|\lambda|\rangle$, where the mean $\langle\ln|\lambda|\rangle$ is calculated from the asymptotic density of states $\rho_\infty(\lambda)$, which is a Wigner semi circle [1–3] of ''radius'' 1, $\rho_\infty(\lambda)=(2/\pi)\sqrt{1-\lambda^2}$ $(|\lambda|\leq1)$, when σ^2 is scaled so that $2N\beta\sigma^2=1$. The large-N mean obtained with the latter scaling up to terms of order $O(1/N)$ from Eq. (12) and Appendix D is

$$\langle V\rangle^{(\beta)}=-N(\tfrac{1}{2}+\ln2)+v^{(\beta)}+O\left(\frac{1}{N}\right). \tag{43}$$

where $v^{(1)}=(\ln2)/2$ and $v^{(2)}=0$.

The eigenvalue scaling obviously has no influence on the values of central moments and of cumulants of $\ln|D|$ for k

TABLE I. Comparison of some calculated moments and cumulants [Eqs. (42) and (44), $\beta=1$] with those obtained from Monte Carlo simulations with 5×10^5 and 2×10^5 real-symmetric matrices for $N=60$ ($\sigma^2=1/120$) and $N=101$ ($\sigma^2=1/202$), respectively.

	N	$\langle V\rangle^{(1)}$	$\mu_2^{(1)}$	$\kappa_3^{(1)}$	$\kappa_4^{(1)}$
Simulation	60	−71.242	5.118	−3.53	8.0
Calculation		−71.2449	5.1310	−3.5259	7.8688
Simulation	101	−120.160	5.639	−3.54	7.4
Calculation		−120.1629	5.6518	−3.5390	7.8695

$\geqslant 2$. Equations (19) and (41) for $\beta=1$ and Eq. (40) with the parameters $b_m^{(2)}=[m/2]$ for $\beta=2$ [15] yield the cumulants

$$\kappa_k^{(\beta)}=\frac{1}{2^k}\psi^{(k-1)}\left(\frac{1}{2}\right)+\alpha_k^{(\beta)}\left(\sum_{j=1}^{p}\psi^{(k-1)}(j+\tfrac{1}{2})\right)+\delta_e\left[-\frac{1}{2^k}\psi^{(k-1)}\left(\frac{N+1}{2}\right)+\delta_{\beta 1}F_k(N)\right] \quad (44)$$

for $k\geqslant 2$, with $\alpha_k^{(1)}=1$, $\alpha_k^{(2)}=2^{1-k}$, where the polygamma functions are defined by $\psi^{(k-1)}(z)=d^k\ln[\Gamma(z)]/dz^k$, $[\psi(z)=\psi^{(0)}(z)]$ [21,25]. Results of Monte Carlo simulations performed with real-symmetric matrices are in very good agreement with the calculated values (Table I). The asymptotic variances deduced from Eqs. (44) and (D6) are

$$\mu_2^{(1)}(\infty)=\log N, \quad \mu_2^{(2)}(\infty)=\tfrac{1}{2}\log N. \quad (45)$$

Equation (D7) gives asymptotic cumulants of order $k\geqslant 3$ independent of N:

$$\kappa_k^{(1)}(\infty)=\frac{1-2^{k-1}}{2^k}\psi^{(k-1)}(\tfrac{1}{2})-(k-1)\psi^{(k-2)}(\tfrac{1}{2}),$$

$$\kappa_k^{(2)}(\infty)=\frac{-(k-1)}{2^{k-1}}\psi^{(k-2)}(\tfrac{1}{2}). \quad (46)$$

The cumulants of order $k\geqslant 3$ of

$$U_N=\frac{\ln[(2\sqrt{e})^N|D|]}{\sqrt{\mu_2^{(\beta)}(\infty)}}=-\frac{F-N(\frac{1}{2}+\ln 2)}{\sqrt{\mu_2^{(\beta)}(\infty)}}$$

are asymptotically equal to 0. The asymptotic distribution of U_N is thus a standard Gauss distribution for $2N\beta\sigma^2=1$ but the convergence is slow (Fig. 3). Baker and Forrester [9] have shown that the potential statistic in circular ensembles also satisfies a central limit theorem as $N\to\infty$ with variances $(1/\beta)\ln N$ (actually valid for general rational β at least [9]) that are identical with those found here [Eq. (45)]. Repulsive interactions between eigenvalues and the resulting rigidity of the eigenvalue distribution produce a change of the width of the asymptotic Gaussian from being proportional to $N^{1/2}$ for uncorrelated eigenvalues for $\beta=0$ to proportional to $(\ln N)^{1/2}$ for correlated eigenvalues for $\beta=1,2$.

IV. SUMMARY AND EXTENSION TO OTHER RME'S

To summarize, the probability distributions $P_{2p+1}^{(\beta)}(D)$ are

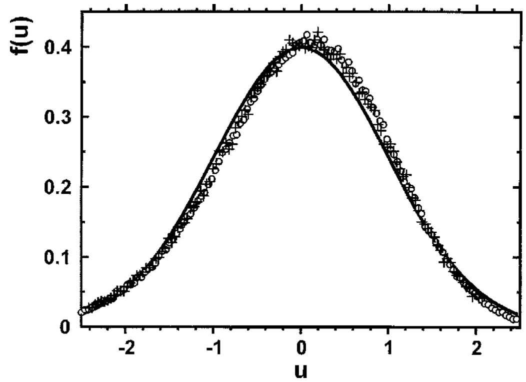

FIG. 3. Distribution $f(u)$ of $u=(\ln|D|-\langle V\rangle^{(1)})/\sqrt{\mu_2^{(1)}}$ ($\langle V\rangle^{(1)}$ and $\mu_2^{(1)}$ are given in Table I) from Monte Carlo simulations with 5×10^5 and 2×10^5 real-symmetric matrices for $N=60$ (empty circles) and $N=101$ (crosses), respectively, and standard Gauss distribution $(1/\sqrt{2\pi})\exp(-u^2/2)$ (solid line).

$$P_{2p+1}^{(\beta)}(D)=K_{2p+1}^{(\beta)}G_{0,2p+1}^{2p+1,0}\left(\frac{D^2}{(2\sigma^2)^N}\middle|0,b_j^{(\beta)},\ \ j=2,\ldots,2p+1\right), \quad (47)$$

$$b_1^{(\beta)}=0,\quad b_j^{(0)}=0,\quad b_j^{(1)}=\frac{1}{2}\left\lfloor\frac{j-1}{2}\right\rfloor+\frac{1}{4},\quad b_j^{(2)}=\left\lfloor\frac{j}{2}\right\rfloor$$

([15] for $\beta=2$), for $j\geqslant 2$. As shown in [27], the distributions found for the Gaussian ensembles can be further used to derive the determinant distributions of fixed-trace ensembles (FTE's) of random matrices of the same symmetry. Rosenzweig and Bronk (see [2] and references therein) defined fixed-trace ensembles by the condition that $\mathrm{tr}(H_N^+H_N)=\mathrm{const}$ with no other constraint (H_N^+ is the Hermitian conjugate of H_N). The latter constant is taken here as 1 without loss of generality. The probability densities $F_N^{(\beta)}(R)$ ($\beta=0,1,2$) of rescaled determinants R of such FTE's,

$$R=N^{N/2}\left[\prod_{k=1}^{N}\lambda_k\right], \quad (48)$$

for which $F_N^{(\beta)}(R)=0$ for $|R|\geqslant 1$ and the determinant densities of the associated Gaussian ensembles are shown to be related by [27]

$$P_N^{(\beta)}(D)=\frac{2}{\Gamma(N_p/2)}\int_{|D|^{1/N}}^{\infty}F_N^{(\beta)}(D/r^N)r^{N_p-N-1}\exp(-r^2)dr \quad (49)$$

for $\sigma=1/\sqrt{2}$ in Eq. (1) and N_p given above Eq. (34). Equation (35) ($\beta=1$, $N=2$), for instance, is derived from Eq. (49) with $F_2^{(1)}(R)=1/2\sqrt{2(1+R)}$ for $|R|<1$. Conversely, the distributions $F_{2p+1}^{(\beta)}(R)$ are obtained from Eqs. (47) and (49) as single Meijer G functions [27] namely,

$$F_{2p+1}^{(\beta)}(R)=D_{2p+1}^{(\beta)}G_{2p+1,2p+1}^{2p+1,0}\times\left(R^2\left|\begin{matrix}\frac{\beta p}{2}+\frac{(j-1)}{(2p+1)}, & j=1,\ldots,2p+1\\ 0,b_j^{(\beta)}, & j=2,\ldots,2p+1\end{matrix}\right.\right). \tag{50}$$

Results of Monte Carlo simulations of the distributions of the determinant of such FTE's for $N\leqslant 11$, to be reported in [27], which are in excellent agreement with the theoretical distributions Eq. (50), as well as the results reported in Table I for $\beta=1$ which agree with Eqs. (42) and (44) provide supplementary confirmation of Eq. (47).

To conclude, the distributions of determinants of diagonal, real-symmetric, and Hermitian [15] random matrices from Gaussian ensembles have complicated forms in most cases. They are, however, conveniently represented by their Mellin transforms, which are known exactly for any finite N for $\beta=0$, 1, 2. The asymptotic Gaussian distributions found for the potential statistic of the log gases associated with the GOE and GUE have variances $(1/\beta)\ln N$ identical with those of the corresponding circular ensembles [9] and fixed-trace ensembles [27].

APPENDIX A: $b_j^{(2)}$

From Eqs. (3) and (4), the Mellin transform $M_N^{(2)+}(s)$ for s real positive is proportional to the normalization constant Z_N of the eigenvalue distribution $P_2(\lambda_1,\ldots,\lambda_N)$ of the unitarily invariant RME:

$$P_2(H_N)\propto|\det(H_N)|^{s-1}\exp[-\mathrm{tr}(H_N^2)] \tag{A1}$$

named the generalized Gaussian ensemble in [13,14]. Z_N is calculated with the classical method of orthogonal polynomials [2]. The structure of the Vandermonde determinant is used to reexpress the eigenvalue distribution $P_2(\lambda_1,\ldots,\lambda_N)$,

$$\begin{aligned}P_2(\lambda_1,\ldots,\lambda_N)&=\frac{1}{Z_N}\exp\left[-\left(\sum_{i=1}^{N}\lambda_i^2\right)\right]\prod_{j=1}^{N}|\lambda_j|^{s-1}\prod_{1\leqslant i<j\leqslant N}(\lambda_i-\lambda_j)^2\\&=\frac{1}{N!}\det(A_NA_N^T),\end{aligned} \tag{A2}$$

as a determinant where the elements of the matrix A_N depend on the considered polynomials [see Eq. (5.2.10) of [2]]. The orthogonal polynomials here are generalized Hermite polynomials [22], p. 156 $H_n^{(\mu)}(x)$ with $\mu=(s-1)/2$, for which

$$\int_{-\infty}^{+\infty}H_n^{(\mu)}(x)H_m^{(\mu)}(x)|x|^{2\mu}\exp(-x^2)dx=h_n\delta_{mn} \tag{A3}$$

with [22]

$$h_n=2^{2n}\left\lfloor\frac{n}{2}\right\rfloor!\,\Gamma\left(\left\lfloor\frac{n+1}{2}\right\rfloor+\frac{s}{2}\right). \tag{A4}$$

The elements of A_N are then $h_n^{-1/2}H_n^{(\mu)}(x)\exp(-x^2/2)$. As monic polynomials are $P_n^{(\mu)}(x)=2^{-n}H_n^{(\mu)}(x)$, the identity between the normalization constants that appear in the two previous expressions for $P_2(\lambda_1,\ldots,\lambda_N)$ [Eq. (A2)] yields Z_n as proportional to the product $\Pi_{n=0}^{N-1}h_n$. The parameters $\lfloor j/2\rfloor$ $(j=1,\ldots,N)$ which are obtained from Eq. (A4) for $j=n+1$ are equal to the $b_j^{(2)}$ [15].

APPENDIX B: $I_N^{(V)}$ [Eq. (11)]

The Vandermonde determinant can be written as

$$\begin{aligned}\prod_{1\leqslant j<i\leqslant N}(x_i-x_j)&=\det(x_j^{i-1})_{1\leqslant i,j\leqslant N}\\&=\det[P_{i-1}(x_j)]_{1\leqslant i,j\leqslant N},\end{aligned} \tag{B1}$$

where $P_{i-1}(x_j)$ is a monic polynomial of degree $i-1$, that is, with a coefficient of the leading term x^{i-1} of 1. The integral

$$I_N^{(V)}=\int_{R^N}\prod_{k=1}^{N}dx_k\exp[-V(x_k)]\prod_{1\leqslant i<j\leqslant N}|x_i-x_j| \tag{B2}$$

is calculated for odd N with the method of alternate variables [2], whose principle is to integrate first over odd variables x_{2k+1} $[k=0,\ldots(N-1)/2]$. The resulting integral can be expressed in terms of functions $\varphi_i(x)$ defined as

$$\varphi_i(x)=\int_{-\infty}^{x}P_i(t)\exp[-V(t)]dt. \tag{B3}$$

Integrating with respect to $x_1,x_3,\ldots,x_N$, we write

$$I_N^{(V)}=N!\int_{-\infty<x_2<x_4<\cdots<x_{N-1}<+\infty}\det[m_{ij}]_{i,j=0,\ldots,N-1}\times\prod_{k=1}^{(N-1)/2}dx_{2k}\exp[-V(x_{2k})] \tag{B4}$$

with $x_0=-\infty$, $x_{N+1}=+\infty$, and

$$m_{i,2k}=P_i(x_{2k}), \tag{B5}$$

$$m_{i,2k+1}=\varphi_i(x_{2k+2})-\varphi_i(x_{2k}) \tag{B6}$$

for $0\leqslant i\leqslant N-1$ and $0\leqslant k\leqslant(N-1)/2$. All second terms in the odd-numbered columns [right member of Eq. (B6)] are eliminated by successive column additions, resulting in

$$I_N^{(V)}=N!\int_{-\infty<x_2<x_4\cdots<x_{N-1}<+\infty}\prod_{k=1}^{(N-1)/2}\exp[-V(x_{2k})]dx_{2k}\times\det\begin{vmatrix}\varphi_0(x_2) & P_0(x_2)\cdots\varphi_0(x_{N-1}) & P_0(x_{N-1}) & \varphi_0(+\infty)\\ & \vdots & & \\ \varphi_{N-1}(x_2) & P_{N-1}(x_2)\cdots\varphi_{N-1}(x_{N-1}) & P_{N-1}(x_{N-1}) & \varphi_{N-1}(+\infty)\end{vmatrix} \tag{B7}$$

We denote as $T(P_i)$ the linear application

$$T(P_i)=\varphi_i(+\infty)=\int_{-\infty}^{+\infty}P_i(t)\exp[-V(t)]dt. \tag{B8}$$

An expansion of the determinant in Eq. (B7) with respect to the last column yields a sum of N integrals:

$$I_N^{(V)}=N!\sum_{k=1}^{N}(-1)^{k-1}T(P_{k-1})M_{k-1} \tag{B9}$$

with

$$M_{k-1}=\int_{-\infty<x_2<x_4<\cdots<x_{N-1}<+\infty}\Delta_{k-1}\times\prod_{n=1}^{(N-1)/2}dx_{2n}\exp[-V(x_{2n})], \tag{B10}$$

where Δ_{k-1} is the minor obtained after deleting the kth row and the last column of the full determinant in Eq. (B7). Using the symmetry and restoring an integration over independent variables makes it possible to express every M_{k-1} with the help of a Pfaffian of a skew-symmetric $(N-1)\times(N-1)$ matrix [2,23]. To obtain a final compact expression, we define an inner product of functions f and g as

$$\langle f,g\rangle=\tfrac{1}{2}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{x}dy[f(y)g(x)-f(x)g(y)]\times\exp[-V(x)-V(y)]. \tag{B11}$$

The inner product is zero when f and g are either both even or both odd. If one function is even and the other one is odd, then

$$\langle f,g\rangle=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{x}dy\, f(y)g(x)\exp[-V(x)-V(y)]. \tag{B12}$$

We define next monic skew-orthogonal polynomials $R_i(x)$ [2,14,23,24]:

$$\langle R_m,R_n\rangle=r_kZ_{mn}$$

with

$$Z_{mn}=\begin{cases}1 & \text{if } m=2k,\ n=2k+1\\ -1 & \text{if } m=2k+1,\ n=2k \quad (k=0,1,2,\ldots)\\ 0, & \text{otherwise.}\end{cases} \tag{B13}$$

The polynomials $R_m(x)$ are even when $m=2k$ while $R_{2k+1}(x)$ can be chosen to be odd. When P_m is chosen as $P_m(x)=R_m(x)$, the last line of the antisymmetric matrix associated with M_{k-1} is zero for $k<N$ and thus $M_{k-1}=0$. From results proven in [2,14,23,24], the elements of the last line are indeed $2\langle R_{N-1},R_m\rangle$ for $0\leqslant m\leqslant N-1$ $(m\neq k-1)$, which is zero from Eq. (B12). The only nonzero minor is therefore M_{N-1}:

$$M_{N-1}=(\det[g_{ij}]_{i,j=0,\ldots,N-2})^{1/2}$$

with $g_{ij}=2\langle R_i,R_j\rangle=2r_kZ_{ij}$, that is, $g_{ij}=0$ except for $g_{2n,2n+1}=-g_{2n+1,2n}=2r_n$ for $n=0,\ldots,(N-3)/2$. Consequently,

$$I_N^{(V)}=N!2^{(N-1)/2}T(R_{N-1})\left(\prod_{k=0}^{(N-3)/2}r_k\right)=N!2^{(N-1)/2}D_T. \tag{B14}$$

Using linear combinations of rows and columns of the second matrix in Eq. (B15), we deduce for odd N

$$D_T=\begin{vmatrix}\langle R_0,R_1\rangle & \langle R_0,R_3\rangle\cdots\langle R_0,R_{N-2}\rangle & T(R_0)\\ \langle R_2,R_1\rangle & \langle R_2,R_3\rangle\cdots\langle R_2,R_{N-2}\rangle & T(R_2)\\ & \vdots & \\ \langle R_{N-1},R_1\rangle & \langle R_{N-1},R_3\rangle\cdots\langle R_{N-1},R_{N-2}\rangle & T(R_{N-1})\end{vmatrix}=\begin{vmatrix}\langle x^0,x^1\rangle & \langle x^0,x^3\rangle\cdots\langle x^0,x^{N-2}\rangle & T(x^0)\\ \langle x^2,x^1\rangle & \langle x^2,x^3\rangle\cdots\langle x^2,x^{N-2}\rangle & T(x^2)\\ & \vdots & \\ \langle x^{N-1},x^1\rangle & \langle x^{N-1},x^3\rangle\cdots\langle x^{N-1},x^{N-2}\rangle & T(x^{N-1})\end{vmatrix}$$

$$=\begin{vmatrix}\langle P_0,P_1\rangle & \langle P_0,P_3\rangle\cdots\langle P_0,P_{N-2}\rangle & T(P_0)\\ \langle P_2,P_1\rangle & \langle P_2,P_3\rangle\cdots\langle P_2,P_{N-2}\rangle & T(P_2)\\ & \vdots & \\ \langle P_{N-1},P_1\rangle & \langle P_{N-1},P_3\rangle\cdots\langle P_{N-1},P_{N-2}\rangle & T(P_{N-1})\end{vmatrix}. \tag{B15}$$

The P_m polynomials that appear in the last matrix are constrained to be monic polynomials of degree m that are even polynomials when $m=2k$ and odd polynomials when $m=2k+1$. When N is even, $I_N^{(V)}$ [Eq. (B2)] is [2,14,15,23]

$$I_N^{(V)}=N!2^{N/2}\begin{vmatrix} \langle P_0,P_1\rangle & \langle P_0,P_3\rangle\cdots\langle P_0,P_{N-1}\rangle \\ \langle P_2,P_1\rangle & \langle P_2,P_3\rangle\cdots\langle P_2,P_{N-1}\rangle \\ & \vdots \\ \langle P_{N-2},P_1\rangle & \langle P_{N-2},P_3\rangle\cdots\langle P_{N-2},P_{N-1}\rangle \end{vmatrix}. \tag{B16}$$

with the same constraints as above on P_m polynomials.

APPENDIX C: MOMENTS OF THE DETERMINANT DISTRIBUTION $P_N^{(1)}(D)$

For $\beta=1$, even moments ($k\geqslant1$) are found from Eqs. (19), (26), (30), and (37) to be

$$\langle D^{2k}\rangle_{N=2p+1}^{(1)}=\sigma^{2Nk}\frac{2^k\Gamma(k+\frac{1}{2})}{\sqrt{\pi}}\prod_{j=1}^{2k}\left(\frac{\Gamma(j+N/2)}{\Gamma(j+1/2)}\right), \tag{C1}$$

$$\langle D^{2k}\rangle_{N=2p}^{(1)}=\sigma^{2Nk}\frac{2^k\Gamma((N+1)/2)\Gamma(k+\frac{1}{2})}{\Gamma(k+(N+1)/2)\sqrt{\pi}}\times F\left(k+\tfrac{1}{2},-k;k+\frac{N+1}{2};\tfrac{1}{2}\right)\times\prod_{j=1}^{2k}\left(\frac{\Gamma(j+(N+1)/2)}{\Gamma(j+\frac{1}{2})}\right). \tag{C2}$$

Odd moments, which are nonzero only for even N, are calculated from Eqs. (26), (30), and (38) ($k\geqslant0$):

$$\langle D^{2k+1}\rangle_{N=2p}^{(1)}=(-1)^{N/2}\sigma^{N(2k+1)}\times\frac{2^k\Gamma((N+1)/2)\Gamma(k+\frac{3}{2})}{\Gamma(k+(N+3)/2)\sqrt{\pi}}\times F\left(k+\tfrac{3}{2},-k;k+\frac{N+3}{2};\tfrac{1}{2}\right)\times\prod_{j=1}^{2k+1}\left(\frac{\Gamma(j+(N+1)/2)}{\Gamma(j+\frac{1}{2})}\right). \tag{C3}$$

The variances $V_N^{(1)}=\langle D^2\rangle_N^{(1)}-(\langle D\rangle_N^{(1)})^2$ are calculated from the previous moments:

$$V_{N=2p+1}^{(1)}=\frac{4\sigma^{2N}(N+2)}{3\pi}\left[\Gamma\left(\frac{N+2}{2}\right)\right]^2, \tag{C4}$$

$$V_{N=2p}^{(1)}=\frac{\sigma^{2N}N(2N+5)}{3\pi}\left[\Gamma\left(\frac{N+1}{2}\right)\right]^2 \tag{C5}$$

Their ratio tends to 1 when $p\to\infty$. For $N=2$, the moments can be calculated directly from Eq. (35), using lemma 6.1 of [28]:

$$\langle D^m\rangle_2^{(1)}=(-1)^m\frac{m!\sigma^{2m}}{\sqrt{\pi}}\left\{\sum_{n=0}^{m}(-1)^n\frac{\Gamma(n+1/2)}{n!}\right\}. \tag{C6}$$

That is,

$$\langle D^m\rangle_2^{(1)}\approx(-1)^m\frac{m!\sigma^{2m}}{\sqrt{2}}$$

for large m. Moments of the determinant distribution are given by Mehta and Normand [15] for the GUE case and

$$\langle D^{2k}\rangle_N^{(0)}=(2\sigma^2)^{Nk}\frac{\Gamma^N(k+1/2)}{\pi^{N/2}} \tag{C7}$$

for $\beta=0$ for any N.

APPENDIX D: ASYMPTOTIC BEHAVIOR OF $\Sigma_k^{(\beta)}(p)$ ($p=\lfloor N/2\rfloor$, $\beta=1,2$)

In addition to terms discussed below, the mean [Eq. (42)] and the kth order cumulant [Eq. (44)] may include a sum $-(1/2^k)\psi^{(k-1)}[(N+1)/2]+\delta_{\beta1}F_k(N)$, where $-\psi^{(k-1)}[(N+1)/2]/2^k$ is $-\frac{1}{2}\ln(N/2)+O(1/N)$ for $k=1$ and $[(-1)^{k-1}(k-2)!]/2N^{k-1}+O(1/N^k)$ for $k\geqslant2$ [25], while

$$F_k(N)=\left[\frac{d^k\ln F((1+t)/2,-t/2;(N+1+t)/2;\frac{1}{2})}{dt^k}\right]_{t=0}$$

is $O(1/N)$ for large N. The asymptotic behaviors of the mean [Eq. (42)] and the cumulants [Eq. (44), $k\geqslant2$] are then obtained from that of

$$\Sigma_k^{(\beta)}(p)=\frac{1}{2^k}\psi^{(k-1)}(\tfrac{1}{2})+\alpha_k^{(\beta)}\left(\sum_{j=1}^{p}\psi^{(k-1)}(j+\tfrac{1}{2})\right) \tag{D1}$$

with $\alpha_k^{(1)}=1$, $\alpha_k^{(2)}=2^{1-k}$. For $k=1$, $\psi(x+1)=\psi(x)+1/x$ [25] allows the deduction

$$\sum_{j=1}^{p}\psi(j+\tfrac{1}{2})=p\psi(\tfrac{1}{2})-p+(2p+1)\left(\sum_{m=1}^{2p}\frac{1}{m}-\frac{1}{2}\sum_{m=1}^{p}\frac{1}{m}\right). \tag{D2}$$

From [21,25],

$$\sum_{m=1}^{n}\frac{1}{m}=C+\ln n+\frac{1}{2n}+O\left(\frac{1}{n^2}\right),$$

where $C\approx0.577\,21$ is the Euler constant, and we deduce finally that

$$\Sigma_1^{(\beta)}(p)=\frac{(2p+1)}{2}\ln p-p+O\left(\frac{1}{p}\right). \tag{D3}$$

From [25],

$$\psi^{(k-1)}(j+\tfrac{1}{2})=(-1)^k(k-1)!2^k\left[\sum_{m=j}^{\infty}\frac{1}{(2m+1)^k}\right], \tag{D4}$$

and we obtain

$$\sum_{j=1}^{p}\psi^{(k-1)}(j+\tfrac{1}{2})=-\tfrac{1}{2}\psi^{(k-1)}(\tfrac{1}{2})+(-1)^k(k-1)!2^{k-1}\times\left(\sum_{m=0}^{p}\frac{1}{(2m+1)^{k-1}}\right)+(-1)^k(k-1)!2^{k-1}(2p+1)\times\left(\sum_{m=p+1}^{\infty}\frac{1}{(2m+1)^{k}}\right). \quad \text{(D5)}$$

The second term in the right-hand side of Eq. (D5) is $\ln p-\psi(\frac{1}{2})+O(1/p)$ for $k=2$ and $-(k-1)\psi^{(k-2)}(\frac{1}{2})+O(1/p)$ for $k\geqslant 3$, while the contribution of the third term, which is of the order of $(-1)^k(k-2)/p^{k-2}$ for large p, can be neglected in the expansion considered except for $k=2$. Finally,

$$\Sigma_2^{(1)}=\ln p+1+C+2\ln 2-\frac{\pi^2}{8}+O\left(\frac{1}{p}\right),$$
$$\Sigma_2^{(2)}=\tfrac{1}{2}\ln p+\frac{1+C+2\ln 2}{2}+O\left(\frac{1}{p}\right), \quad \text{(D6)}$$

and

$$\Sigma_k^{(1)}(p)=\frac{1-2^{k-1}}{2^k}\psi^{(k-1)}(\tfrac{1}{2})-(k-1)\psi^{(k-2)}(\tfrac{1}{2})+O\left(\frac{1}{p}\right),$$
$$\Sigma_k^{(2)}(p)=\frac{-(k-1)}{2^{k-1}}\psi^{(k-2)}(\tfrac{1}{2})+O\left(\frac{1}{p}\right) \quad \text{(D7)}$$

for $k\geqslant 3$.

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