Numerical verification of bona fide stochastic resonance

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The notion of stochastic resonance as an input/output synchronization mechanism is reestablished quantitatively, based on accurate numerical simulation of a simple two-state model, namely, the Schmitt trigger. The corresponding phenomenological criterion known as bona fide stochastic resonance is thus proven applicable throughout the entire range of the input parameters. Claims to the contrary are briefly discussed.

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I. INTRODUCTION

Stochastic resonance (SR) is an intriguing phenomenon of nonlinear physics, where an optimal amount of noise has the capability of enhancing the rate of synchronous switches between the local minima $\pm y_m$ of a bistable system driven by a weak periodic modulation (or forcing signal) with period T_Ω and amplitude A_0 [1]. Stochastic resonance rests upon a noise controlled synchronization mechanism which can be vividly illustrated in terms of switch-time distributions. The most convenient choice is provided by the distribution $N_1(T)$ of the residence times, commonly defined as the time intervals between any two subsequent switches in *opposite* directions [see Fig. 1(a)]. Since the modulation favors the switching events by perturbing the symmetry of the system, $N_1(T)$ peaks at (close to) $T_n=(n-1/2)T_\Omega$ with $n=1,2,\ldots$; the input/output synchronization sets in when the first peak of $N_1(T)$ (at $T_\Omega/2$) dominates the remaining peaks and the exponential random-switch background; each distribution peak exhibits its own maximum for a certain value of the noise strength given by the phenomenological condition [2]

$$T_n\mu_0(D)\simeq 1, \tag{1}$$

where $\mu_0(D)$ is the Kramers escape rate out of one stable minimum, D denotes the noise strength, and n labels the N_1 peak centered at T_n. Equation (1) for $n=1$ corresponds to the optimal synchronization (or SR) condition, which can be attained on varying either Ω or D. Such a characterization of the SR phenomenon is often referred to as bona fide SR [2]. It should be noticed that in the more popular spectral representation [1], no SR condition sets in as a function of the forcing frequency, the only control parameter being the noise strength. This difference makes the notion of bona fide SR particularly attractive to experimentalists.

Recently, the interpretation of SR as a bona fide resonance has been questioned [3] on two accounts. (a) For *strongly* driven devices $A_0y_m/D\gg 1$ the peak resonance condition (1) would be inaccurate; as a consequence, doubts are cast on the validity of the underlying resonant synchronization argument, mostly motivated by an approximate analysis of the escape process in a periodically tilted bistable potential (where the A_0 dependence of the escape rates between tilted minima is neglected [4]). (b) For *weakly* driven devices $A_0y_m/D\ll 1$, the switch-time distribution peaks become hardly detectable, so that a synchronization condition would make no sense. This is not the SR regime best investigated experimentally in the current literature [6,7]; nevertheless, the limit of vanishingly small input amplitudes may have a bearing on the SR phenomenology in that it embodies the approximations leading to the linear response theory for the process under study [1].

In order to sort out the ensuing controversy we planned an extensive simulation project aimed at gathering high statistics residence time distributions for the archetypal two-state model, namely, the symmetric Schmitt trigger (Sec. II). Preliminary results for the strongly driven regime (and details of our simulation code) were published in Refs. [4] and [5]. In the present paper we focus on the case of a weakly driven trigger where criticism (b) seems to be more serious. This work should be regarded as a numerical experiment, whose outcome provides a benchmark for further analytic studies. For the time being, we conclude that numerical simulation does vindicate the notion of bona fide SR; moreover, the synchronization condition (1) can be implemented quantitatively more effectively than originally presented [2].

II. THE SCHMITT TRIGGER

The symmetric Schmitt trigger (ST) provides the simplest example of a bistable system exhibiting SR [6]. Our numerical investigation was limited to such a two-state model, although the conclusions discussed below are of wider application to the theory of SR. As a major advantage with respect to the continuous bistable systems studied in the earlier literature [1], the output of a ST is entirely controlled by the switching mechanism, whereas in a continuous bistable system interwell and intrawell dynamics are hard to unravel.

Typically, the ST [8,9] input consists of two components, whose amplitudes greatly depend on the experimental circumstances: (i) a noisy signal with zero mean and finite correlation time; (ii) one or more embedded periodic signals with arbitrary wave forms. Let us consider for simplicity input signals $x(t)$ of the form

$$x(t)=\xi(t)+A_0\sin(\Omega t+\phi), \tag{2}$$

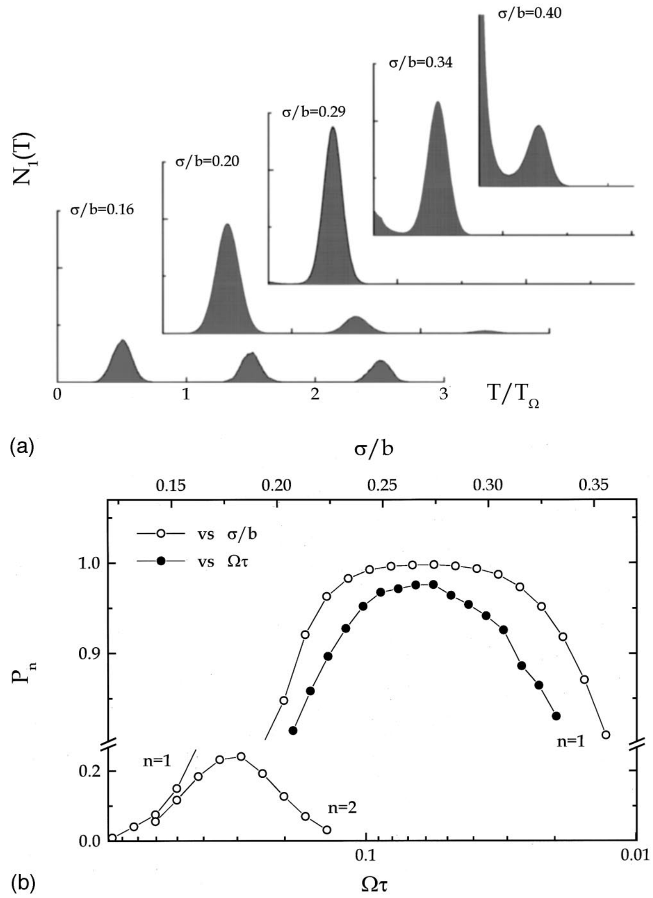

FIG. 1. (a) Normalized residence time distributions $N_1(T)$ (arbitrary units) for $\Omega\tau = 1.25\times10^{-2}$, $A_0/b = 0.5$, and different values of σ/b. (b) Peak strengths P_1 and P_2 versus σ/b (at $\Omega\tau = 1.25\times10^{-2}$) and P_1 versus $\Omega\tau$ (at $\sigma/b = 0.37$) for $\alpha = 0.25$ and after background subtraction. The nth peak strength P_n is defined explicitly in Eq. (6). Time and signal scales: $\tau = 2\times10^{-5}$ s and $b = 200$ (arbitrary units). The choice of the σ/b axis origin and the orientation of the logarithmic $\Omega\tau$ axis are motivated by purely typographical reasons.

where $\xi(t)$ is a zero-mean stationary Gaussian noise with intensity (i.e., standard deviation) σ, strength $D = \sigma^2\tau$, and autocorrelation function

$$\langle \xi(t)\xi(0)\rangle = \sigma^2 e^{-|t|/\tau}. \tag{3}$$

Throughout this work $\xi(t)$ is assumed to be short-time correlated in comparison with the modulation, that is, $\Omega\tau \ll 1$ (steady-state limit). The trigger output rests in state $-$ as long as the input $x(t)$ is smaller than a threshold value b. As $x(t)$ crosses b the trigger switches (almost) instantaneously into state $+$ and remains there as long as $x(t) > -b$. The trigger output is a dichotomic signal with values $\pm y_m$; in the following y_m is set to 1 for convenience. Of course, the modulation of the input signal (2) drives a periodic output component $\langle y(t)\rangle$ with period $T_\Omega = 2\pi/\Omega$.

In the *weakly driven* regime $A_0/b \ll (\sigma/b)^2 \ll 1$ the trigger switches from the $\mp$ to the $\pm$ state are noise-assisted random events that occur with time constants $T_\mp(A_0) \simeq T_0(b)$, where $T_0(b)$ is the spontaneous switch time [4]

$$T_0(b) = \frac{\tau\sqrt{\pi}}{1+\Phi(\bar{b})} \int_{-\infty}^{\bar{b}} e^{x^2}[1+\Phi(x)]^2 dx, \tag{4}$$

with $\Phi(x) = (2/\sqrt{\pi})\int_0^x e^{-z^2}dx$ and $\bar{b} = b/\sqrt{2\sigma^2}$. As the noise intensities are small, $\sigma \ll b$, Eq. (4) may be approximated to $T_0(b) = \tau\sqrt{2\pi}(\sigma/b)\exp(b^2/2\sigma^2)$. Note that in this limit $\tau \ll T_0(b)$.

In the adiabatic limit $(A_0 b/\sigma^2)\Omega T_0(b) \ll 1$ [10], the amplitude $\langle y\rangle$ of the periodic component of the trigger output reads

$$\langle y\rangle = y_m\left(\frac{A_0 b}{\sigma^2}\right)\frac{\mu_0}{\sqrt{\Omega^2+\mu_0^2}} \tag{5}$$

with $\mu_0 = 2/T_0(b)$. This leads to the popular spectral characterization of SR [1], where the system response exhibits a maximum only as a function of the noise intensity σ (at fixed Ω).

The synchronization mechanism underlying the SR phenomenon has been investigated extensively in the opposite regime of *strongly driven* devices [1] only; for a ST this corresponds to choosing $(\sigma/b)^2 \ll A_0/b \ll 1$ [6,7]. On looking at the residence time distributions of Fig. 1(a), it is apparent that the strength P_n of the nth N_1 peak,

$$P_n(\sigma,\Omega) = \int_{T_n-\alpha T_\Omega}^{T_n+\alpha T_\Omega} \tilde{N}_1(T)dT, \quad n = 1,2,\ldots, \tag{6}$$

with $0 < \alpha \leqslant 1/4$ [2], exhibits SR behavior as a function of both σ (at fixed Ω) and Ω (at fixed σ). The σ and Ω dependence of P_1 is shown in Fig. 1(b). Here, the (negligibly small) exponential random-switch background has been first blown up by taking the logarithm of the residence time distribution and then subtracted by means of a standard linear fitting algorithm; the *subtracted* distribution is denoted by $\tilde{N}_1(T)$. Note that our background subtraction procedure has been carried out in the presence of modulation, although in the weakly driven regime the background does not depend appreciably on the modulation amplitude [1].

In view of the scales chosen for σ/b and $\Omega\tau$ in Fig. 1(b), the time constant matching condition (1) implied by SR as well as the existence of a maximum synchronization distribution $N_1(T)$ in Fig. 1(a) are apparent. Therefore, as detailed in [4], for a strongly driven subthreshold ST the scheme of bona fide SR is sound both qualitatively and quantitatively—i.e., the criticisms of Ref. [3] do not apply.

III. SIMULATION RESULTS

The input/output synchronization effect in a weakly driven ST is, of course, much harder to quantify. In Fig. 2 the residence time distribution $N_1(T)$ is plotted for two small values of the input amplitude A_0 and fixed noise intensity σ. It is apparent that the N_1 peak structure tends to merge into the random-switch background, to the point that for the smallest amplitude we managed to simulate, $A_0/b = 0.025$, the fraction of synchronous switches amounts to a few percent of the sample. In order to appreciate the magnitude of the synchronization effect, we verified first that the random-switch background closely fits an exponential curve (Fig. 2,

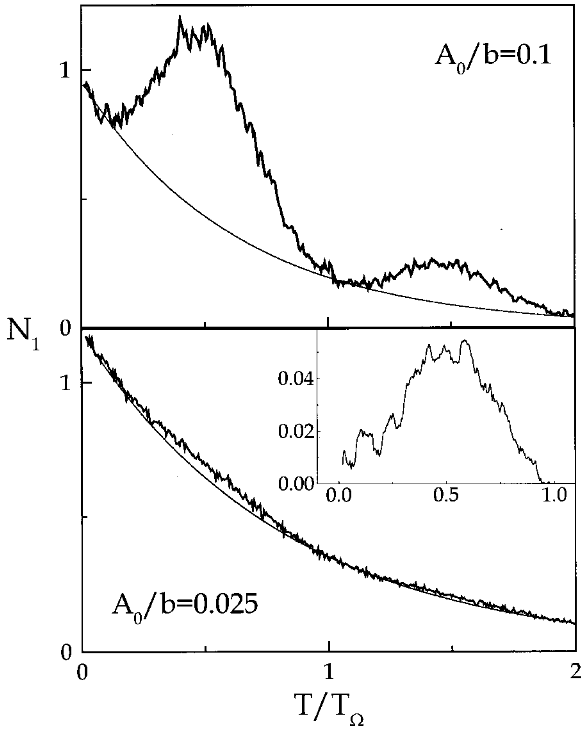

FIG. 2. Residence time distributions $N_1(T)$ for $A_0b/\sigma^2=1.0$ (top), 0.3 (bottom). Other simulation parameter values are as in Fig. 1. Inset: subtracted distribution $\tilde{N}_1(T)$; the area enclosed under the numerical curve corresponds to P_1. The solid lines represent the fitted exponential curves of our subtraction procedure.

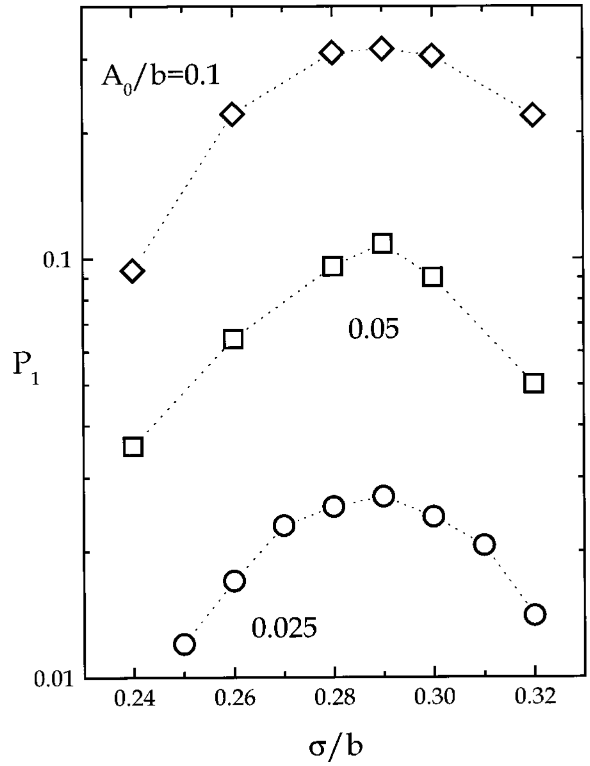

FIG. 3. Curves of P_1 versus σ/b for different values of A_0/b and $\Omega\tau=1.25\times10^{-2}$. Other simulation parameter values are as in Fig. 1. Note that each value of P_1 has been computed from the corresponding switch-time distribution after subtraction.

top panel) and then we subtracted it from $N_1(T)$ so as to extract the residual peak structure $\tilde{N}_1(T)$ (Fig. 2, inset); finally, the first peak strength P_1 was computed according to our definition (6). For this subtraction procedure to make sense, one not only requires very high statistics [our $N_1(T)$ distributions were computed over samples of up to 10^5 recorded switches], but also stipulates the exponential separability of peaks from background (in our numerical experiment such a property was checked on a run-by-run basis).

The dependence of P_1 on the noise intensity is plotted in Fig. 3 for three values of the forcing amplitude. A few important properties of the synchronization mechanism in the weakly driven regime $A_0b/\sigma^2\ll1$ can easily be pointed out by inspection: (i) after background subtraction, all curves of P_1 versus σ/b attain a distinct maximum, even for vanishingly small values of A_0/b; (ii) within the accuracy of our numerics, the position of the SR maxima is rather insensitive to the value of A_0/b; (iii) the maxima of the curves $P_1(\sigma)$ depend *quadratically* on A_0 [as suggested also by Eq. (40) of Ref. [3]]. Properties (i) and (ii) suggest that the synchronization based definition of SR is consistently applicable, at least in principle, no matter how small the input amplitude. The observation that the SR value of the noise intensity tends to a unique limit for $A_0/b\rightarrow0$ is compatible with the linear response theory description of the SR phenomenon [perhaps more familiar to the reader from the spectral representation of Eq. (5)].

The dependence of P_1 on the input frequency Ω, plotted in Fig. 4, reveals a few further properties of the SR synchronization mechanism. (i) The peaks of the curves $P_1(\sigma/b)$ shift toward lower σ/b values with decreasing Ω, as expected. However, the peak values of $P_1(\sigma/b)$ decrease monotonically with Ω—of course, $P_1<1$ for any choice of σ and Ω. This trend confirms the spectral representation result that the synchronization effect underlying SR is most pronounced at vanishingly low forcing frequencies [1]. (ii) In the upper inset of Fig. 4 P_1 is plotted versus $\Omega\tau$ for the

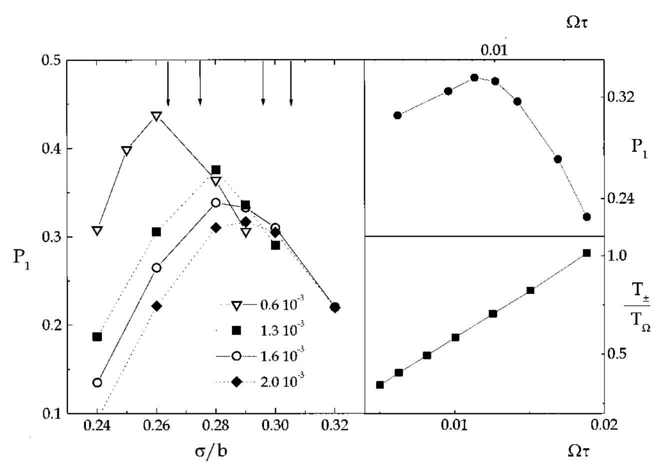

FIG. 4. Curves of P_1 versus σ/b for different values of $\Omega\tau$ and $A_0/b=0.1$. Other simulation parameter values are as in Fig. 1. Vertical arrows denote the optimal value of σ/b according to condition (1) on increasing $\Omega\tau$ from the left to the right. Upper inset: peak strength P_1 versus $\Omega\tau$. Lower inset: escape times $T_\pm$ versus Ω. In both insets the noise intensity σ/b was set to 0.29 (estimated position of the P_1 peaks in Fig. 3). The magnitude of the statistical error falls within the data-point thickness.

(constant) σ/b value corresponding to the maxima of P_1 versus σ/b in Fig. 3. As the maximum of this curve shifts toward lower frequencies, it is clear that under the conditions of Fig. 3 the synchronization rule (1) is not exact. However, our simulation shows that condition (1) works better and better as the $P_1(\sigma/b)$ peaks shift toward lower σ/b values, namely, at low frequencies. (iii) In the lower inset of Fig. 4 we plotted $T_\pm/T_\Omega$ versus $\Omega\tau$ for the same simulation parameter values as in the upper inset (the definition of $T_\pm$ is given in Sec. II). As the strength P_1 versus Ω attains its maximum, the ratio $T_\pm/T_\Omega$ approaches 1/2, as required by Eq. (1).

The results of Fig. 4 lead to a better assessment of the synchronization condition (1). In contrast to Ref. [3] (see Sec. IV E there) we find that the notion of bona fide SR applies reasonably well throughout the parameter value range explored numerically in the current literature. The synchronization rule (1) is strictly obeyed at low frequencies, namely, when the $P_1(\sigma/b)$ peaks pile up in the low noise intensity domain and the quantity A_0b/σ^2 tends to grow larger than unity (strongly driven devices [4]). For practical uses, we notice that condition (1) works better to locate the maxima of P_1 versus Ω (at fixed σ/b), than vice versa. Finally, it should be pointed out that Eq. (37) of Ref. [3] fails, too, to reproduce this complicated dependence of P_1 on σ and Ω. [For the ST of Eqs. (3) and (4) Choi *et al.* would have concluded that P_1 was a function of α and β alone, with $\alpha = A_0b/\sigma^2$ and $\beta = \mu_0/\Omega$.]

IV. DISCUSSION

The results of the foregoing section clearly support the notion of bona fide SR. We make now a few concluding remarks.

(1) As pointed out in Ref. [3], the N_1 peak strengths P_n do grow vanishingly small for $A_0/b \ll 1$. However, the subtraction of the exponential background allowed us to show that P_1 (and also the remaining P_n) decreases quadratically with A_0/b. It should be noticed that this is precisely the same difficulty we are confronted with in spectral representation. There one must compute the subtracted power spectral density of the output $y(t)$ at the forcing frequency; the strength of the δ-like spectral spike corresponding to the periodic output component $\langle y(t)\rangle$ [see Eq. (5)], turns out to be quadratic in A_0/b [1]. The applicability of the bona fide SR criterion to everyday laboratory practice is confirmed by recent experimental reports also [11].

(2) We chose to simulate SR in a modulated ST because this problem is numerically more tractable than the continuous bistable process addressed in Ref. [3]. This corresponds to filtering out the unimportant details of the intrawell dynamics so that high statistics switch-time distributions become accessible by means of a personal computer. We believe that such a simplification does not affect the validity of our conclusions. The output of a continuous bistable process can anyway be fed through a two-state filter, thus producing a stochastic signal that is hardly distinguishable from the output of a subthreshold ST.

(3) To our knowledge, there are no existing theories that account for the de-synchronization mechanism that occurs at high noise intensities. For instance, on increasing σ/b above the relevant SR value, the switch phase distributions develop a doubly peaked profile, being now dominated by in-phase and π out-of-phase switch events. This numerical observation, reported in Ref. [5], calls for an even more systematic theory of the residence time distributions in the presence of a periodic modulation.

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[10] For the *adiabatic condition*, the starting point is represented by the ansatz $\partial_t p(x,t) = -\mu p(x,t)$ for the time relaxation of the process probability density $p(x,t)$; hence, $p(x,t) = e^{-\mu t}p(x)$. Suppose now that a time perturbation is added that leaves the structure of the problem eigenvalues unchanged, i.e., such that $\{\mu_n\} \to \{\mu_n(t)\}$. This may happen only under the condition that $|\dot{\mu}/\mu^2| \ll 1$, so that $\partial_t p(x,t) = -(\mu + \dot{\mu}t)p(x,t) \simeq -\mu(t)\exp[-\mu(1-\dot{\mu}/\mu^2)t]p(x) \simeq -\mu(t)p(x,t)$. The restriction $\alpha\beta \ll 1$ is implicit also in Eq. (37) of Ref. [3], where, to make contact with the parameters of the present problem, $\alpha = A_0b/\sigma^2$ and $\beta = \mu_0/\Omega$.

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