Stable solitons of quadratic Ginzburg-Landau equations

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We present a physical model based on coupled Ginzburg-Landau equations that supports *stable* temporal solitary-wave pulses. The system consists of two parallel-coupled cores, one having a quadratic nonlinearity, the other one being effectively linear. The former core is active, with bandwidth-limited amplification built into it, while the latter core has only losses. Parameters of the model can be easily selected so that the zero background is stable. The model has nongeneric exact analytical solutions in the form of solitary pulses (''dissipative solitons''). Direct numerical simulations, using these exact solutions as initial configurations, show that they are unstable; however, the evolution initiated by the exact unstable solitons ends up with nontrivial stable localized pulses, which are very robust attractors. Direct simulations also demonstrate that the presence of group-velocity mismatch (walkoff) between the two harmonics in the active core makes the pulses move at a constant velocity, but does not destabilize them.

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I. INTRODUCTION

Propagation of localized pulses (solitons) in many nonlinear physical systems is governed by various form of the complex Ginzburg-Landau (GL) equations which include dispersive and nonlinear effects in both conservative and dissipative forms [1–3]. The GL equation and its different modifications describe various effects in laser physics [4], fluid dynamics [5] and nonlinear optics [6]. Different types of solutions to the one-dimensional complex GL equation, such as pulselike, shocklike, sources, sinks, and periodic and quasiperiodic solutions, have been analyzed. A fundamental issue, from the standpoint of possible applications, is the stability relative to small perturbations of the localized pulselike solutions of these complex GL equations. The simplest form of this kind of nonlinear Schrödinger equation with frequency and intensity-dependent gain and loss is the cubic GL equation which includes only cubic terms. This equation has been analyzed mainly in the context of plasma physics [7] and its exact solitary wave solutions are known to be unstable in general. Later on, the cubic-quintic GL equation was put forward as it admits stable pulselike solutions [8–11].

Recently, a great deal of attention has been attracted to solitons in optical media with a quadratic ($\chi^{(2)}$) nonlinearity that consist of two mutually locked components, viz., the fundamental (FH) and second harmonic (SH) fields [12]. Experimental observations of the solitons in the spatial [13], temporal [14], and spatiotemporal [15] domains were reported. Because any optical medium has intrinsic losses, stable propagation or circulation of solitons requires a compensating gain. The action of pure losses on $\chi^{(2)}$ solitons was considered in [16], and adiabatic amplification of spatial solitons was analyzed numerically in [17]. Recently, an adiabatic perturbation theory was developed for quadratic soliton propagation in nonconservative media [18]. Actually, the losses are insignificant for the spatial solitons, as the size of the experimental sample is usually much smaller than the attenuation length [13]. Recently, robust *temporal* solitons have been predicted in an externally pumped $\chi^{(2)}$cavity with parametric gain and an additional cubic nonlinearity present [19]. Other situations can be expected too where there is no synchronized pump but the solitons circulate in a cavity, accumulating loss and gain effects as a result of many round trips, thus making them important. Moreover, for narrow temporal solitons, circulating in the cavity, dispersive losses due to the finite amplifier bandwidth will be especially important.

The objective of this work is to analyze a physical configuration with a quadratic nonlinearity, losses, and compensating gain, modeled by linearly coupled one-dimensional complex GL equations, that gives rise to *stable* temporal solitons. The model is introduced, and its special exact solitary wave solutions are obtained, in Sec. II. In Sec. III we present detailed numerical simulations of the soliton stability. The results of this work are briefly summarized in the final section.

II. THE MODEL AND ITS EXACT SOLITARY PULSELIKE SOLUTIONS

First of all, it is natural to assume that dominating losses are those at SH, while losses at FH are negligible (note that the most prominent contribution to the losses, the Rayleigh scattering, has its intensity decaying as λ^{-4} with the wavelength). On the other hand, the amplifier (that we assume to be integrated with the nonlinear $\chi^{(2)}$ crystal, by means of doping the crystal with a resonant impurity, which is pumped by an external source of light [20]) should operate at FH, as, otherwise, the amplification will be inefficient (for quadratic solitons, the SH component is usually weaker than the FH one). So, we adopt a model combining losses at SH and bandwidth-limited gain at FH.

It is well known that solitons in the models of this type are unstable, because the linear gain makes the trivial zero state, i.e., the soliton's background, unstable (see, e.g., [21]). A general and experimentally feasible way to stabilize the background instability was proposed in the context of solitons in optical fibers with a cubic nonlinearity and bandwidth-limited amplification [22]: the nonlinear waveguide has to be linearly coupled to a parallel core with sufficiently strong losses. Because the coupling decays exponentially with the inverse wavelength, it is reasonable to take into regard only the coupling in the FH component. Then, the linearly coupled complex GL equations that are expected to admit stabilized solutions takes the form

$$iA_z+(1/2)DA_{\tau\tau}+A^*B=i\gamma_0A+i\gamma_1A_{\tau\tau}+KA', \quad (1)$$

$$iB_z-i\delta B_\tau+(\sigma/2)DB_{\tau\tau}-\beta B+2A^2=-i\Gamma_0B+i\Gamma_1B_{\tau\tau}, \quad (2)$$

$$iqA'_z+(i\alpha+\nu)A'=KA. \quad (3)$$

Here A, B, and A' are, respectively, the FH and SH amplitudes in the main core and the FH amplitude in the added lossy one; z and τ are, as usual, the propagation distance and reduced time; $D>0$ is the coefficient of the chromatic dispersion at FH; δ is a group-velocity mismatch between FH and SH; σ is the relative SH/FH dispersion coefficient; β is the phase-velocity mismatch; γ_0 is the FH gain; an effective filtering coefficient γ_1 accounts for the finite size of the gain band; Γ_0 and Γ_1 control the losses at SH; K is the FH coupling constant; while ν, q, and α are the mismatch, propagation, and loss constants in the added core. The parameters σ and β may have any sign, while all the parameters on right-hand side of the equations are positive (Γ_1 may also be zero). We set, by means of obvious rescaling, $D\equiv K\equiv 1$, which implies that all the lengths are measured in units of the intercore coupling length. An explicit dependence of the solutions, to be obtained below, on the size of the coupling constant can be readily analyzed by reversing the rescaling.

The model assumes that the losses (along with the possible phase-velocity mismatch) dominate in the additional core, while the dispersion, nonlinearity, and dispersive losses may be neglected in it. Note that two versions of the model with the cubic ($\chi^{(3)}$) nonlinearity, one neglecting these terms and the other one taking them all into account, were numerically studied and compared in full detail in Refs. [22]. It was found that there is no essential difference, whatsoever, between the properties of exact solitary pulses in the two versions of the model. There is no reason to assume that the situation would be drastically different in the case of the $\chi^{(2)}$ nonlinearity.

As concerns the group-velocity mismatch between FH and SH in the main core, accounted for by the parameter δ in Eq. (2), it is well known that for conservative second-harmonic generation (SHG) the corresponding term can always be eliminated by means of a formal phase transformation. Physically, a sufficient suppression of the group-velocity mismatch between the FH and SH waves is a crucial condition that must be met in order to provide for the creation of SHG solitons, and there are various experimental techniques that make it possible to achieve this purpose [12,14]. However, including dispersive losses and gain, a formal elimination of δ is impossible, which also has a direct physical purport: it is harder to provide for the full SHG resonance in the presence of losses and gain. On the other hand, if a stable balance between losses and gain is possible for a soliton, it may then help to stabilize the soliton against various nondissipative perturbations (such as the group-velocity mismatch) which would otherwise be able to destroy it. Thus, the parameter δ is quite important in the model, and it will be taken into regard in the numerical simulations reported below.

Lastly, our approach implies one more assumption, viz., that the inverse group-velocity mismatch c between FH waves in the active and passive cores is negligible, too. To estimate physical conditions that justify the latter assumption, we note that temporal $\chi^{(2)}$ solitons dealt with in experiments [14] have a very small temporal width, $T\sim 50$ fs, and they propagate at a strong anomalous FH dispersion, $D\sim 1$ ps^2/m, which gives the soliton's dispersion length $z_D\sim T^2/D\sim 1$ mm. On the other hand, the inclusion of the parallel core produces a nontrivial effect if the corresponding coupling length, z_c [$\sim 1/K$, in terms of Eqs. (1) and (3)], is about z_D. We note that it is quite possible technologically to fabricate a dual-core waveguide with $z_c\sim 1$ mm; this circumstance makes the proposed scheme realistic, because $\chi^{(2)}$ optical crystals available for the experiment have a size of a few cm [14,15]. Now, Eq. (3) tells us that the walkoff length has to be much larger than the coupling length, i.e., $c\ll T/z_c\sim 50$ ps/m. It is necessary to add that, in reality, there is no need to make the two cores of different materials—it is sufficient to have the cores identically doped, only the active one being externally pumped. In this case, the cores have identical refractive indices and thus the pulses propagate with almost identical group velocities if the shape of the cores is similar. To conclude the discussion of the possible experimental implementation of the proposed scheme, we note that the analysis of the fluctuations of the coupling constant induced by inevitable irregularities in the separation between the cores shows that solitons have a strong immunity against such fluctuations [23].

Proceeding to the mathematical analysis of the model, we notice that the first necessary condition for soliton stability is the stability of the zero background, $A=B=A'=0$. The derivation of this condition is straightforward: one substitutes into the linearized equations a small perturbation of the form $A,A'\sim\exp(gz-i\omega\tau)$, $B\sim\exp(2gz-2i\omega\tau)$, where ω is an arbitrary frequency of the perturbation and λ is the instability growth rate. This yields a dispersion equation,

$$qg^2+(\alpha-q\gamma_0+q\gamma_1\omega^2-i\nu+\tfrac{1}{2}iq\omega^2)g+\left\{1-\alpha\gamma_0+\left(\alpha\gamma_1+\frac{\nu}{2}\right)\omega^2+i\left[\gamma_0\nu+\left(\frac{\alpha}{2}-\nu\gamma_1\right)\omega^2\right]\right\}=0. \quad (4)$$

Obviously, the necessary stability condition is $\mathrm{Re}\,g(\omega)\leq 0$ for all real ω. The corresponding algebraic problem following from Eq. (4) can be easily solved in a numerical form. A typical background-stability region on the parametric plane (α, γ_0) at fixed values of other parameters is shown in Fig. 1. One can check that a corollary of the stability condition is

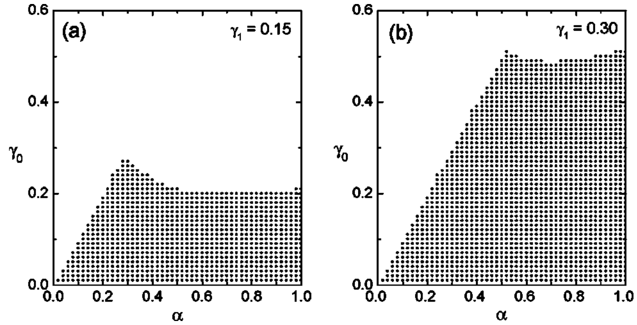

FIG. 1. The stable-background domain for $q=1$, $\sigma=2$, $\Gamma_0=0.2$, $\Gamma_1=0.1$, $\gamma_1=0.15$ (a), and $\gamma_1=0.3$ (b).

$\alpha>\gamma_0$, i.e., the loss in the added core must be *stronger* than the linear gain in the main one.

A remarkable property of Eqs. (1)–(3) is that they admit an *exact* solitary-pulse solution in a form suggested by the works [7,24,25] provided that the walkoff is disregarded ($\delta=0$),

$$A=a[\mathrm{sech}(\lambda\tau)]^{2+i\mu}e^{ikz}, \quad B=b[\mathrm{sech}(\lambda\tau)]^{2+2i\mu}e^{2ikz},$$

$$A'=a'[\mathrm{sech}(\lambda\tau)]^{2+i\mu}e^{ikz}, \tag{5}$$

with the pulse's inverse width λ, *chirp* μ, and wave-number shift k. These parameters are real, while the amplitudes a, a', and b may be complex. However, a can be made real too through a trivial phase shift, therefore the complex unknowns are a' and b. Substituting the ansatz (5) into Eqs. (1)–(3), we arrive at five complex equations for four real unknowns λ, μ, k, a and two complex ones a' and b. Obviously, a solution may exist if two constraints are imposed on the eight real parameters of the model (σ, β, γ_1, Γ_0, Γ_1, q, α, and ν; recall that we are now dealing with the case $\delta=0$). First, we can derive a relatively simple equation for the chirp:

$$4\mu^4+30d\mu^3-80\mu^2-90d\mu+36=0, \tag{6}$$

where $d\equiv\frac{1}{2}(\epsilon-4\gamma_1)/(1+\epsilon\gamma_1)$ and $\epsilon\equiv\sigma/\Gamma_1$. Assuming that real solutions to Eq. (6) have been found, we set $\tau_1\equiv 90\mu-30\mu^3$ and cast the subsequent results into the form

$$\lambda=[\Gamma_0/(4\Gamma_1-4\sigma\mu-4\Gamma_1\mu^2)]^{1/2},$$

$$b=\lambda^2(\tfrac{1}{2}-i\gamma_1)(2+i\mu)(3+i\mu),$$

$$a=(\lambda/\sqrt{2})[(\tfrac{1}{2}\sigma-\Gamma_1)(2+2i\mu)(3+2i\mu)b]^{1/2},$$

$$a'=a/[(\nu-qk)+i\alpha],$$

and

$$\beta=-2k+2\lambda^2(\sigma+4\mu\Gamma_1-\sigma\mu^2).$$

Additionally, as both $|a|$ and λ must be real, the parameters of the model have to fulfill some inequalities, $\tau_1(\Gamma_1+\sigma\gamma_1)>0$ and $\Gamma_0/(4\Gamma_1-4\sigma\mu-4\Gamma_1\mu^2)>0$.

Introducing $r_1\equiv(2+4\gamma_1\mu-\frac{1}{2}\mu^2)\lambda^2$, $r_2\equiv(-4\gamma_1+2\mu+\gamma_1\mu^2)\lambda^2$, and $r_3\equiv\gamma_0-r_2$, the equation for k can be written, after some algebra, as

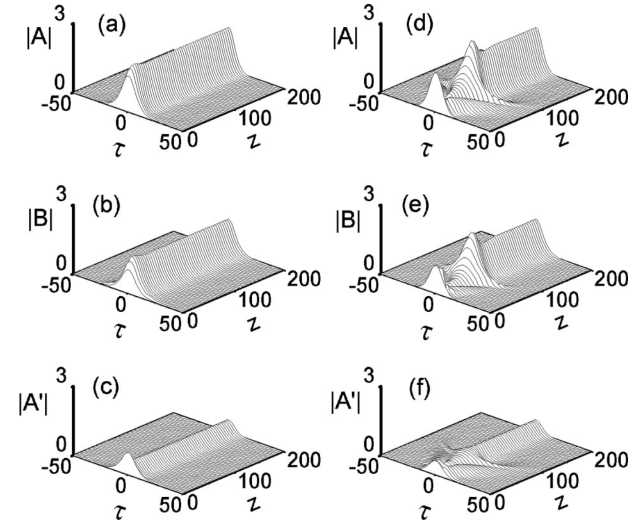

FIG. 2. The evolution of the exact pulse [(a), (b), and (c)] and of the Gaussian pulse [(d), (e), and (f)] in the absence of walkoff. Here $q=1$, $\gamma_0=0.3$, $\gamma_1=0.3$, $\Gamma_0=0.2$, $\Gamma_1=0.1$, $\sigma=1.4$, $\alpha=0.330\,06$, $\beta=-1.557\,89$, and $\nu=0$. The Gaussian pulse has the amplitudes $a_0=1.7$, $b_0=1.2$, $a_0'=0.6$, and the inverse width $\lambda_0=0.12$.

$$qk^3-(\nu+2qr_1)k^2+(2\nu r_1+qr_1^2+qr_3^2-1)$$
$$\times k-(r_1^2\nu+r_3^2\nu-r_1)=0. \tag{7}$$

This equation has one or three real roots for k. For each root, $\alpha=r_3(\nu-qk)/(r_1-k)$. Thus, the exact solitary-pulse solution exists, provided that the values of the mismatch β and loss α are selected by the above constraints.

III. STABILITY OF PULSELIKE SOLITARY WAVES

The background stability is not sufficient for the full dynamical stability of the solitary pulse. Therefore, we have numerically tested the stability of the exact solution found above, arriving at results which are different from those for the similar model with the cubic nonlinearity [22]: the analytical solution is *unstable* in *all* the cases considered, but there is another *stable* pulse in all the cases, including those when the exact solution does not exist (recall that the exact pulse solution was not generic, depending on two extra conditions; in contrast with this, the new numerically found stable pulse appears to be a fairly generic solution). We observed that the initial exact pulse reshapes into the new stable pulse, a typical example being displayed in Fig. 2 [panels (a), (b), and (c)]. The same stable pulse can also be generated using, instead of the exact analytical solution, an input in the form of rather arbitrary Gaussians: $A=a_0\exp(-\lambda_0^2\tau^2)$, $B=b_0\exp(-\lambda_0^2\tau^2)$, and $A'=a_0'\exp(-\lambda_0^2\tau^2)$, with some a_0, b_0, a_0', and λ_0 [see panels (d), (e), and (f) in Fig. 2]. It should be mentioned here that such stable pulses can be generated from a variety of inputs, including, e.g., sech^2 intensity profile input pulses. Thus, we conclude that the nontrivial stationary pulses are very robust *attractors*. It is noteworthy that the rearrangement of the Gaussian into the stable pulse is much more violent than in the case of the exact initial pulse, see Fig. 2, i.e., the exact pulse, although being unstable, is rather close in shape to the stable one. We have studied in detail the structure of the stable pulse. Its real and

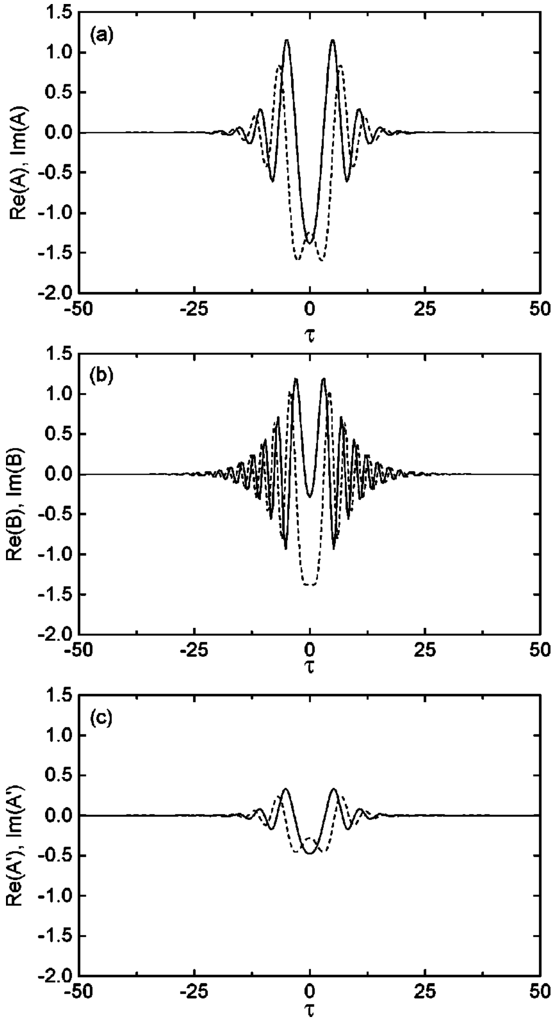

FIG. 3. The real and imaginary parts (full and dashed lines) of the stable-pulse solution in the case corresponding to Fig. 2. Here $z=200$.

imaginary parts are even functions of τ, and it is strongly chirped, see a typical example in Fig. 3.

Now considering the generic case, when the exact pulse solution is not available, the stable pulse can be obtained by launching a Gaussian solely in the FH component, the two other ones being empty, see Fig. 4. Because of the bistability, there is a threshold for the generation of the stable pulse. In Figs. 4(a) and 4(b), we show that the same stable pulse is generated from different initial Gaussians, provided that their intensity is above the threshold. Figure 4(c) shows that, with an insufficient intensity, i.e., below the threshold, the input decays into the zero solution.

The present model is bistable: coexisting attractors are the nontrivial pulse and the stable zero solution. According to the general principles, a *separatrix*, in the form of an unstable stationary solution, must exist between two attractors, cf. a similar situation found in Refs. [22]. It is obvious that our unstable exact pulse is, when it exists, just the separatrix. In the generic case, when the exact solution is not available, an unstable soliton must exist, too. It can be found from a numerical solution of the stationary version of Eqs. (1)–(3), but is of no interest.

As it was mentioned above, it is quite important to include in the consideration the group-velocity mismatch (walkoff) between FH and SH, accounted for by the parameter δ in Eq. (2). Our simulations have demonstrated that the pulse that was stable at $\delta=0$ readily survives in the presence of the walkoff, just acquiring some constant velocity. Note that in the case of conservative quadratically nonlinear media the properties of this kind of moving solitons (usually termed *walking solitons*) were analyzed systematically in Ref. [26]. In Fig. 5 [panels (a)–(c)] we show the contour plots of the evolution of the walking soliton generated by the initial condition in the form of the exact (for the case $\delta=0$) pulse

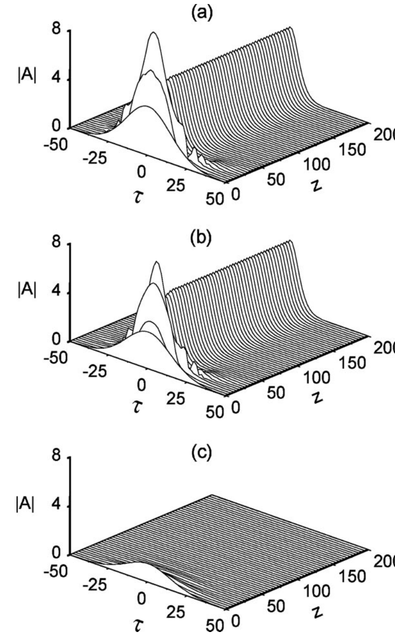

FIG. 4. Evolution of the Gaussian launched in the FH component in the absence of walkoff. The initial amplitudes are $a_0=4$ (a), $a_0=3$ (b), and $a_0=1.5$ (c). The inverse width of the initial pulse is $\lambda_0=0.05$, and the other parameters are $q=1$, $\gamma_0=0.3$, $\gamma_1=0.2$, $\Gamma_0=0.2$, $\Gamma_1=0.2$, $\sigma=3.1$, $\alpha=0.45$, and $\beta=-20$. Only the evolution of the FH absolute value is shown.

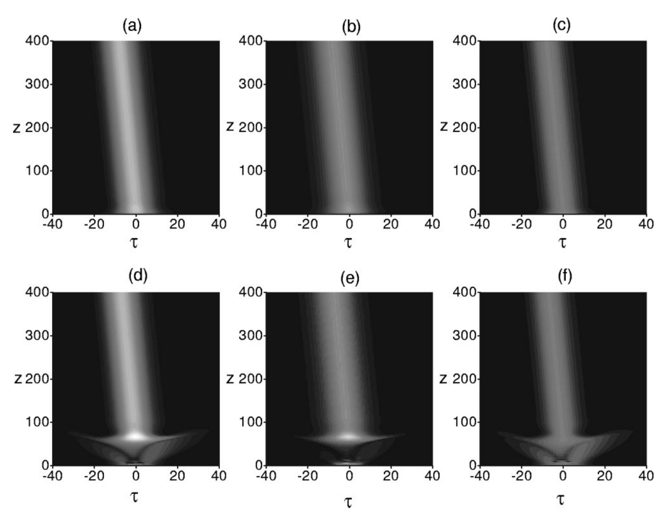

FIG. 5. The gray-scale plots of the evolution of the exact pulse [(a)$|A|$, (b) $|B|$, and (c) $|A'|$] and of the Gaussian pulse [(d) $|A|$, (e) $|B|$, and (f) $|A'|$] for the same parameters as in Fig. 2 but with a nonzero walkoff parameter $\delta=0.2$.

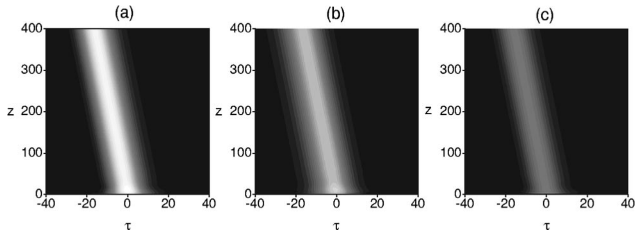

FIG. 6. The gray-scale plots of the evolution of the exact pulse for the same parameters as in Fig. 2 but for the walkoff parameter $\delta=0.4$. (a) $|A|$,(b) $|B|$, and (c) $|A'|$.

solution (5) in the presence of a moderate mismatch $\delta=0.2$. Recall that the results of simulations of the same case but with $\delta=0$ were displayed above in Figs. 2(a)–2(c). The only essential difference introduced by the nonzero group-velocity mismatch is a finite velocity of the soliton (represented by the slope clearly seen in Fig. 5). More detailed analysis of the numerical data shows that, while the moving soliton becomes asymmetric, its amplitude and energy are practically identical to those of the quiescent one shown for the same values of the parameters and $\delta=0$ in Fig. 2. Moreover, the simulations with $\delta\neq 0$, starting with the Gaussian input, rather than the former exact pulse solution, produce exactly the same stable moving solitary wave, see Figs. 5(d)–5(f).

The evolutions of the exact pulse solution (5) corresponding to the parameters given in the caption of Fig. 2 and of the Gaussian pulse launched only in the FH component corresponding to the parameters given in the caption of Fig. 4(a), at still larger values of the walkoff parameter $\delta=0.4$, are shown in Figs. 6 and 7, respectively. Numerical measurement of the amplitude and energy of the emerging stable moving solitons again yields results nearly coinciding with those found in the absence of walkoff and for the same values of the other parameters.

Detailed numerical studies revealed that for the walkoff parameter $\delta\lesssim 4$ the solitons maintain their integrity. So, by increasing the walkoff parameter, we found that the propagation displays, after a transient regime, persistent oscillations similarly to those reported for quadratic solitons in nondissipative systems [27]. The outcomes of our simulations for $\delta=1$ and $\delta=2$ are shown in Figs. 8 and 9, respectively. One should note that the greater the walkoff parameter, the longer is the transient regime. Moreover, the soliton transverse velocity increases with the walkoff parameter δ. Thus, the stable pulses supported by the present system are rather

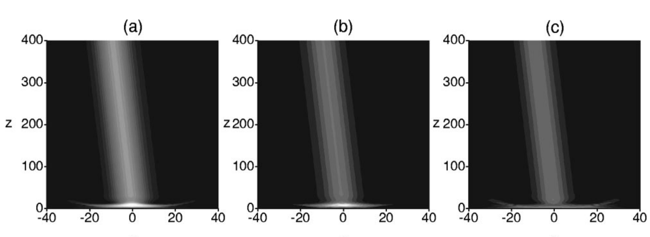

FIG. 7. The gray-scale plots of the evolution of a Gaussian pulse launched in the FH component for the same parameters as in Fig. 4(a) but for the walkoff parameter $\delta=0.4$. (a) $|A|$, (b) $|B|$, and (c) $|A'|$.

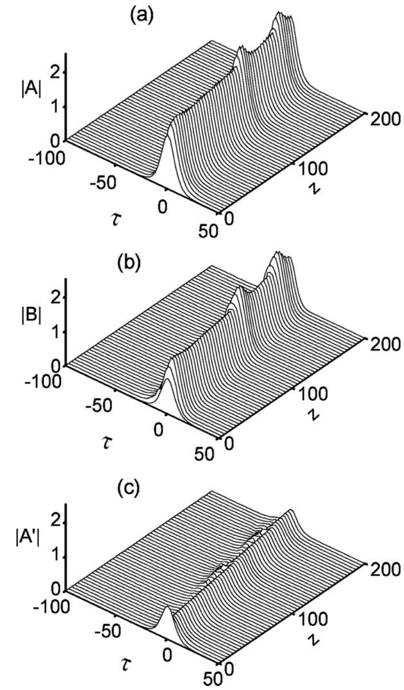

FIG. 8. The evolution of the exact pulse for the same parameters as in Fig. 2 but for the walkoff parameter $\delta=1$. (a) $|A|$, (b) $|B|$, and (c) $|A'|$.

robust objects, whose characteristics do not depend on the particular initial conditions.

IV. CONCLUSION

In conclusion, we have proposed a physical configuration, modeled by linearly coupled one-dimensional complex Ginzburg-Landau equations, consisting of two parallel-

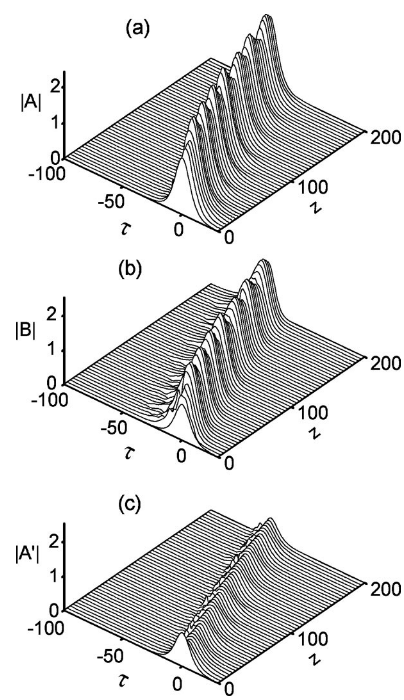

FIG. 9. The same as in Fig. 8 but for the walkoff parameter $\delta=2$. (a) $|A|$, (b) $|B|$, and (c) $|A'|$.

coupled waveguide cores, one having the quadratic nonlinearity, the other one being linear. The former core is active, with bandwidth-limited amplification integrated into it, while the latter one is lossy, which provides for complete stability of the zero background. The proposed model has nongeneric exact solitary-pulse solutions. Direct simulations demonstrate that they are always unstable; however, there always exist stable pulses, which are robust attractors, that can be generated from various initial conditions, including the generic case when the exact-pulse solution does not exist. If the group-velocity mismatch between the two harmonics in the main core is taken into regard, the soliton acquires a constant velocity, remaining fairly stable during propagation, thus constituting a walking soliton in media with gain and losses. Because the recently observed temporal [14] $\chi^{(2)}$ solitons have a very small dispersion length (of a few mm), and it is technologically possible to fabricate a dual-core waveguide whose coupling length is on the same order of magnitude, the proposed model might find a direct physical realization.

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