Exact soliton solutions for a family of *N* coupled nonlinear Schrödinger equations in optical fiber media

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(Received 8 November 1999)

We consider a family of both homogeneous and inhomogeneous *N*-coupled nonlinear Schrödinger equations which govern simultaneous propagation of *N* fields in an optical fiber with various important physical effects. The eigenvalue problem associated with homogeneous equations is constructed with the help of the Ablowitz-Kaup-Newell-Segur method. Using the Bäcklund transformation method, one-soliton solutions are explicitly derived.

PACS number(s): 42.81.Dp, 02.30.Jr, 05.45.Yv, 42.65.Tg

I. INTRODUCTION

The all-soliton-optical communication link is going to play a vital role in the rapidly growing information technology. The principle of solitons in optical fiber is based on the exact balance between the effects, group velocity dispersion (GVD) (linear effect), and self-phase modulation (SPM) (nonlinear effect). This was theoretically reported by Hasegawa and Tappert [1] and experimentally proved by Mollenauer *et al.* [2]. Propagation of optical solitons in a single mode fiber is governed by the famous nonlinear Schrödinger (NLS) equation of the form [1,3,4]

$$q_z = i\left[\frac{1}{2}q_{tt} + |q|^2 q\right], \tag{1}$$

where q is the slowly varying envelope of the axial field, and subscripts z and t denote spatial and temporal partial derivatives.

For handling more channels it is necessary to propagate more than one field simultaneously. Transmission of many fields simultaneously in a fiber is called wavelength division multiplexing (WDM) (i.e., fields with slightly different frequencies). In 1974, Manakov [5] derived the coupled NLS (CNLS) equations from the NLS equation by considering that the total field is comprised of two fields (left and right polarizations). In the same work he presented the linear eigenvalue problem associated with the CNLS equations and the soliton solutions using the inverse scattering transform (IST). The Painlevé analysis of the CNLS equations was carried out by Sahadevan *et al.* [6]. Soliton solutions using the Hirota bilinear method for CNLS equations were presented by Radhakrishnan and Lakshmanan [7]. In [8] we have generated the soliton solutions using the Bäcklund transformation method.

When we consider the simultaneous propagation of N nonlinear waves in a fiber, the wave dynamics of the system will be governed by N-CNLS equations of the form

$$q_{jz} = i\left[\frac{1}{2}q_{jtt} + \left(\sum_{n=1}^{N}|q_n|^2\right)q_j\right], \quad j=1,2,\ldots,N. \tag{2}$$

The Painlevé analysis of N-CNLS equations has been carried out in [6]. The Lax pair for N-CNLS equations have been presented by Fordy and Kulish [9].

For transmitting pulses at a high bit rate, it is necessary to propagate ultrashort pulses. Ultrashort pulses suffer from higher-order effects such as higher-order dispersion (HOD), Kerr dispersion (also called self-steepening), and delayed nonlinear response [3,4,10]. HOD is a linear effect but, unlike GVD, it broadens the pulses asymmetrically in the time domain. Kerr dispersion is due to the intensity dependence of the group velocity. This forces the peak of the pulse to travel faster than wings, which causes asymmetrical spectral broadening.

If we consider only the effects of HOD and Kerr dispersion, the wave dynamics of simultaneous propagation of N fields is governed by the N-coupled Hirota (N-CH) equations of the form

$$\begin{aligned} q_{jz} = {} & i\left[\frac{1}{2}q_{jtt} + \left(\sum_{n=1}^{N}|q_n|^2\right)q_j\right] \\ & + \epsilon\left[q_{jttt} + 3\left(\sum_{n=1}^{N}|q_n|^2\right)q_{jt} + 3\left(\sum_{n=1}^{N}q_n^* q_{nt}\right)q_j\right], \\ & j=1,2,\ldots,N. \end{aligned} \tag{3}$$

The Hirota equation was first considered by Hirota himself in [11]. Two coupled Hirota equations were first considered by Tasgal and Potasek [12]. In that they have constructed the Lax pair and obtained the soliton solutions using IST. Radhakrishnan *et al.* [13] have performed the Painlevé analysis and generated the soliton solutions for the coupled Hirota equations using the bilinear transformation method. Using the Bäcklund transformation method, we have generated the soliton solutions for the same [8].

With all the higher-order effects, simultaneous N-nonlinear waves propagation is governed by the N-coupled higher-order nonlinear Schrödinger (N-CHNLS) equations of the form

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$$q_{jz}=i\left(\frac{1}{2}q_{jtt}+\sum_{n=1}^{N}|q_n|^2q_j\right)-\epsilon\left[q_{jttt}+6\sum_{n=1}^{N}|q_n|^2q_{jt}+3q_j\left(\sum_{n=1}^{N}|q_n|^2\right)_t\right],\quad j=1,2,\ldots,N. \tag{4}$$

Here in Eq. (4), we consider only the real part of the last term. The imaginary part is related to the stimulated Raman scattering effect. The inverse scattering transform scheme for the HNLS equation [i.e., $j=1$ in Eq. (4)] was applied by Sasa and Satsuma [14]. Painlevé analysis and other related integrable properties of HNLS equation were carried out in [15,16].

Coupled HNLS (CHNLS) [i.e., $j=2$ in Eq. (4)] equations have been proposed and have shown that the system equation is integrable for a particular form using Painlevé analysis [17]. The linear eigenvalue problem for CHNLS equations and the exact one-soliton solutions generated using the Bäcklund transformation are given in [18]. Similar analyses were extended to simultaneous propagation of three fields also. The bilinear form for CHNLS equations and the associated soliton solutions were constructed in [19]. In [20], Painlevé analysis and the inverse scattering transform scheme for *N*-CHNLS equations have been presented.

In real fiber, the core medium is not homogeneous [21,22]. There will always be some nonuniformity due to many factors, and important among them are (i) a variation in the lattice parameters of the fiber medium, so that the distance between two neighboring atoms is not constant throughout the fiber, and (ii) the variation of the fiber geometry (diameter fluctuations, etc.). These nonuniformities influence various effects such as loss (or gain), dispersion, phase modulation, etc. [23]. When considering the inhomogeneities in the fiber, the dynamics of the optical pulse propagation is governed by the following equation:

$$iq_z+\frac{1}{2}q_{tt}+|q|^2q+iF(z)q+M(z)t^2q=0, \tag{5}$$

where $F(z)$ and $M(z)$ are inhomogeneous parameters related to gain (or loss) and phase modulation, respectively.

Recently, the application of Eq. (5) with various forms of inhomogeneities has been studied in various papers. The possibility of clean and efficient nonlinear compression of chirped solitary waves with appropriate tailoring of the gain or dispersion as a function of distance and with optional phase modulation have been studied by Moores [24]. Kumar and Hasegawa derived the chirped stationary solutions of Eq. (5) with $F(z)=0$ and $M(z)=\text{const}$ [25]. Clarkson carried out the Painlevé analysis of the inhomogeneous NLS (INLS) equation [26] and Balakrishnan discussed the inverse-scattering scheme for the INLS equation [27]. Equation (5) with $M(z)=0$ and $F(z)=1/2z$ was studied by Burstev *et al.* [28] from the soliton point of view. In that, they have presented the Lax pair for the system with a nonisospectral eigenvalue parameter (i.e., an eigenvalue parameter as a function of time and space). The soliton solution and the possibility of amplification of soliton pulses using a rapidly increasing distributed amplification with scale lengths comparable to the characteristic dispersion length have also been reported by Quiroga-Teixeiro *et al.* [29].

For simultaneous propagation of *N* fields, system Eq. (5) becomes

$$iq_{jz}+\frac{1}{2}q_{jtt}+q_j\left(\sum_{n=1}^{N}|q_n|^2\right)+iF(z)q_j+M(z)t^2q_j=0,\quad j=1,2,\ldots,N. \tag{6}$$

For the propagation of two orthogonally polarized optical fields in a nonuniform fiber media, coupled INLS equations of a particular form have been considered and have shown that with suitable variable transformation, the system equation can be transformed to coupled NLS equations [30]. Similarity reduction for variable-coefficient coupled NLS equations of different form has also been studied in [31].

In this paper, we consider the *N*-coupled nonlinear Schrödinger (*N*-CNLS) equations (2), *N*-coupled Hirota (*N*-CH) equations (3), *N*-coupled higher-order nonlinear Schrödinger (*N*-CHNLS) equations (4), and *N*-coupled inhomogeneous nonlinear Schrödinger (*N*-CINLS) equations (6) [for case (i) $M(z)=0$ and $F(z)=1/[2(z+z_0)]$, where z_0 is a constant, and case (ii) $M(z)=F(z)=1$], which govern the simultaneous propagation of *N* fields in an optical fiber with various important physical effects. The eigenvalue problem associated with the homogeneous equations is constructed with the help of the Ablowitz-Kaup-Newell-Segur (AKNS) method [32]. Using the Bäcklund transformation method, one-soliton solutions are explicitly derived.

II. *N*-CNLS EQUATIONS

The wave dynamics of simultaneous propagation of *N* fields in an optical fiber with only the effects of GVD and SPM is governed by the *N*-CNLS equations (2). The linear eigenvalue problem for Eq. (2) can be written as [9]

$$\frac{\partial\Psi}{\partial t}=U_1\Psi,\quad \Psi=(\psi_1\psi_2\psi_3\cdots\psi_{N+1})^T, \tag{7}$$

where

$$U_1=\begin{pmatrix} -i\lambda & q_1 & q_2 & \cdots & q_N\\ -q_1^* & i\lambda & 0 & \cdots & 0\\ -q_2^* & 0 & i\lambda & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -q_N^* & 0 & 0 & \cdots & i\lambda \end{pmatrix}. \tag{8}$$

λ is the spectral parameter. Space evolution of eigenfunction Ψ is given by

$$\frac{\partial\Psi}{\partial z}=V_1\Psi, \tag{9}$$

$$V_1=i\lambda^2\begin{pmatrix}-1&0&0&\cdots&0\\0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\end{pmatrix}+\lambda\begin{pmatrix}0&q_1&q_2&\cdots&q_N\\-q_1^*&0&0&\cdots&0\\-q_2^*&0&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\-q_N^*&0&0&\cdots&0\end{pmatrix}+\frac{i}{2}\begin{pmatrix}A&q_{1t}&q_{2t}&\cdots&q_{Nt}\\q_{1t}^*&-|q_1|^2&-q_2q_1^*&\cdots&-q_Nq_1^*\\q_{2t}^*&-q_1q_2^*&-|q_2|^2&\cdots&-q_Nq_2^*\\\vdots&\vdots&\vdots&\ddots&\vdots\\q_{Nt}^*&-q_1q_N^*&-q_2q_N^*&\cdots&-|q_N|^2\end{pmatrix}, \tag{10}$$

where

$$A=\sum_{n=1}^{N}|q_n|^2. \tag{11}$$

Equation (2) can be obtained from the compatibility condition $U_{1z}-V_{1t}+[U_1,V_1]=0$.

In order to construct the Bäcklund transformation of Eq. (2), let us write down the linear eigenvalue problem in terms of the Riccati equation. For this purpose, we introduce new variables (or pseudopotentials),

$$\Gamma_j=\frac{\psi_j}{\psi_{N+1}},\quad j=1,2,\ldots,N. \tag{12}$$

Inserting Eq. (12) into Eq. (7), we get

$$\Gamma_{1t}=-2i\lambda\Gamma_1+q_j\sum_{n=1}^{N-1}\Gamma_{j+1}+q_N+q_N^*\Gamma_1^2, \tag{13}$$

$$\Gamma_{jt}=-q_{j-1}^*\Gamma_1+q_N^*\Gamma_1\Gamma_j\quad\text{for } j=2,3,\ldots,N. \tag{14}$$

Similarly, equations for Γ_{jz} can be obtained from Eq. (9). Now, to construct the Bäcklund transformation, we define the new transformations in the form $\Gamma_j\to\Gamma_j'$, $\lambda\to\lambda'$, $q_j\to q_j'$, which keeps the form of Eqs. (13) and (14) invariant. The simplest transformation can be tried by setting $\Gamma_j'=\Gamma_j$, $\lambda'=\lambda^*$, and after some simplifications the Bäcklund transformation for Eq. (2) is obtained in the form

$$q_j-q_j'=\begin{cases}\dfrac{2i(\lambda-\lambda^*)\Gamma_1\Gamma_{j+1}^*}{1+\sum_{n=1}^{N}|\Gamma_n|^2} & \text{for } j=1,2,\ldots,N-1,\\[2ex] \dfrac{2i(\lambda-\lambda^*)\Gamma_1}{1+\sum_{n=1}^{N}|\Gamma_n|^2} & \text{for } j=N.\end{cases} \tag{15}$$

In Eq. (15), the primed quantities refer to N-soliton solutions and the unprimed quantities refer to $(N-1)$-soliton solutions. Using Eq. (15), one can in principle generate N-soliton solutions.

For instance, the trivial solutions of Eq. (2), $q_j=0$, correspond to the following pseudopotentials:

$$\Gamma_1=a_1\exp[-2i(\lambda t+\lambda^2 z)], \tag{16}$$

$$\Gamma_j=a_j\quad\text{for } j=2,3,\ldots,N, \tag{17}$$

where a_j's are arbitrary integration constants. So, we can find new solutions of Eq. (2) from Eq. (15), which is generated from the trivial one (with $\lambda=i\beta$),

$$q_j=\frac{2\beta a_{j+1}^*}{a_1^*}\operatorname{sech}(2\beta t)\exp(2i\beta^2 z),\quad j=1,2,\ldots,N-1, \tag{18}$$

$$q_N=\frac{2\beta}{a_1^*}\operatorname{sech}(2\beta t)\exp(2i\beta^2 z), \tag{19}$$

with the condition $1+\Sigma_{n=2}^{N}|a_n|^2=|a_1|^2$.

From Eqs. (18) and (19), one can generate the N-soliton solutions in a recursive manner. From the one-soliton solution, one can calculate the pulse width, amplitude, and shape of the pulses for WDM communication.

III. *N*-CH EQUATIONS

The nonlinear wave propagation of simultaneous N fields in an optical fiber with the effects of GVD, SPM, HOD, and Kerr dispersion is governed by the N-CH equations (3). The linear eigenvalue problem for Eq. (3) can be obtained using AKNS method as

$$\frac{\partial\Psi}{\partial t}=U_2\Psi,$$

$$\Psi=(\psi_1\psi_2\psi_3\cdots\psi_{N+1})^T, \tag{20}$$

where

$$U_2=\begin{pmatrix}-i\lambda&q_1&q_2&\cdots&q_N\\-q_1^*&i\lambda&0&\cdots&0\\-q_2^*&0&i\lambda&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\-q_N^*&0&0&\cdots&i\lambda\end{pmatrix}. \tag{21}$$

Space evolution of eigenfunction Ψ is given by

$$\frac{\partial \Psi}{\partial z} = V_2 \Psi, \tag{22}$$

$$V_2 = -(4i\epsilon\lambda^3 - i\lambda^2)\begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} - (4\epsilon\lambda^2 - \lambda)\begin{pmatrix} 0 & q_1 & q_2 & \cdots & q_N \\ -q_1^* & 0 & 0 & \cdots & 0 \\ -q_2^* & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_N^* & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$-\left(2i\epsilon\lambda - \frac{i}{2}\right)\begin{pmatrix} A & q_{1t} & q_{2t} & \cdots & q_{Nt} \\ q_{1t}^* & -|q_1|^2 & -q_2 q_1^* & \cdots & -q_N q_1^* \\ q_{2t}^* & -q_1 q_2^* & -|q_2|^2 & \cdots & -q_N q_2^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{Nt}^* & -q_1 q_N^* & -q_2 q_N^* & \cdots & -|q_N|^2 \end{pmatrix}$$

$$+\epsilon\begin{pmatrix} \sum_{n=1}^{N}(q_{nt}q_n^* - q_n q_{nt}^*) & q_{1tt} + 2Aq_1 & q_{2tt} + 2Aq_2 & \cdots & q_{Ntt} + 2Aq_N \\ -q_{1tt}^* - 2Aq_1^* & -(q_{1t}q_1^* - q_1 q_{1t}^*) & -(q_{2t}q_1^* - q_2 q_{1t}^*) & \cdots & -(q_{Nt}q_1^* - q_N q_{1t}^*) \\ -q_{2tt}^* - 2Aq_2^* & -(q_{1t}q_2^* - q_1 q_{2t}^*) & -(q_{2t}q_2^* - q_2 q_{2t}^*) & \cdots & -(q_{Nt}q_2^* - q_N q_{2t}^*) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_{Ntt}^* - 2Aq_N^* & -(q_{1t}q_N^* - q_1 q_{Nt}^*) & -(q_{2t}q_N^* - q_2 q_{Nt}^*) & \cdots & -(q_{Nt}q_N^* - q_N q_{Nt}^*) \end{pmatrix}, \tag{23}$$

where $A = \Sigma_{n=1}^{N}|q_n|^2$. Equation (3) can be obtained from the compatibility condition $U_{2z} - V_{2t} + [U_2, V_2] = 0$.

Using the same procedure, the Bäcklund transformation for Eq. (3) is found to be

$$q_j - q_j' = \begin{cases} \dfrac{2i(\lambda - \lambda^*)\Gamma_1 \Gamma_{j+1}^*}{1 + \sum_{n=1}^{N}|\Gamma_n|^2} & \text{for } j = 1,2,\ldots,N-1, \\ \dfrac{2i(\lambda - \lambda^*)\Gamma_1}{1 + \sum_{n=1}^{N}|\Gamma_n|^2} & \text{for } j = N. \end{cases} \tag{24}$$

Similarly, the one-soliton solutions for the N-CH equations are generated as

$$q_j = \frac{2\beta a_{j+1}^*}{a_1^*}\operatorname{sech}(2\beta t + 8\epsilon\beta^3 z)\exp(2i\beta^2 z),$$

$$j = 1,2,\ldots,N-1, \tag{25}$$

$$q_N = \frac{2\beta}{a_1^*}\operatorname{sech}(2\beta t + 8\epsilon\beta^3 z)\exp(2i\beta^2 z), \tag{26}$$

with the condition $1 + \Sigma_{n=2}^{N}|a_n|^2 = |a_1|^2$.

IV. N-CHNLS EQUATIONS

In order to analyze the N-CHNLS equations (4), it is rather convenient to introduce variable transformations,

$$u_j(x,T) = q_j(t,z)\exp\left[\frac{-i}{6\epsilon}\left(t - \frac{z}{18\epsilon}\right)\right],$$

$$T = z, \tag{27}$$

$$x = t - \frac{z}{12\epsilon}.$$

Then, Eq. (4) reduces to N coupled complex modified Korteweg–deVries (KdV)-type equations,

$$u_{jT} + \epsilon\left[u_{jxxx} + 6\sum_{n=1}^{N}|u_n|^2 u_{jx} + 3u_j\left(\sum_{n=1}^{N}|u_n|^2\right)_x\right] = 0. \tag{28}$$

The Lax pair for N coupled complex modified KdV equations (28) is derived as

$$\frac{\partial \Psi}{\partial x} = U_3 \Psi,$$

$$\Psi = (\psi_1 \psi_2 \psi_3 \ldots \psi_{2N+1})^T, \tag{29}$$

where

$$U_3=\begin{pmatrix} -i\lambda & 0 & \cdots & 0 & 0 & 0 & 0 & u_N \\ 0 & -i\lambda & \cdots & 0 & 0 & 0 & 0 & u_N^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -i\lambda & 0 & 0 & 0 & u_2 \\ 0 & 0 & \cdots & 0 & -i\lambda & 0 & 0 & u_2^* \\ 0 & 0 & \cdots & 0 & 0 & -i\lambda & 0 & u_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -i\lambda & u_1^* \\ -u_N^* & -u_N & \cdots & -u_2^* & -u_2 & -u_1^* & -u_1 & i\lambda \end{pmatrix}. \tag{30}$$

Space evolution of eigenfunction Ψ is given by

$$\frac{\partial\Psi}{\partial T}=V_3\Psi, \tag{31}$$

$$V_3=-4i\epsilon\lambda^3\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 \end{pmatrix}+4\epsilon\lambda^2\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_N \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_N^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_2 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_2^* \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & u_1^* \\ -u_N^* & -u_N & \cdots & -u_2^* & -u_2 & -u_1^* & -u_1 & 0 \end{pmatrix}$$

$$+2i\epsilon\lambda\begin{pmatrix} |u_N|^2 & u_N^2 & \cdots & u_2^*u_N & u_2u_N & u_1^*u_N & u_1u_N & u_{Nx} \\ u_N^{*2} & |u_N|^2 & \cdots & u_2^*u_N^* & u_2u_N^* & u_1^*u_N^* & u_1u_N^* & u_{Nx}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_2u_N^* & u_2u_N & \cdots & |u_2|^2 & u_2^2 & u_2u_1^* & u_2u_1 & u_{2x} \\ u_2^*u_N^* & u_2^*u_N & \cdots & u_2^{*2} & |u_2|^2 & u_2^*u_1^* & u_2^*u_1 & u_{2x}^* \\ u_1u_N^* & u_1u_N & \cdots & u_1u_2^* & u_1u_2 & |u_1|^2 & u_1^2 & u_{1x} \\ u_1^*u_N^* & u_1^*u_N & \cdots & u_1^*u_2^* & u_1^*u_2 & u_1^{*2} & |u_1|^2 & u_{1x}^* \\ u_{Nx}^* & u_{Nx} & \cdots & u_{2x}^* & u_{2x} & u_{1x}^* & u_{1x} & -2B \end{pmatrix}$$

$$+\epsilon\begin{pmatrix} u_{Nx}^*u_N-u_N^*u_{Nx} & 0 & \cdots & u_{2x}^*u_N-u_2^*u_{Nx} & u_{2x}u_N-u_2u_{Nx} & u_{1x}^*u_N-u_1^*u_{Nx} & u_{1x}u_N-u_1u_{Nx} & -4Bu_N-u_{Nxx} \\ 0 & u_N^*u_{Nx}-u_{Nx}^*u_N & \cdots & u_{2x}^*u_N^*-u_2^*u_{Nx}^* & u_{2x}u_N^*-u_2u_{Nx}^* & u_{1x}^*u_N^*-u_1^*u_{Nx}^* & u_{1x}u_N^*-u_1u_{Nx}^* & -4Bu_N^*-u_{Nxx}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_2u_{Nx}^*-u_{2x}u_N^* & u_2u_{Nx}-u_{2x}u_N & \cdots & u_{2x}^*u_2-u_2^*u_{2x} & 0 & u_2u_{1x}^*-u_{2x}u_1^* & u_2u_{1x}-u_{2x}u_1 & -4Bu_2-u_{2xx} \\ u_2^*u_{Nx}^*-u_{2x}^*u_N^* & u_2^*u_{Nx}-u_{2x}^*u_N & \cdots & 0 & u_2^*u_{2x}-u_{2x}^*u_2 & u_2^*u_{1x}^*-u_{2x}^*u_1^* & u_2^*u_{1x}-u_{2x}^*u_1 & -4Bu_2^*-u_{2xx}^* \\ u_1u_{Nx}^*-u_{1x}u_N^* & u_1u_{Nx}-u_{1x}u_N & \cdots & u_1u_{2x}^*-u_{1x}u_2^* & u_1u_{2x}-u_{1x}u_2 & u_{1x}^*u_1-u_1^*u_{1x} & 0 & -4Bu_1-u_{1xx} \\ u_1^*u_{Nx}^*-u_{1x}^*u_N^* & u_1^*u_{Nx}-u_{1x}^*u_N & \cdots & u_1^*u_{2x}^*-u_{1x}^*u_2^* & u_1^*u_{2x}-u_{1x}^*u_2 & 0 & u_1^*u_{1x}-u_{1x}^*u_1 & -4Bu_1^*-u_{1xx}^* \\ 4Bu_N^*+u_{Nxx}^* & 4Bu_N+u_{Nxx} & \cdots & 4Bu_2^*+u_{2xx}^* & 4Bu_2+u_{2xx} & 4Bu_1^*+u_{1xx}^* & 4Bu_1+u_{1xx} & 0 \end{pmatrix}, \tag{32}$$

where $B=\Sigma_{n=1}^{N}|u_n|^2$. Equation (28) can be obtained from the compatibility condition $U_{3z}-V_{3t}+[U_3,V_3]=0$ [and Eq. (4) simultaneously].

Using the same procedure, the Bäcklund transformation for Eq. (28) is found to be

$$u_j-u_j'=\begin{cases}\dfrac{2i(\lambda-\lambda^*)\Gamma_1\Gamma_{2j+1}^*}{1+\sum_{n=1}^{2N}|\Gamma_n|^2} & \text{for } j=1,2,\ldots,N-1,\\[2ex] \dfrac{2i(\lambda-\lambda^*)\Gamma_1}{1+\sum_{n=1}^{2N}|\Gamma_n|^2} & \text{for } j=N.\end{cases} \tag{33}$$

Similarly, the one-soliton solutions for Eq. (28) (simultaneously for N-CHNLS equations) are generated as

$$u_j=\frac{2\beta a_{2j+1}^*}{a_1^*}\text{sech}(2\beta t-8\epsilon\beta^3 z),\quad j=1,2,\ldots,N-1, \tag{34}$$

$$u_N=\frac{2\beta}{a_1^*}\text{sech}(2\beta t-8\epsilon\beta^3 z) \tag{35}$$

with the condition $1+\Sigma_{n=2}^{2N}|a_n|^2=|a_1|^2$.

V. N-CINLS EQUATIONS

With the effects of inhomogeneities and phase modulation, wave dynamics of the simultaneous propagation of N optical pulses in a fiber medium with the effects of GVD and SPM alone is given by

$$iq_{jz}+\frac{1}{2}q_{jtt}+q_j\left(\sum_{n=1}^{N}|q_n|^2\right)+iF(z)q_j+M(z)t^2q_j=0,$$
$$j=1,2,\ldots,N. \tag{36}$$

Here we consider the following two cases for which the system equation (36) is completely integrable and possesses Lax pair and exact soliton solutions through Bäcklund transformation.

Case (i): $M(z)=0$ and $F(z)=1/[2(z+z_0)]$, where z_0 is a constant. With these conditions, Eq. (36) will become

$$iq_{jz}+\frac{1}{2}q_{jtt}+q_j\left(\sum_{n=1}^{N}|q_n|^2\right)+\frac{i}{2(z+z_0)}q_j=0,$$
$$j=1,2,\ldots,N. \tag{37}$$

The Lax pair associated with Eq. (37) is contructed as

$$\frac{\partial\Psi}{\partial t}=U_4\Psi,$$
$$\Psi=(\psi_1\psi_2\psi_3\cdots\psi_{N+1})^T, \tag{38}$$

where

$$U_4=\begin{pmatrix}-i\lambda & q_1 & q_2 & \cdots & q_N\\ -q_1^* & i\lambda & 0 & \cdots & 0\\ -q_2^* & 0 & i\lambda & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -q_N^* & 0 & 0 & \cdots & i\lambda\end{pmatrix}. \tag{39}$$

λ is the nonisospectral parameter given by

$$\lambda=-\frac{2\mu+t}{2(z+z_0)}, \tag{40}$$

where μ is the hidden spectral parameter.

The space evolution of eigenfunction Ψ is given by

$$\frac{\partial\Psi}{\partial z}=V_4\Psi, \tag{41}$$

$$V_4=i\lambda^2\begin{pmatrix}-1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}+\lambda\begin{pmatrix}0 & q_1 & q_2 & \cdots & q_N\\ -q_1^* & 0 & 0 & \cdots & 0\\ -q_2^* & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -q_N^* & 0 & 0 & \cdots & 0\end{pmatrix}+\frac{i}{2}\begin{pmatrix}A & q_{1t} & q_{2t} & \cdots & q_{Nt}\\ q_{1t}^* & -|q_1|^2 & -q_2q_1^* & \cdots & -q_Nq_1^*\\ q_{2t}^* & -q_1q_2^* & -|q_2|^2 & \cdots & -q_Nq_2^*\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ q_{Nt}^* & -q_1q_N^* & -q_2q_N^* & \cdots & -|q_N|^2\end{pmatrix}, \tag{42}$$

where

$$A=\sum_{n=1}^{N}|q_n|^2. \tag{43}$$

Equation (37) can be obtained from the compatibility condition $U_{4z}-V_{4t}+[U_4,V_4]=0$.

Using the same procedure, the Bäcklund transformation for Eq. (37) is found to be

$$q_j-q_j'=\begin{cases}\dfrac{2i(\lambda-\lambda^*)\Gamma_1\Gamma_{j+1}^*}{1+\sum_{n=1}^{N}|\Gamma_n|^2} & \text{for } j=1,2,\ldots,N-1,\\[2ex] \dfrac{2i(\lambda-\lambda^*)\Gamma_1}{1+\sum_{n=1}^{N}|\Gamma_n|^2} & \text{for } j=N.\end{cases}\tag{44}$$

Similarly, the one-soliton solutions for the N-CINLS equations (37) are generated as

$$q_j=\frac{2\mu a_{j+1}^*}{a_1^*(z+z_0)}\operatorname{sech}\left(\frac{2\mu t}{z+z_0}\right)\exp\left(\frac{i}{z+z_0}(t^2/2-2\mu^2)\right),$$

$$j=1,2,\ldots,N-1,\tag{45}$$

$$q_N=\frac{2\mu}{a_1^*(z+z_0)}\operatorname{sech}\left(\frac{2\mu t}{z+z_0}\right)\exp\left(\frac{i}{z+z_0}(t^2/2-2\mu^2)\right),\tag{46}$$

with the condition $1+\Sigma_{n=2}^{N}|a_n|^2=|a_1|^2$.

It is interesting to mention that under the variable transformations

$$q_j(z,t)=\frac{\sqrt{2}z_0}{z+z_0}Q_j(Z,T)\exp\left(\frac{it^2}{2(z+z_0)}\right),$$

$$Z=\frac{zz_0}{z+z_0},\quad T=\frac{\sqrt{2}tz_0}{z+z_0},\tag{47}$$

the N-CINLS equations (37) can be transformed into N-CNLS equations of the form

$$iQ_{jZ}+Q_{jTT}+2Q_j\sum_{n=1}^{n}|Q_n|^2=0.\tag{48}$$

Case (ii): $M(z)=F(z)=1$. With this choice, the N-CINLS equations (36) become

$$iq_{jz}+q_{jtt}+2q_j\left(\sum_{n=1}^{N}|q_n|^2\right)+iq_j+t^2q_j=0,$$

$$j=1,2,\ldots,N.\tag{49}$$

The Lax pair associated with Eq. (49) is constructed as

$$\frac{\partial\Psi}{\partial t}=U_5\Psi,$$

$$\Psi=(\psi_1\psi_2\psi_3\cdots\psi_{N+1})^T,\tag{50}$$

where

$$U_5=\begin{pmatrix}-i\lambda & Q_1 & Q_2 & \cdots & Q_N\\ -Q_1^* & i\lambda & 0 & \cdots & 0\\ -Q_2^* & 0 & i\lambda & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -Q_N^* & 0 & 0 & \cdots & i\lambda\end{pmatrix},\tag{51}$$

where $Q_j=q_j\exp(-it^2/2)$ and λ is the nonisospectral parameter given by

$$\lambda=\mu\exp(-2z),\tag{52}$$

where μ is the hidden spectral parameter.

The space evolution of eigenfunction Ψ is given by

$$\frac{\partial\Psi}{\partial z}=V_5\Psi,\tag{53}$$

$$V_5=2i\lambda^2\begin{pmatrix}-1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}+2\lambda\begin{pmatrix}it & Q_1 & Q_2 & \cdots & Q_N\\ -Q_1^* & -it & 0 & \cdots & 0\\ -Q_2^* & 0 & -it & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -Q_N^* & 0 & 0 & \cdots & -it\end{pmatrix}$$

$$+i\begin{pmatrix}\sum_{n=1}^{n}|Q_n|^2 & Q_{1t}+2itQ_1 & Q_{2t}+2itQ_2 & \cdots & Q_{Nt}+2itQ_N\\ Q_{1t}^*-2itQ_1^* & -|Q_1|^2 & -Q_2Q_1^* & \cdots & -Q_NQ_1^*\\ Q_{2t}^*-2itQ_2^* & -Q_1Q_2^* & -|Q_2|^2 & \cdots & -Q_NQ_2^*\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ Q_{Nt}^*-2itQ_N^* & -Q_1Q_N^* & -Q_2Q_N^* & \cdots & -|Q_N|^2\end{pmatrix}.\tag{54}$$

Equation (49) can be obtained from the compatibility condition $U_{5z}-V_{5t}+[U_5,V_5]=0$.

Using the same procedure, the Bäcklund transformation for Eq. (49) is found to be

$$Q_j-Q_j'=\begin{cases}\dfrac{2i(\lambda-\lambda^*)\Gamma_1\Gamma_{j+1}^*}{1+\sum_{n=1}^{N}|\Gamma_n|^2} & \text{for } j=1,2,\ldots,N-1,\\[2ex] \dfrac{2i(\lambda-\lambda^*)\Gamma_1}{1+\sum_{n=1}^{N}|\Gamma_n|^2} & \text{for } j=N.\end{cases} \tag{55}$$

Similarly, the one-soliton solutions for the N-CINLS equations (49) are generated as

$$q_j=\frac{2\alpha_2 a_{j+1}^*}{a_1^*}\text{sech}\left(2\alpha_2 t+8\int^z\alpha_1\alpha_2 dz\right)\times\exp\left[-2i\alpha_1 t-4i\int^z(\alpha_1^2-\alpha_2^2)dz+it^2/2\right],\quad j=1,2,\ldots,N-1, \tag{56}$$

$$q_N=\frac{2\alpha_2}{a_1^*}\text{sech}\left(2\alpha_2 t+8\int^z\alpha_1\alpha_2 dz\right)\times\exp\left[-2i\alpha_1 t-4i\int^z(\alpha_1^2-\alpha_2^2)dz+it^2/2\right], \tag{57}$$

where $\alpha_1=k_1\exp(-2z)$, $\alpha_2=k_2\exp(-2z)$ (k_1,k_2 are real and imaginary parts of μ), and with the condition $1+\Sigma_{n=2}^N|a_n|^2=|a_1|^2$.

VI. DISCUSSIONS AND CONCLUSION

In Sec. II, we found that Manakov model N-CNLS equations allow soliton-type pulse propagation in optical fibers. From the complete integrability of the system equation, it is clear that for the simultaneous propagation of N nonlinear optical fields, there exists exact balancing between the physical effects of GVD and SPM (with the inclusion of cross-phase modulation also).

N-CH equations include the higher-order effects due to HOD and Kerr dispersion. For the simultaneous propagation of N fields, the existence of the Lax pair proves the exact balancing between the asymmetrical temporal broadening by HOD and the asymmetrical spectral broadening by Kerr dispersion. Similarly, in Sec. IV, we find the possibility of soliton-type pulse propagation for the fiber system described by N-CHNLS equations. Here also the asymmetrical temporal broadening of optical pulses due to HOD is counterbalanced by the asymmetrical spectral broadening due to the combined effects of Kerr dispersion and delayed nonlinear effects. We have already shown through Painlevé analysis that only for this form of N-CHNLS equations does the fiber system allow soliton-type pulse propagation [20].

Finally, in Sec. V, simultaneous propagation of N nonlinear pulses in inhomogeneous optical fibers was considered. The first case of integrable N-CINLS equations dealt with the spatial inhomogeneity. A similar kind of system equation in an erbium-doped optical fiber system for single field propagation has been discussed in [33]. The second case is with phase modulation. This integrable case is also related to the dispersion-managed solitons. In [34], one can see the relationship between the system equation (49) (for the single-field case) and the dispersion-managed fiber system equation.

Thus, in this paper, we have considered the N-CNLS, N-CH, N-CHNLS, and two cases of N-CINLS equations which govern simultaneous propagation of N fields in a fiber medium with various important physical effects. Then, with the help of the respective linear eigenvalue problem, exact one-soliton solutions have been generated from the Bäcklund transformations.

ACKNOWLEDGMENT

Financial support from the Conseil Régional de Bourgogne is gratefully acknowledged.

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