Random walks in logarithmic and power-law potentials, nonuniversal persistence, and vortex dynamics in the two-dimensional XY model

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The Langevin equation for a particle (''random walker'') moving in d-dimensional space under an attractive central force and driven by a Gaussian white noise is considered for the case of a power-law force, $F(r) \sim -r^{-\sigma}$. The ''persistence probability,'' $P_0(t)$, that the particle has not visited the origin up to time t is calculated for a number of cases. For $\sigma > 1$, the force is asymptotically irrelevant (with respect to the noise), and the asymptotics of $P_0(t)$ are those of a free random walker. For $\sigma < 1$, the noise is (dangerously) irrelevant and the asymptotics of $P_0(t)$ can be extracted from a weak noise limit within a path-integral formalism employing the Onsager-Machlup functional. The case $\sigma = 1$, corresponding to a logarithmic potential, is most interesting because the noise is exactly marginal. In this case, $P_0(t)$ decays as a power law, $P_0(t) \sim t^{-\theta}$ with an exponent θ that depends continuously on the ratio of the strength of the potential to the strength of the noise. This case, with $d=2$, is relevant to the annihilation dynamics of a vortex-antivortex pair in the two-dimensional XY model. Although the noise is multiplicative in the latter case, the relevant Langevin equation can be transformed to the standard form discussed in the first part of the paper. The mean annihilation time for a pair initially separated by r is given by $t(r) \sim r^2 \ln(r/a)$ where a is a microscopic cutoff (the vortex core size). Implications for the nonequilibrium critical dynamics of the system are discussed and compared to numerical simulation results.

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I. INTRODUCTION

This paper deals with two seemingly distinct topics: persistence and nonequilibrium critical dynamics. We show that they are related in the context of the nonequilibrium critical dynamics of the two-dimensional (2D) XY model with nonconserved order parameter. The link is established through the study of a d-dimensional random walker moving in a logarithmic potential.

Persistence phenomena are related to first-passage problems for stochastic processes. Consider a stochastic process $x(t)$. The ''persistence probability,'' $P_0(t)$, is the probability that $x(t)$ has not crossed some given level, x_c (often taken to be zero), up to time t. The probability distribution, $P_1(t)$, of the first-passage time [i.e., the first time t for which $x(t) = x_c$] is $P_1 = -dP_0/dt$. A familiar example is the 1D random walk, with Langevin equation $dx/dt = \xi(t)$, where $\xi(t)$ is a Gaussian white noise. For initial condition $x(0) = x_0$, the probability that the walker has not crossed the origin ($x = 0$) up to time t decays as $P_0(t) \sim x_0/t^{1/2}$ for $t \gg x_0^2$. The power entering this algebraic decay has been termed the ''persistence exponent,'' θ, i.e., $\theta = 1/2$ for the 1D random walk.

Persistence phenomena have been widely studied in recent years [1–11]. Theoretical and computational studies include spin systems in one [2] and higher [3] dimensions, diffusion fields [4], fluctuating interfaces [5], phase-ordering dynamics [6], and reaction-diffusion systems [7]. Experimental studies include the coarsening dynamics of breath figures [8], soap froths [9], and twisted nematic liquid crystals [10]. Persistence in nonequilibrium critical phenomena has been studied in the context of the global order parameter, $M(t)$, (e.g., the total magnetization of a ferromagnet), regarded as a stochastic process [11], but in the present work we will address a different, and more fundamental, aspect of nonequilibrium critical dynamics.

If a system at its critical temperature evolves from a nonequilibrium initial state, critical correlations develop over a length scale, $\xi(t)$, which increases with time. According to the standard theory [12], $\xi \sim t^{1/z}$ for large t, where z is the critical exponent for *equilibrium* critical dynamics. This result, which is in accord with simple dynamical scaling, has been demonstrated within a field-theoretic framework [12]. This approach also shows that the result $\xi(t) \sim t^{1/z}$ is independent of the initial conditions.

In the present work we challenge this picture for the specific case of the 2D XY model with nonconserved order parameter, and show that $\xi(t) \sim t^{1/2}$ if there are no free vortices present in the initial state, while $\xi(t) \sim (t/\ln t)^{1/2}$ if free vortices are present. Physically, these two cases correspond to ordered initial states (e.g., the equilibrium state at $T=0$), and disordered initial states (e.g., the equilibrium state at $T = \infty$), respectively. Furthermore, since the 2D XY model is described, through the Kosterlitz-Thouless (KT) theory [13], by a critical line, $T \leqslant T_{KT}$, rather than a single critical point, the above dependence on initial conditions will persist throughout the KT phase.

The connection with persistence arises as follows. For an initial condition containing free vortices and antivortices, vortex-antivortex annihilation is the slowest relaxation process. Much can be learned by studying the annihilation of a single, initially widely separated, vortex-antivortex pair. The probability that they have not annihilated up to time t defines a persistence problem. By a series of transformations, this can be mapped onto a random walk in a logarithmic potential. Analysis of this problem shows that the persistence exponent θ is a continuous function of the ratio of the strength of the potential to the strength of the thermal noise, i.e., θ is

nonuniversal. In the context of the vortex problem, we find $\theta=\pi\rho_s(T)/k_BT$, where $\rho_s(T)$ is the large-scale spin-wave stiffness at temperature T. Since the ratio ρ_s/k_BT takes the universal value $2/\pi$ at T_{KT}, we obtain $\theta(T_{KT})=2$, while $\theta(T)\to\infty$ as $T\to 0$.

Although the vortex problem in two dimensions was the initial motivation for the present study, for pedagogical reasons we will begin (Sec. II) by discussing the d-dimensional random walk in a logarithmic potential, and deriving the persistence properties. We show that the exponent θ depends continuously on the ratio of the strength of the potential to the strength of the noise [14], i.e., the logarithmic potential is *marginal* in the renormalization-group sense. In Sec. III we consider the case of a general attractive power-law force, $\mathbf{F}(\mathbf{r})=(-A/r^\sigma)\hat{\mathbf{r}}$, where the logarithmic potential corresponds to $\sigma=1$. We show that for $\sigma>1$ the force is irrelevant (relative to the noise) as far as asymptotic persistence properties are concerned, so the results of the free random walk still hold. For $\sigma<1$, by contrast, the force is relevant and the noise becomes irrelevant. For zero noise, however, the dynamics are deterministic, so the persistence asymptotics are governed by rare fluctuations of the noise. In this sense, the noise is *dangerously* irrelevant. The asymptotic persistence follows from an optimal path (or steepest descent) approach formally valid in the weak noise limit [15,16]. The result is the ''stretched-exponential'' decay, $P_0(t)\sim\exp[-\text{const}\times t^{(1-\sigma)/(1+\sigma)}]$.

Section IV deals with the application of the previous results to the problem of vortex-antivortex annihilation in the KT phase of the 2D XY model. The potential energy of a free vortex-antivortex pair is logarithmic in their separation, so this is the marginal case with a nonuniversal θ. An additional complication is that the effective vortex mobility is scale (i.e., separation) dependent, implying, via the Einstein relation between mobility and diffusion constant, that the noise strength is also scale dependent or, equivalently, that the noise is multiplicative. However, this system can be transformed to an additive noise problem identical to that discussed in Sec. II. Although the persistence exponent is nonuniversal, the dependence of the typical annihilation time on the initial separation r has the universal form $t\sim r^2\ln(r/a)$, where a is a (nonuniversal) short-distance cutoff, for all $T\leqslant T_{KT}$. Standard scaling arguments then suggest that, for an initial condition containing many free vortices and antivortices, the characteristic length scale describing the approach to equilibrium will grow as $\xi(t)\sim(t/\ln t)^{1/2}$ throughout the KT phase.

II. RANDOM WALKER IN A LOGARITHMIC POTENTIAL

The Langevin equation for an isotropic, continuous-time random walker moving in d dimensions in the central potential $V(r)=A\ln r$ is

$$\frac{d\mathbf{r}}{dt}=-\frac{A}{r}\hat{\mathbf{r}}+\boldsymbol{\xi}(t), \tag{1}$$

where the Gaussian white noise has the correlation function $(i,j=1,\ldots,d)$

$$\langle\xi_i(t)\xi_j(t')\rangle=2D\delta_{ij}\delta(t-t'). \tag{2}$$

A. Reduction to a one-dimensional problem

The analysis starts from the equivalent Fokker-Planck equation for the probability distribution, $P(\mathbf{r},t)$, for the position of the particle at time t:

$$\frac{\partial P}{\partial t}=\boldsymbol{\nabla}\cdot\left(\frac{A}{r}\hat{\mathbf{r}}+D\boldsymbol{\nabla}P\right). \tag{3}$$

This equation can be reduced to an effectively one-dimensional equation in terms of the radial coordinate r by integrating over the angle variables. Writing $\mathbf{r}=(r,\Omega)$, and defining the radial probability distribution $Q(r,t)=r^{d-1}\int d\Omega P(r,\Omega,t)$, gives

$$\frac{\partial Q}{\partial t}=D\partial_r\left(\frac{b}{r}Q+\partial_rQ\right), \tag{4}$$

where $\partial_r\equiv\partial/\partial r$ and

$$b=\frac{A}{D}+1-d. \tag{5}$$

This is equivalent to the radial Langevin equation

$$\frac{dr}{dt}=-\frac{A}{r}+\frac{(d-1)D}{r}+\xi(t), \tag{6}$$

where $\langle\xi(t)\xi(t')\rangle=2D\delta(t-t')$, i.e., projecting Eq. (1) onto the radial direction leads to an additional repulsive force $(d-1)D/r$, proportional to the noise strength. This means [see Eq. (5)] that a particle moving in a logarithmic potential in space dimension d has the same radial distribution function as a free particle (random walker) moving in dimension $d'=d-A/D$ (which need not be an integer, or even positive).

B. The persistence probability

In this subsection we derive an exact expression for the persistence probability $P_0(t)$. First we make the change of variable $Q=r^{(1-b)/2}R$ in Eq. (4). Then the equation governing the relaxation modes $R(r,t)=R_k(r)\exp(-Dk^2t)$ becomes

$$\frac{d^2R_k}{dr^2}+\frac{1}{r}\frac{dR_k}{dr}+\left(k^2-\frac{\nu^2}{r^2}\right)R_k=0, \tag{7}$$

where

$$\nu=(1+b)/2. \tag{8}$$

The solutions are the Bessel functions $J_\nu(kr)$ and $J_{-\nu}(kr)$, so the general solution is

$$Q(r,t)=r^{(1-b)/2}\int_0^\infty dk[\alpha(k)J_\nu(kr)+\beta(k)J_{-\nu}(kr)]\times\exp(-Dk^2t). \tag{9}$$

The desired solution has to satisfy the initial condition $Q(r,0)=\delta(r-r_0)$. To determine $P_0(t)$ we impose an absorbing boundary at $r=0$, such that the particle is removed if it reaches the origin. To determine the appropriate solution we note from Eq. (4) that the probability current is

$$j(r) = -D(\partial_r Q + bQ/r). \tag{10}$$

Consider separately the two terms in the general solution (9). The term involving J_ν behaves as r for $r \to 0$, while the term in $J_{-\nu}$ behaves as r^{-b}. Both terms have corrections which multiply the leading terms by power series in r^2. Inserting both forms into Eq. (10) shows that the first term gives a finite (and negative) current at $r=0$, so that for this solution the origin is an absorbing point, or current sink. The second term, on the other hand, gives zero current and the origin is a not a special point: This solution therefore gives the spherically averaged Greens function for diffusion in $d' = 1-b$ dimensions.

Using the orthogonality properties of Bessel functions to determine the functions $\alpha(k)$ and $\beta(k)$ in each case, we find that the Greens function is given by

$$\begin{aligned} Q_G(r,t) &= r_0 \left(\frac{r}{r_0}\right)^{(1-b)/2} \int_0^\infty dk\, k J_{-\nu}(kr_0) J_{-\nu}(kr) \\ &\quad \times \exp(-Dk^2 t) \\ &= \frac{r_0}{2Dt} \left(\frac{r}{r_0}\right)^{(1-b)/2} \exp\left(-\frac{r^2+r_0^2}{4Dt}\right) I_{-\nu}\left(\frac{rr_0}{2Dt}\right), \end{aligned} \tag{11}$$

where $I_\nu(z)$ is the modified Bessel function, while the relevant solution for an absorbing point at the origin is

$$\begin{aligned} Q_{abs}(r,t) &= r_0 \left(\frac{r}{r_0}\right)^{(1-b)/2} \int_0^\infty dk\, k J_{\nu}(kr_0) J_{\nu}(kr) \\ &\quad \times \exp(-Dk^2 t) \\ &= \frac{r_0}{2Dt} \left(\frac{r}{r_0}\right)^{(1-b)/2} \exp\left(-\frac{r^2+r_0^2}{4Dt}\right) I_{\nu}\left(\frac{rr_0}{2Dt}\right). \end{aligned} \tag{12}$$

At this point a comment on the possible values of b is needed. The leading term in Q_G for $r \to 0$ is $O(r^{-b})$, and this gives rise to zero current at $r=0$ as discussed above. The next-to-leading term, however, is $O(r^{2-b})$ leading to a current of order r^{1-b}. This vanishes as $r \to 0$ only for $b<1$. Thus the Greens function is ill defined for $b \geqslant 1$: The nonzero current at $r=0$ in this regime means the radial probability distribution $Q_G(r,t)$ collapses onto the origin. In terms of the dimension d' of a free random walk we have $b = 1-d'$, so the requirement $b<1$ means $d'>0$, which makes physical sense. A physically reasonable Greens function can be restored for the case $d>0$, $A \geqslant dD$, where $b \geqslant 1$, by regulating the $r=0$ singularity of the force, $F(r) = -A/r$. For example, if $F = -A/(r+\epsilon)$ we expect the Greens function to have a width which vanishes with ϵ.

We turn now to the more interesting case, for present purposes, of the distribution $Q_{abs}(r,t)$, appropriate to an absorbing point at $r=0$. Equation (12) gives $Q_{abs} \sim r$ for $r \to 0$, so the current (10) at the origin is $j(0) = -D(1+b)(Q_{abs}/r)_{r=0}$. This requires $b>-1$, since $j(0)<0$ for an absorbing boundary, and Q is necessarily non-negative. In terms of the equivalent free random walk with dimension d', the condition $b>-1$ requires $d'<2$, i.e., the probability to reach $r=0$ for a random walk in $d' \geqslant 2$ dimensions is zero.

For $b>-1$, the persistence probability, $P_0(t)$, can be readily calculated from Eq. (12) by first computing the current (10) at $r=0$. This current gives the rate of change of the persistence,

$$j(0) = \frac{dP_0}{dt} = -P_1(t), \tag{13}$$

where $P_1(t)$ is the probability distribution for the time at which the particle first reaches the origin (first passage time). Using Eqs. (10) and (12) gives the final result:

$$P_1(t) = \frac{1}{\Gamma[(1+b)/2]} \frac{4D}{r_0^2} \left(\frac{r_0^2}{4Dt}\right)^{(3+b)/2} \exp\left(-\frac{r_0^2}{4Dt}\right). \tag{14}$$

The persistence probability is $P_0(t) = \int_t^\infty ds\, P_1(s)$. Using the large-$t$ behavior of Eq. (14) gives

$$P_0(t) \to \frac{1}{\Gamma[(3+b)/2]} \left(\frac{r_0^2}{4Dt}\right)^{(1+b)/2} \tag{15}$$

for $t \to \infty$. Thus the ''persistence exponent,'' θ, is given by

$$\theta = (1+b)/2, \tag{16}$$

and is nonuniversal. For the free random walk in dimension d', this translates to $\theta = (2-d')/2$.

The nonuniversality of θ with respect to the strength, A, of the potential is special to the case of a logarithmic potential, for which the Langevin equation (1) is invariant under the rescalings $\mathbf{r} \to a\mathbf{r}$, $t \to a^2 t$ of space and time. This means that the potential is a *marginal* perturbation with respect to the equation with $A=0$, and it is this marginality which is responsible for the continuous variation of θ with A (actually, with the ratio A/D) through its dependence on b, which we recall is defined by Eq. (5). The condition $b>-1$ for the particle to visit the origin with probability one is equivalent to $A>(d-2)D$. Note that for $d=1$ this even allows a (sufficiently weak) repulsive potential, whereas for $d \geqslant 2$ a guaranteed visit to the origin requires a sufficiently strong attractive potential.

In the following section we discuss the case of a general power-law potential, corresponding to a force $F(r) = -A/r^\sigma$. We show that the force is irrelevant for $\sigma>1$, and the asymptotic persistence probability is that of a free random walker. For $\sigma<1$ the force is a relevant perturbation to the free random walker. In this case the noise term is irrelevant, but dangerously irrelevant as far as the calculation of $P_0(t)$ is concerned. We show that in this case $P_0(t)$ decays as a stretched exponential.

III. RANDOM WALKER IN A POWER-LAW POTENTIAL

After reducing the problem, as in the previous section, to an effectively one-dimensional problem for the distance r of the particle from the origin, the radial Langevin equation

reads

$$\frac{dr}{dt} = -\frac{A}{r^{\sigma}} + \frac{(d-1)D}{r} + \xi(t). \tag{17}$$

A. Scaling analysis

Under the scale transformations $r \to ar$, $t \to a^z t$, the Langevin equation (17) retains the same form, but with potential strength A and noise strength D rescaled to

$$A' = a^{z-1-\sigma} A, \tag{18}$$

$$D' = a^{z-2} D. \tag{19}$$

For $\sigma \neq 1$ these equations have two nontrivial fixed points:

(i) $A>0$, $D=0$, with $z=1+\sigma$, which is stable (D scales to zero at large time) for $\sigma<1$, and

(ii) $A=0$, $D>0$, with $z=2$, which is stable for $\sigma>1$. For the special case $\sigma=1$, there is a line of fixed points with $z=2$ and A/D arbitrary. In this last case, as we have seen, the exponent θ depends continuously on A/D.

At the second fixed point, with $\sigma>1$, the force is irrelevant: it falls off too rapidly with distance to affect the asymptotic large-time behavior. In particular, the exponent θ is given by the zero-force value, $\theta=(2-d)/2$ $(0<d<2)$.

At the first fixed point, with $\sigma<1$, the *noise* is irrelevant. If we set D to zero, the process (17) is *deterministic*: A particle staring at r_0 reaches the origin in a time $t = r_0^{1+\sigma}/[(1+\sigma)A]$. The limiting behavior of $P_0(t)$ at large t is, therefore, dominated by rare events: the noise is a *dangerously* irrelevant variable in this context, and we cannot simply set it to zero. Instead, we have to examine the limit of small but nonzero D.

We can argue for the asymptotic form of $P_0(t)$ as follows. A long survival time of the particle is a rare event, dominated by an activated process where the particle initially moves *away from the origin*. The potential corresponding to the force $-A/r^{\sigma}$ is $V(r)=Ar^{1-\sigma}/(1-\sigma)$. Suppose the particle is driven (by the noise) to a point r_1. The time for a subsequent deterministic descent to the origin is $t = r_1^{1+\sigma}/[A(1+\sigma)]$. The activation barrier for the ''uphill'' process is $\Delta V = [A/(1-\sigma)](r_1^{1-\sigma} - r_0^{1-\sigma})$, so the probability to reach r_1 before the origin is of order $\exp(-[A/(1-\sigma)]r_1^{1-\sigma}/D)$, where the term in r_0 in ΔV has been taken out and absorbed into a pre-exponential factor. Using the time for the subsequent free descent to estimate r_1, i.e., $r_1 \simeq [A(1+\sigma)t]^{1/(1+\sigma)}$, gives

$$P_0(t) \sim \exp\left(-\frac{A}{(1-\sigma)D} [A(1+\sigma)t]^{(1-\sigma)/(1+\sigma)} \right), \tag{20}$$

a stretched-exponential form, $P_0(t) \sim \exp(-\text{const} \times t^{\beta})$, with $\beta = (1-\sigma)/(1+\sigma)$. Note that the coefficient of t^{β} does not depend on the initial displacement r_0. The reason is that the rare trajectories for which the particle survives a long time take the particle far from its original position.

Despite its crudeness, this argument gives the correct result up to a constant of order unity in the exponent. This is shown in the next subsection using a path-integral formalism, augmented by a steepest descent calculation valid for $D \to 0$. Finally, we note that the crucial factor $(At)^{(1-\sigma)/(1+\sigma)}/D$ in the exponent of Eq. (20), and in particular the value of the exponent β, can be deduced immediately from dimensional analysis once one recognizes that (i) the result must be independent of r_0, and (ii) the factor $1/D$ is a necessary consequence of activated dynamics.

B. Path-integral formulation for $\sigma<1$

We begin from the probability distribution functional for the noise history $\xi(t)$. Since the noise is Gaussian and white, this functional is

$$P[\xi(t)] = N \exp\left(-\frac{1}{4D} \int dt \, \xi^2(t) \right), \tag{21}$$

where N is a normalization constant. This can be transformed to a probability distribution functional for $r(t)$ using the Langevin equation (17):

$$P[r(t)] = N \, J[r(t)] \exp\left(-\frac{1}{D} S[r(t)] \right), \tag{22}$$

where $J[r]$ is the Jacobian of the transformation from $\xi(t)$ to $r(t)$, whose precise form will not concern us, and $S[r]$ is the Onsager-Machlup functional (or action),

$$S[r] = \frac{1}{4} \int dt \left(\frac{dr}{dt} + \frac{A}{r^{\sigma}} - (d-1)\frac{D}{r} \right)^2. \tag{23}$$

It is convenient to compute $P_1(t)$, the probability density for the first visit to the origin (recall that $P_1 = -dP_0/dt$), given that the particle starts from r_0 at $t=0$. This is given by the path integral

$$P_1(t) \sim \int dr(t) J[r(t)] \exp\left(-\frac{1}{D} S[r(t)] \right), \tag{24}$$

where the time integral in Eq. (23) now runs from 0 to t (and we introduce a dummy time integration variable s), and the path integral is over all paths $r(s)$ which satisfy $r(0)=r_0$ and $r(t)=0$. The use of $\sim$ in Eq. (24) means we are concerned only with the leading exponential terms and not with the prefactors.

In the limit $D \to 0$, the path integral can be evaluated by steepest descents. To leading order the Jacobian, $J[r(t)]$, can be replaced by $J[r_c(t)]$, where $r_c(t)$ is the ''classical'' path which minimizes the action $S[r]$. The Jacobian therefore contributes to the prefactor, and we will not consider it further. In a similar way, the term $(d-1)D/r$ in S is subdominant for $D \to 0$ and can be dropped to leading order.

As noted above, the dominant path (path of least action) is the one which keeps $\xi(t)$ as small as possible for as long as possible. This is achieved by having the particle move initially away from the origin (''uphill''), where a smaller noise force is needed to overcome the deterministic force driving the particle towards the origin. The path $r_c(t)$ therefore consists of two parts: an uphill part to a maximum displacement r_1, followed by a deterministic ($\xi=0$) downhill part. Only the uphill path has a nonzero action associated with it.

The variational problem for the uphill path is simplified [16] by introducing the velocity $v=dr/ds$ and parametrizing the path by $v(r)$ instead of $r(s)$. The action then becomes

$$S[v]=\frac{1}{4}\int_{r_0}^{r_1}\frac{dr}{v}\left(v+\frac{A}{r^{\sigma}}\right)^2. \tag{25}$$

The variational equation $\delta S/\delta v=0$ becomes $1-A^2/(vr^{\sigma})^2=0$, with solutions $v=\pm A/r^{\sigma}$. The minus sign corresponds to the deterministic downhill path, with zero action, the plus sign to the uphill path. The action for the latter is

$$S=A\int_{r_0}^{r_1}\frac{dr}{r^{\sigma}}=\frac{A}{1-\sigma}(r_1^{1-\sigma}-r_0^{1-\sigma}). \tag{26}$$

The times t_u, t_d associated with the uphill and downhill paths, respectively, are

$$t_u=\int_{r_0}^{r_1}\frac{dr}{v}=\frac{1}{A(1+\sigma)}[r_1^{1+\sigma}-r_0^{1+\sigma}], \tag{27}$$

$$t_d=\frac{r_1^{1+\sigma}}{A(1+\sigma)}. \tag{28}$$

The final step is to eliminate r_1 in Eq. (26) in favor of the total time $t=t_u+t_d$. For $t\to\infty$, $r_1\to\infty$ and $r_0\ll r_1$ in Eq. (27). Dropping the terms in r_0 in Eqs. (26) and (27) gives $r_1\simeq[A(1+\sigma)t/2]^{1/(1+\sigma)}$ and

$$P_1(t)\sim\exp(-S/D)\simeq\exp\left(-\frac{A}{(1-\sigma)D}\left[\frac{A(1+\sigma)}{2}t\right]^{(1-\sigma)/(1+\sigma)}\right). \tag{29}$$

The persistence probability, $P_0(t)=\int_t^{\infty}dt\,P_1(t)$, clearly has the same asymptotic form. It differs from Eq. (20) only by factors of order unity, as promised. It is easy to show that the corrections due to keeping the r_0 term in Eq. (27) vanish for $t\to\infty$, while the correction associated with the r_0 term in Eq. (26) represents a time-independent prefactor. This prefactor, $\exp[Ar_0^{1-\sigma}/(1-\sigma)D]$ can, of course, be very large (and very sensitive to r_0) for small D.

It is important to note that while the asymptotic form (29) was derived in the limit $D\to 0$, it actually holds as an asymptotic result for all D, since D is an irrelevant variable and scales to zero as $t\to\infty$.

IV. VORTEX-ANTIVORTEX ANNIHILATION IN THE 2D *XY* MODEL

A. Nonequilibrium critical dynamics

As a final application of these methods, we consider vortex dynamics in the two-dimensional (2D) *XY* model, which was in fact the motivation for the present study. At all temperatures $T\leqslant T_{KT}$, where T_{KT} is the Kosterlitz-Thouless (KT) transition temperature, a system prepared in a nonequilibrium initial state will approach the equilibrium state through a coarsening mechanism in which local equilibrium is established over a length scale $\xi(t)$ which grows with time. For example, the spin-spin correlation function has, according to the KT theory [13], the equilibrium form $C(r)\sim r^{-\eta(T)}$, for all $T\leqslant T_{KT}$. Consider now a system prepared in a random initial state, with only short-range spatial correlations, and allowed to evolve in contact with a heat bath at temperature $T\leqslant T_{KT}$. According to the conventional theory of nonequilibrium critical phenomena [12], the system will approach equilibrium via a dynamical scaling state, characterized by a growing length scale $\xi(t)$. For example, the scaling form for the spin-spin correlation function reads

$$C(r,t)=\frac{1}{r^{\eta}}f\left(\frac{r}{\xi(t)}\right). \tag{30}$$

The scaling function $f(x)$ has the limiting behavior $f(0)=\text{const}$ [so that equilibrium is recovered for $\xi(t)\to\infty$], while $f(x)$ falls off rapidly for $x\gg 1$, representing the fact that the spins are uncorrelated on length scales large compared to $\xi(t)$.

The standard theory of nonequilibrium critical dynamics predicts that the length scale $\xi(t)$ should grow as $\xi(t)\sim t^{1/z}$, where z is the *equilibrium* dynamical exponent. This should hold independent of the initial conditions, though the scaling function $f(x)$ in Eq. (30) can depend on initial conditions. A commonly considered case is uniform initial conditions (all spins parallel). For this case one requires that $f(x)\sim x^{\eta}$ for $x\to\infty$, since the long-range order present in the initial condition will persist at any finite time.

In a recent paper, however, Bray *et al.* [17] have argued, on the basis of numerical simulations and physical arguments, that this picture breaks down for the 2D *XY* model with nonconserved dynamics (model A of the Hohenberg-Halperin [18] classification). Specifically, the growth of $\xi(t)$ will depend on whether the initial state contains free vortices and antivortices. In particular, they argue that for a uniform initial condition, for which there are no free vortices, $\xi(t)$ is determined by spin-wave theory to be $\xi(t)\sim t^{1/2}$ [19]. On the other hand, for a random initial condition there are many free vortices and antivortices present. The dominant coarsening mechanism in this case is vortex-antivortex annihilation, and this leads to $\xi(t)\sim(t/\ln t)^{1/2}$.

B. Vortex-antivortex annihilation

Physical arguments for $\xi(t)\sim(t/\ln t)^{1/2}$ have been given previously for the coarsening dynamics at $T=0$ from a random initial condition [20,21]. The basic idea [20] is to consider a single vortex-antivortex pair, and to derive expressions for the energy, $E_p(r)$ of the pair, and hence the force, $F(r)=-dE_p/dr$, between them, as a function of their separation r. The result is $E_p\sim\ln(r/a)$ (where a, the vortex core scale, is a microscopic length), and hence $F\sim -1/r$. Crucially it is found [20] that the vortex mobility μ, which relates the force F, to the velocity v, via $v=\mu F$, also depends logarithmically on the pair separation: $\mu\sim 1/\ln(r/a)$. At $T=0$, therefore, the variable r obeys the deterministic equation

$$\ln\left(\frac{r}{a}\right)\frac{dr}{dt}=-\frac{1}{r}, \tag{31}$$

up to an overall constant which can be absorbed into the timescale. Integrating this equation gives the annihilation time for a pair initially separated by a distance $r_0 \gg a$, namely $t \sim r_0^2 \ln(r_0/a)$, which one can invert to obtain $r_0 \sim [t/\ln(t/a^2)]^{1/2}$. This already suggests that, in a many vortex situation, lengths and time are related by $\xi(t) \sim (t/\ln t)^{1/2}$.

This result can be motivated in another way using scaling arguments. If the characteristic scale in a many-vortex system is ξ, the typical force on a vortex scales as $1/\xi$, the typical mobility as $1/\ln(\xi/a)$ and the typical velocity as $d\xi/dt$. This gives $\ln(\xi/a)d\xi/dt \sim 1/\xi$, i.e., $\xi \sim (t/\ln t)^{1/2}$ as before. We now discuss the influence of thermal fluctuations on the annihilation of a single vortex-antivortex pair at nonzero temperatures.

C. Vortex-antivortex annihilation at $T>0$

As a first step, we present a detailed and quantitative treatment of the $T=0$ arguments employed in the previous subsection. It is convenient to adopt a continuum approach based on the nonlinear σ model Hamiltonian

$$H = \frac{\rho_s}{2} \int d^2 r \, (\nabla \vec{\phi})^2, \tag{32}$$

where $\vec{\phi}$ is the two-component order-parameter field, subject to a local constraint $\vec{\phi}^2 = 1$, and ρ_s is the spin-wave stiffness. For a field configuration describing a single free vortex, $\vec{\phi} = \vec{r}/|\vec{r}|$, one has $(\nabla \vec{\phi})^2 = 1/r^2$, leading to an energy $E_v = (\rho_s/2) \int (d^2 r/r^2) = \pi \rho_s \ln(L/a)$, where L and a are the system size and microscopic cutoff, respectively. A vortex-antivortex pair, separated by distance r, screen each other's far fields at scales larger than r, leading to a pair energy $E_p(r) \simeq 2\pi\rho_s \ln(r/a)$, and an attractive force $F = -dE_p/dr = -2\pi\rho_s/r$ between the vortex and the antivortex.

The corresponding continuum description of the nonconserved (model A) dynamics is given, at $T=0$, by the Langevin equation [18]

$$\frac{\partial \vec{\phi}}{\partial t} = -\Gamma \frac{\delta H}{\delta \vec{\phi}}. \tag{33}$$

This equation can be used [20] to compute the effective friction constant $\gamma(r) = 1/\mu(r)$, where μ is the mobility, associated with the motion of the vortex and antivortex under the force F. An isolated vortex moving at speed v in the x direction has field configuration $\vec{\phi}(x,y,t) = \vec{\phi}_v(x - vt, y)$. Energy is dissipated at a rate

$$\begin{aligned} \frac{dE}{dt} &= \int d^2 r \left(\frac{\delta H}{\delta \vec{\phi}} \right) \cdot \left(\frac{\partial \vec{\phi}}{\partial t} \right) \\ &= -\frac{1}{\Gamma} \int d^2 r \left(\frac{\partial \vec{\phi}}{\partial t} \right)^2 \\ &= -\frac{v^2}{\Gamma} \int d^2 r \left(\frac{\partial \vec{\phi}_v}{\partial x} \right)^2 \\ &= -\gamma_v v^2, \end{aligned} \tag{34}$$

where the notation γ_v indicates the residual dependence of γ on v at noninfinitesimal velocities. Inserting the equilibrium vortex configuration, which is isotropic, gives the limiting zero-velocity friction constant: $\gamma_0 = E_v/\rho_s\Gamma = (\pi/\Gamma)\ln(L/a)$, i.e., γ_0, like the vortex energy E_v, diverges logarithmically with the system size, L. For a vortex-antivortex pair, this translates into a logarithmic dependence on the separation [20],

$$\gamma(r) \simeq \left(\frac{\pi}{\Gamma} \right) \ln \left(\frac{r}{a} \right). \tag{35}$$

The effect of thermal fluctuations, neglected up to now, is twofold. First, as in the equilibrium theory, thermally activated bound vortex-antivortex pairs lead to a renormalization of the spin-wave stiffness, ρ_s, and kinetic coefficient, Γ, to temperature-dependent functions, $\rho_s(T)$ and $\Gamma(T)$, that describe the large length-scale properties of the system. In equilibrium, however, there are no *free* vortices at any temperature below the KT transition temperature, T_{KT}. This means that the large-scale properties are described by the spin-wave theory. In this theory all vortices are neglected and the angle representation, $\vec{\phi}(\mathbf{r}) = [\cos\theta(\mathbf{r}), \sin\theta(\mathbf{r})]$ is employed, with the angles $\theta(\mathbf{r})$ defined on the interval $(-\infty, \infty)$. The effective Hamiltonian for the long-wavelength degrees of freedom is $H = [\rho_s(T)/2] \int d^2 r (\nabla\theta)^2$, and the equation of motion is $\partial\theta/dt = -\Gamma(T)(\delta H/\delta\theta) + \xi(\mathbf{r},t) = \Gamma(T)\rho_s(T)\nabla^2\theta + \xi$, where $\xi(\mathbf{r},t)$ is a Gaussian white noise with correlator given by the fluctuation-dissipation theorem, $\langle \xi(\mathbf{r},t)\xi(\mathbf{r}',t') \rangle = 2\Gamma(T) k_B T \delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$.

Since the equation of motion is linear, it can be solved exactly. The dynamic exponent is $z=2$ (for all $T \leqslant T_{KT}$). Nonequilibrium properties can also be evaluated exactly in the absence of free vortices [19]. For nonequilibrium situations where free vortices and antivortices are present, e.g., after a quench into the KT phase from a disordered (high-temperature) initial condition, one can argue as follows. In the late stages of coarsening, when the remaining free vortices and antivortices are widely separated, with a typical spacing $\xi(t)$, the spin-wave theory can be used on scales much larger than the microscopic scale a but smaller than $\xi(t)$. For example, the calculation of the dynamics of a single, widely separated vortex-antivortex pair would proceed as at $T=0$, but using the temperature-dependent function $\rho_s(T)$ and $\Gamma(T)$ that incorporate the effect of thermal fluctuations on smaller length scales. [Strictly one should use a scale-dependent spin-wave stiffness and kinetic coefficient, evaluated at scale $\xi(t)$, but we are interested in the limit $\xi(t) \to \infty$ where these functions can be replaced by their infinite-scale limits, $\rho_s(T)$ and $\Gamma(T)$.]

In this approach the only modification of the $T=0$ results would be the replacement of ρ_s and Γ, in the expression for the annihilation time of a vortex-antivortex pair, by their T-dependent generalizations, as follows. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the positions of the vortex and antivortex, and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ be their relative separation. The equations of motion for $\mathbf{r}_1$ and $\mathbf{r}_2$, at $T=0$, read

$$\gamma(r) \frac{d\mathbf{r}_1}{dt} = \frac{2\pi\rho_s}{r} \hat{\mathbf{r}},$$

$$\gamma(r)\frac{d\mathbf{r}_2}{dt}=-\frac{2\pi\rho_s}{r}\hat{\mathbf{r}}, \tag{36}$$

where $\gamma(r)\simeq(\pi/\Gamma)\ln(r/a)$ is the vortex mobility. Subtracting these gives the equation for the relative separation $\mathbf{r}$:

$$\gamma(r)\frac{d\mathbf{r}}{dt}=-\frac{4\pi\rho_s}{r}\hat{\mathbf{r}}. \tag{37}$$

Notice that the force $-4\pi\rho_s/r$ corresponds to an effective potential $V_{\rm eff}=2V(r)$, where $V(r)=2\pi\rho_s\ln(r/a)$ is the energy of a vortex-antivortex pair.

How do we generalize this equation to $T>0$? There are two distinct effects. First as we have discussed, thermally induced bound vortex-antivortex pairs renormalize the large-scale spin-wave stiffness and kinetic coefficient to temperature-dependent functions $\rho_s(T)$ and $\Gamma(T)$, with nonzero values $\rho_s(T_{KT})$ and $\Gamma(T_{KT})$ at the transition, i.e., the spin-wave theory describes the large-distance, large-time behavior at and below T_{KT}. Above T_{KT}, the spin-wave theory breaks down due to the thermal nucleation of free vortices and antivortices.

If the renormalization of ρ_s and Γ were the only effects of thermal fluctuations, we would conclude immediately that deterministic annihilation of a single vortex-antivortex pair would occur, as at $T=0$, on a time scale $t\sim r_0^2\ln(r_0/a)$, where r_0 is the initial separation. A second consequence of thermal noise, however, is diffusion of the vortex and antivortex. This means we have to add Langevin noise terms to Eq. (36). We anticipate that the strength of the noise will be proportional to $\sqrt{\gamma(r)}$ on account of the fluctuation-dissipation theorem (Einstein relation). Therefore we write a stochastic version of Eq. (37) in the form

$$\gamma(r)\frac{d\mathbf{r}}{dt}=-\boldsymbol{\nabla}U+\sqrt{\gamma(r)}\boldsymbol{\xi}(t), \tag{38}$$

where

$$\langle\xi_i(t)\xi_j(t')\rangle=4D\delta_{ij}\delta(t-t'), \tag{39}$$

and $U(r)$ is a central potential which, it turns out, will differ by terms of order D from the effective potential $V_{\rm eff}(r)=4\pi\rho_s\ln r$ suggested by Eq. (37). We will determine $U(r)$ from the condition that the stationary distribution $Q(r)$ derived from Eq. (38) satisfies $Q(r)\propto 2\pi r\exp[-V_{\rm eff}(r)/2D]$. The strength of the noise in Eq. (38) is $4D$, rather than the usual $2D$ (where $D=k_BT$), since the noise acting on $\mathbf{r}$ is the sum of independent noises of strength $2D$ acting on $\mathbf{r}_1$ and $\mathbf{r}_2$.

The noise term in Eq. (38) is multiplicative noise in the Stratonovich sense. To determine the stationary distribution, and hence infer the form of $U(r)$, we first write Eq. (38) in the canonical form

$$\frac{d\mathbf{r}}{dt}=\mathbf{A}(r)+g(r)\boldsymbol{\xi}(t), \tag{40}$$

where

$$g=\frac{1}{\sqrt{\gamma}}, \tag{41}$$

$$\mathbf{A}=-\frac{1}{\gamma}\boldsymbol{\nabla}U=-g^2\boldsymbol{\nabla}U. \tag{42}$$

The corresponding Fokker-Planck equation is

$$\frac{\partial P}{\partial t}=\boldsymbol{\nabla}\cdot[-\mathbf{A}P+2Dg\boldsymbol{\nabla}(gP)]. \tag{43}$$

Noting that $\mathbf{A}(\mathbf{r})$ is in the radial direction, i.e., $\mathbf{A}(\mathbf{r})=-g^2(r)(\partial_rU)\hat{\mathbf{r}}$, this equation can be reduced to a one-dimensional equation for the angle-averaged radial distribution function, $Q(r,t)=r^{d-1}\int d\Omega P(r,\Omega,t)$, just as in Sec. II A:

$$\partial_tQ=\partial_r[g^2(\partial_rW)Q+2Dg\partial_r(gQ)], \tag{44}$$

where

$$W(r)=U(r)-2D(d-1)\ln r. \tag{45}$$

Equation (44) can be recast as the one-dimensional Langevin equation

$$\frac{dr}{dt}=-g^2(r)\frac{dW}{dr}+g(r)\,\xi(t), \tag{46}$$

with $\langle\xi(t)\xi(t')\rangle=4D\delta(t-t')$, which in turn can be written as a Langevin equation with *additive* noise, via the change of variable $dy=dr/g(r)$:

$$\frac{dy}{dt}=-\frac{dW}{dy}+\xi(t). \tag{47}$$

We can now determine the potential $U(r)$ that corresponds to a ''physical'' potential $V(r)$. Since the effective potential for the dynamics of y is $W(y)$, the stationary distribution for y is $P_y(y)\propto\exp[-W(y)/2D]$, implying that the radial distribution function, $Q(r)=P_y(y)(dy/dr)$, is given by $Q(r)\propto[g(r)]^{-1}\exp[-W(r)/2D]$ which, using Eq. (45), implies $Q(r)\propto r^{d-1}\exp\{-[U(r)-2D\ln g(r)]/2D\}$. The prefactor r^{d-1} is just the phase-space factor for the d-dimensional space: the required stationary distribution function for motion in a potential $V_{\rm eff}(r)$ $[=2V(r)]$ is $Q(r)\propto r^{d-1}\exp[-V_{\rm eff}(r)/2D]$. Equating these two results for $Q(r)$, and using $V_{\rm eff}(r)=2V(r)$, where $V(r)$ is the vortex-antivortex potential energy, determines $U(r)$:

$$U(r)=2V(r)+2D\ln g(r). \tag{48}$$

The final equation for $y(t)$ becomes

$$\frac{dy}{dt}=\left(-2\frac{dV}{dr}-\frac{2D}{g}\frac{dg}{dr}+\frac{2(d-1)D}{r}\right)\frac{dr}{dy}+\xi(t). \tag{49}$$

The final step is to apply this (rather general) result to the 2D XY model. Inserting $\gamma(r)=\gamma_0\ln(r/a)$ and $V(r)=\alpha\ln(r/a)$, where $\gamma_0=\pi/\Gamma$ and $\alpha=2\pi\rho_s$, in Eq. (49), using $dr/dy=g(r)=1/\sqrt{\gamma(r)}$, and setting $d=2$, gives

$$\frac{dy}{dt}=\frac{2}{\sqrt{\gamma(r)}}\left(\frac{D-\alpha}{r}-\frac{D}{g}\frac{dg}{dr}\right)+\xi(t). \tag{50}$$

Consider now the size of the second term in the large bracket. Using Eq. (41) gives

$$-\frac{1}{g}\frac{dg}{dr}=\frac{1}{2}\frac{d\ln\gamma}{dr}=\frac{1}{2r\ln(r/a)}\ll\frac{1}{r}$$

for $r\to\infty$. This means that this term can be dropped in the calculation of the large-distance behavior of the system. Furthermore, integrating the relation $dy/dr=\sqrt{\gamma(r)}$ gives, to leading order for large r, $y=r\sqrt{\gamma(r)}=r\sqrt{\gamma_0\ln(r/a)}$, with corrections which are again of relative order $1/\ln(r/a)$. To leading order, therefore, Eq. (50) reduces to (inserting $\alpha=2\pi\rho_s$)

$$\frac{dy}{dt}=\frac{2D-4\pi\rho_s}{y}+\xi(t), \tag{51}$$

with $\langle\xi(t)\xi(t')\rangle=4D\delta(t-t')$.

Apart from a doubling of the noise strength, $D\to 2D$, associated with the fact that we are dealing here with a two-body problem, the result (51) is identical to Eq. (6) derived in Sec. II A for a particle moving in a logarithmic potential, except that the coordinate y corresponds to $r\sqrt{\gamma_0\ln(r/a)}$ in the present context as a result of the scale-dependent mobility. Vortex-antivortex annihilation can be deemed to have occurred when $r=a$, the vortex core size, which corresponds to $y=0$, so our previous results on persistence in a logarithmic potential can be applied directly to the persistence problem for a vortex-antivortex pair. In particular the parameter b, which controls the first-passage time distribution, $P_1(t)$, and persistence probability, $P_0(t)$, according to Eqs. (14) and (15), respectively, is obtained by setting $A=4\pi\rho_s$, $d=2$ and $D\to 2D=2k_BT$ in Eq. (5) to give

$$b=\frac{2\pi\rho_s(T)}{k_BT}-1. \tag{52}$$

The ratio $\rho_s(T)/k_BT$ is a decreasing function of T, and approaches the universal limit $2/\pi$ for $T\to T_{KT}$. It is related to the exponent $\eta(T)=k_BT/2\pi\rho_s(T)$, i.e., $b=(1-\eta)/\eta$. It follows that b is a decreasing function of T, diverging to infinity for $T\to 0$, and approaching the limiting value 3 for $T\to T_{KT}$, where $\eta=1/4$. Recall that the persistence exponent is $\theta=(1+b)/2$, so θ varies continuously between $\theta(0)=\infty$ and $\theta(T_{KT})=2$, where the infinite limiting value at $T=0$ simply reflects the deterministic collapse in the absence of thermal noise. The persistence exponent for vortex-antivortex annihilation is therefore nonuniversal, depending continuously on T:

$$\theta=\frac{\pi\rho_s(T)}{k_BT}=\frac{1}{2\eta(T)}, \tag{53}$$

for $T\leqslant T_{KT}$. From Eq. (14) we see that the mean annihilation time is finite for all $T\leqslant T_{KT}$ (since $b\geqslant 3$), and the characteristic annihilation time scales as $t\sim y_0^2\sim r_0^2\ln(r_0/a)$ where r_0 is the initial separation. We conclude that thermal fluctuations do not change the fundamental relation between length scales and time scales deduced earlier from zero-temperature considerations, and that the coarsening growth law, $\xi(t)\sim(t/\ln t)^{1/2}$, holds for all $T\leqslant T_{KT}$.

To end this section we briefly discuss the equilibration of the vortex density. We have argued (see also [20]) that for the usual ''downquench,'' into the KT phase from the paramagnetic phase, the typical vortex separation increases as $\xi(t)\sim(t/\ln t)^{1/2}$, as free vortices and antivortices annihilate. So the density, ρ_V, of free vortices decreases as $\rho_V\sim 1/\xi(t)^2\sim(\ln t)/t$, a result confirmed numerically by Yurke *et al.* [20]. What about the corresponding upquench, from $T=0$ to another temperature in the KT phase? In this case no free vortices appear, but one can still discuss [22] the time scale for the equilibration of bound pairs with given separation. These pairs are created at the core scale a. Dimensional analysis of Eq. (51) suggests $t\sim y^2\sim r^2\ln(r/a)$ as the relevant timescale for pairs at scale r. As far as the vortices are concerned, therefore, both types of quench involve similar relations between time scales and length scales. The main point is that bound pairs do not change qualitatively the large-distance form of spin-correlation functions, whereas free vortices do. The relaxation of any quantity expressible in terms of spin correlations only, such as the Binder parameter (see the following section), will therefore be different for downquenches and upquenches.

D. Comparison with simulation data

Recent simulation data [17] support the conclusions of the above analysis. Since the method and results have already been presented in a short paper [17], we will just briefly recall the salient features here. The nearest-neighbor lattice Hamiltonian $H=-\Sigma_{\langle i,j\rangle}\cos(\theta_i-\theta_j)$ was simulated on square lattices of size $L\times L$, for $12\leqslant L\leqslant 48$. Both uniform (all spins parallel) and disordered (randomly orientated spins) initial conditions were employed. In both cases, the ''time-dependent Binder cumulant'' [23,24]

$$g(L,t)=2-\frac{\langle[\vec{M}(t)^2]^2\rangle}{[\langle\vec{M}(t)^2\rangle]^2}, \tag{54}$$

where $\vec{M}(t)$ is the total magnetization at time t, was measured, and a finite-size scaling, $g(L,t)=f[\xi(t)/L]$, attempted, where $\xi(t)$ is a characteristic length scale at time t.

In [17], all simulations were at the KT temperature $T=0.90$ [25]. In earlier work, Luo *et al.* [26] have additionally studied some lower temperature within the KT phase. For uniform initial conditions, a good scaling collapse was obtained using a characteristic length scale $\xi(t)\sim t^{1/z}$, with an exponent z that tends to 2 (from slightly smaller values) for large L in accordance with the predictions of spin-wave theory. This is entirely reasonable since the uniform initial conditions contain no free vortices and subsequent thermal fluctuations can only produce bound vortex-antivortex pairs.

For disordered initial conditions, a naive collapse using $\xi(t)\sim t^{1/z}$ gives unreasonably large values of z, in the range 2.3–2.4, [26,17], for all temperatures studied. Disordered initial conditions seed the system with free vortices and antivortices, so the analysis presented above suggests that a characteristic length scale $\xi(t)\sim(t/\ln t)^{1/2}$ is appropriate for this case. This form indeed yields an excellent data collapse.

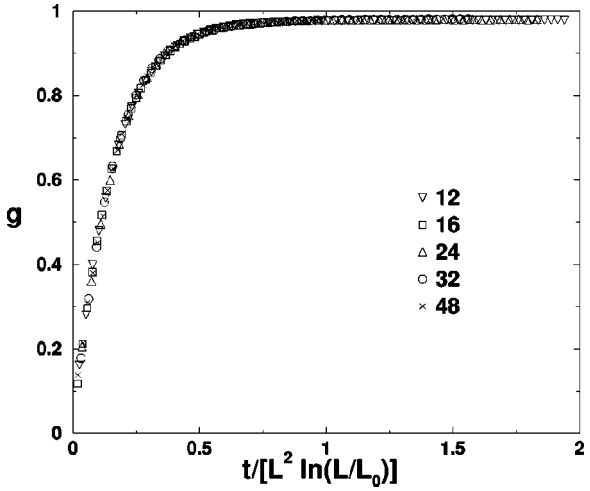

FIG. 1. Scaling plot for the time-dependent Binder parameter, starting from a disordered initial condition, for system sizes $L=12,16,24,32,48$, and $L_0=1.4$

In practice, the form $\xi(t)=[t/\ln(t/t_0)]^{1/2}$ is used, with t_0 a fitting parameter whose value is of order unity [17].

The data can, however, be presented in another way. The logic of finite-size dynamical scaling requires that the scaling variable can be written as $t/\tau(L)$, where $\tau(L)$ is the relaxation time of the finite-size system. For uniform initial conditions the spin-wave theory, which determines the relaxation of spin correlations in the absence of free vortices, gives $\tau(L)\sim L^2$, equivalent to the previous scaling form, whereas for disordered initial conditions $\tau(L)$ is given by the time for the last free vortex-antivortex pair, which are typically separated by a distance of order L, to annihilate, i.e., $\tau(L)\sim L^2\ln L$. A scaling collapse using $t/[L^2\ln(L/L_0)]$ as scaling variable is presented in Fig. 1, with $L_0=1.4$ where L_0 is a short distance cutoff expected (and found) to be of order unity. The scaling collapse is excellent. For large enough L and t, this method of plotting the data and the previous method, using $t/[L^2\ln(t/t_0)]$ as scaling variable, will be indistinguishable. For finite times and sizes, however, they differ slightly, and the new method gives an improved collapse.

E. Discussion

The fact that $\xi(t)$ depends on the initial conditions is surprising from the viewpoint of conventional nonequilibrium critical dynamics [12], according to which $\xi(t)\sim t^{1/z}$, where z is the dynamical exponent for equilibrium critical dynamics. An equivalent statement in the context of finite-size dynamical scaling is that the relaxation time grows with L as $\tau(L)\sim L^z$, independent of initial conditions. While this seems to be true for most phase transitions (e.g., the 2D Ising model [27]) we have argued that it does not hold for spin correlations in the 2D XY model, at and below T_{KT}, for disordered initial conditions. We conjecture that the breakdown of the standard field-theoretic methods [12] in this case is due to the key role played by vortex configurations which, due to their non-perturbative character, are not accessible to perturbative methods based on a $4-\epsilon$ expansion. The latter involves perturbing around a Gaussian theory, which cannot support topological defects. It would be interesting to investigate whether similar results are obtained in other defect-driven phase transitions.

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