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RAPID COMMUNICATIONS

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Excess noise in intracavity laser frequency modulation

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The response to perturbations and to stochastic noise of a laser below threshold subjected to an intracavity periodic frequency modulation is theoretically studied. It is shown that, when the modulation frequency is close to the cavity axial mode separation but yet detuned from exact resonance, the laser exhibits a strongly enhanced sensitivity to external noise, which includes large transient energy amplification of perturbations in the deterministic case and enhancement of field fluctuations in presence of a continuous stochastic noise. This large excess noise is due to the nonorthogonality of Floquet laser modes which makes it possible continuous energy transfer from the forcing noise to transient growing perturbations.

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In recent years an increasing and considerable interest has been devoted to the study of noise in non-Hermitian physical systems, and many theoretical and experimental works in different physical contexts, including hydrodynamics [1–4] and nonlinear optics [5–7], have revealed a rather universal feature of these systems to show some form of ''excess noise'' as compared to common normal systems. In the optical context, much attention has been payed to the study of non-Hermitian laser cavities, where the nonorthogonality of resonator cavity eigenmodes (longitudinal or transverse) [5,6] or of polarization eigenmodes [7] leads to an enhancement of spontaneous-emission noise in the lasing mode, expressed by the Petermann excess noise factor [8]. A different manifestation of excess noise in non-normal systems, which has been recently recognized in the hydrodynamic context but not yet in nonlinear optics, is the enhancement of variance levels sustained by a non-normal system close to an instability when subjected to a continuous stochastic forcing [1,3,4]. It is well known that any physical system close to a bifurcation point shows some universal dynamical features, such as critical slowing down, spectral narrowing, and noise amplification [9]. In particular, amplification of noise in a typical stochastically driven normal system near an instability basically depends on the decay rate of the most unstable normal mode, thus resulting in large noise variance levels close to the instability [see [3,9] and Eq. (6) below]. In non-normal systems, amplification of noise may largely exceed the expected level of a normal system, as shown in recent works on Couette and Poiseuille hydrodynamic flows [1,3]. The enhanced amplification of noise can be traced to the nonorthogonality of modes; this circumstance, however, as shown in a very general framework by Farrell and Ioannou [3,4], is a necessary but not a sufficient condition to observe excess variance levels, which requires, besides lack of mode orthogonality, also the capability of the system to support transient growth of perturbations despite the asymptotic linear stability of the system. In this case, a dynamical amplification of noise, with a variance level well above that anticipated by the decay rate of modes, is possible indeed. This variance enhancement effect should be therefore observable also in non-normal active optical systems capable of supporting transient growth of perturbations and should be regarded as a further and distinctive feature of non-Hermitian cavity optics.

In this Rapid Communication we demonstrate that a laser with an internal periodic modulation of the optical frequency [10–13] operated just below threshold for oscillation is an example of a non-normal system in nonlinear optics that exhibits large excess of variance for the field fluctuations when subjected to a continuous noise. This excess noise is a signature of the nonorthogonality of laser modes and is physically due to an excess gain that makes possible transient amplification of spectrally narrow perturbations despite the linear stability of the nonlasing solution. As compared to previous analyses on excess noise in lasers (see [6] and references therein), in this work the nonorthogonality of modes does not involve transverse, longitudinal, nor polarization eigenmodes of the laser, but time-dependent periodic Floquet modes that are the natural eigenmodes of a frequency-modulated laser [13].

The starting point of our analysis is provided by a rather standard model of intracavity laser frequency modulation (FM) in a laser cavity with a spectral gainline much broader than the cavity free spectral range [10–13]. We consider a ring cavity of length L containing a gain medium, a frequency limiter that determines the gain bandwidth of the cavity, and an electro-optic phase modulator that varies periodically the optical cavity length at a frequency ω_m close to the longitudinal mode separation $\omega_c = 2\pi c/L$ of the cavity. After expanding the intracavity electric field F on the basis of longitudinal ring cavity eigenmodes by setting $F(z,t) = \Sigma_n F_n(t)\exp(-2\pi int)\exp(2\pi inz)$, where z is the longitudinal spatial coordinate along the cavity, scaled to the cavity length L, and t is time normalized to the modulation period $T_m = 2\pi/\omega_m$, the semiclassical coupled-mode equations for the amplitudes F_n read [10,11]

$$\dot{F}_n = (-2\pi in\gamma + g_n - l)F_n + \frac{i\Delta}{2}(F_{n+1} + F_{n-1}) + \xi_n \quad (1)$$

$(n = 0, \pm 1, \pm 2, \dots)$, where $\gamma \equiv (\omega_c - \omega_m)/\omega_m$ is the frequency detuning parameter $(|\gamma| \ll 1)$, g_n is the round-trip gain for the nth mode, l is the cavity loss, Δ is the single-pass modulation index introduced by the phase modulator, and the dot stands for the derivative with respect to time. A simple model for the spectral gain g_n is provided by $g_n = g - n^2(\omega_c/\omega_g)^2$, where g is the gain of the central mode $n = 0$, which is assumed to be tuned at the center of the gainline, and ω_g is the spectral bandwidth of the gainline $(\omega_c/\omega_g \ll 1)$, which is assumed to be independent of the gain parameter g (see, for instance, [12]). Other effects, such as frequency pulling and cavity dispersion effects, will be not considered here for the sake of simplicity, although their effects would not change substantially the basic dynamics of the frequency-modulated laser [13]. In Eq. (1), $\xi_n(t)$ are independent Gaussian complex stochastic variables with zero mean and correlation $\langle \xi_n(t')\xi_l^*(t'')\rangle = \epsilon\delta_{n,l}\delta(t'-t'')$, which provide a standard semiclassical model of noise in a laser, such as the spontaneous emission noise of the laser (see, for instance, [5]). In absence of the stochastic noise sources, the Langevin equations (1) have the deterministic zero solution $F_n = 0$ $(n = 0, \pm 1, \pm 2, \dots)$, corresponding to the laser being below threshold. This solution is stable provided that the matrix $\mathcal{A}$ in the linearized dynamics, given by

$$\mathcal{A}_{n,m} = \left[-2\pi in\gamma + g - l - n^2\left(\frac{\omega_c}{\omega_g}\right)^2\right]\delta_{n,m} + i\frac{\Delta}{2}(\delta_{n,m+1} + \delta_{n,m-1}) \quad (2)$$

is asymptotically stable. This implies $g < g_{th}$, where the threshold for laser oscillation, g_{th}, is attained when the real part of the most unstable eigenvalue of $\mathcal{A}$ crosses zero. Notice that the Langevin equations (1), as derived by field expansion in terms of longitudinal cavity eigenmodes, are coupled in the presence of the intracavity phase modulation, i.e., when $\Delta \neq 0$. Diagonalization is possible by introduction of the set of Floquet modes, which represent the natural basis of modes of the frequency-modulated laser [14]; these are defined by

$$|\alpha(z,t)\rangle = \sum_n v_n^{(\alpha)}\exp[2\pi in(z-t)], \quad (3)$$

where $v_n^{(\alpha)}$ is the αth eigenvector of $\mathcal{A}$ corresponding to the eigenvalue μ_α. After expanding the intracavity field $F(z,t)$ on the basis of Floquet modes, $F(z,t) = \Sigma_\alpha f_\alpha(t)|\alpha\rangle$, from Eqs. (1) it follows that the expansion coefficients $f_\alpha(t)$ satisfy the uncoupled Langevin equations

$$\dot{f}_\alpha = \mu_\alpha f_\alpha + \eta_\alpha \quad (4)$$

where $\langle \eta_\alpha(t)\rangle = 0$, $\langle \eta_\alpha^*(t')\eta_\alpha(t'')\rangle = K_\alpha\epsilon\delta(t'-t'')$, and $K_\alpha \geq 1$ is the excess noise factor for the mode $|\alpha\rangle$. As usual [5], it turns out that $K_\alpha = \langle \alpha^\dagger|\alpha^\dagger\rangle$, where $|\alpha^\dagger\rangle$ is the adjoint mode, given by $|\alpha^\dagger\rangle = \Sigma_n v_n^{\dagger(\alpha)}\exp[2\pi in(z-t)]$, $v_n^{\dagger(\alpha)}$ is the eigenvector of the adjoint matrix $\mathcal{A}^\dagger$ corresponding to the eigenvalue μ_α^*, and $\langle f|g\rangle$ stands for $\int_0^1 dz f^*(z)g(z)$; normalization has been chosen such that $\langle \alpha|\alpha\rangle = 1$ and $\langle \alpha^\dagger|\beta\rangle = \delta_{\alpha,\beta}$ [14]. In presence of a phase modulation, the Floquet modes $|\alpha\rangle$ are in general not orthogonal. This can be seen, for instance, by observing that $\langle \alpha|\beta\rangle = \Sigma_n v_n^{(\alpha)*}v_n^{(\beta)}$ and that, for $\Delta \neq 0$, the matrix $\mathcal{A}$ is not normal, i.e., $\mathcal{A}$ does not commute with its adjoint $\mathcal{A}^\dagger$. A direct calculation of the commutator $[\mathcal{A},\mathcal{A}^\dagger]$ shows in fact that it scales like $\sim\Delta(\omega_c/\omega_g)$, vanishing when no modulation is applied $(\Delta = 0)$ or when the spectral bandwidth of the gain is infinite $(\omega_c/\omega_g \rightarrow 0)$. The former case is trivial and describes the dynamics of the longitudinal normal modes of the free-running laser. The latter case corresponds to the laser operated in the ideal FM regime, where the normal modes $|\alpha\rangle$ reduce to the ideal Bessel modes $|\alpha\rangle = \exp[i\Gamma\sin(2\pi z - 2\pi t)]\exp[2\pi\alpha iz - 2\pi\alpha i(1+\gamma)t]$ $(\alpha = 0, \pm 1, \pm 2, \dots)$, $\Gamma = \Delta/(2\pi\gamma)$ being the effective modulation index [10,12,13]. Note that the spectrum of a Bessel normal mode extends over an interval of frequencies of width $\sim 2\Gamma\omega_m$, which diverges as γ approaches zero, i.e., when the synchronous modulation regime is attained. This implies that, near the zero frequency detuning, finite spectral bandwidth effects of the laser gain may not be neglected, and loss of mode orthogonality occurs. The singular behavior of the system near $\gamma \sim 0$ is responsible for the well-known transition of laser operation from the FM regime to the pulsed FM mode-locking that occurs as the frequency detuning parameter $|\gamma|$ is decreased toward zero, the frequency detuning at which the transition takes place being given by $\gamma_T \sim (2\pi)^{-1}N^{-3/2}[-\Delta^{1/2} + (\Delta + 2\Delta^2N^2)^{1/2}]^{1/2}$, where $N \equiv (\omega_g/\omega_c)$ is the normalized spectral gain bandwidth [13]. A typical behavior of the laser threshold g_{th} as a function of the frequency detuning parameter γ near the transition region is shown in Fig. 1. Note that, far enough from the zero frequency detuning, the laser

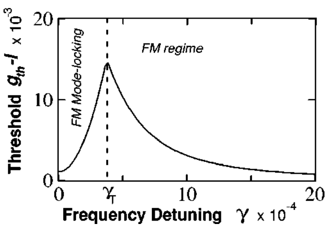

FIG. 1. Behavior of the laser threshold as a function of the frequency detuning γ. The vertical dashed line at $\gamma=\gamma_T\sim 3.7\times 10^{-4}$ marks the transition from FM oscillation to the pulsed FM mode-locking. The threshold curve turns out to be symmteric around the synchronous modulation condition $\gamma=0$. Parameter values are δ=0.05, $\omega_g/\omega_c=100$.

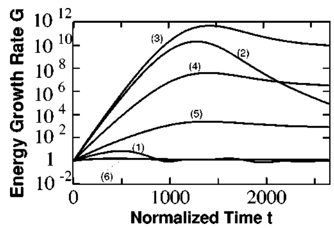

FIG. 2. Energy growth rate as a function of normalized time t for a few values of the frequency detuning γ. Curve (1), $\gamma=10^{-3}$; curve (2), $\gamma=4\times 10^{-4}$; curve (3), $\gamma=3.7\times 10^{-4}$; curve (4), $\gamma=3\times 10^{-4}$; curve (5), $\gamma=2\times 10^{-4}$; curve (6), $\gamma=0$. For each value of the frequency detuning the laser gain has been chosen 3% below its threshold value. The other parameter values are: $\Delta=0.05$, $\omega_g/\omega_c=100$.

threshold approaches that of the free-running laser, i.e., $g_{th}\sim l$, and in this case the laser is operated in the undistorted FM regime, where the finite bandwidth of the spectral gain is negligible. However, as $|\gamma|$ is decreased toward γ_T, there is an increase of the lasing threshold, which is due to the spectral broadening of the FM modes. If the laser is operated below threshold, i.e., if the gain parameter g is chosen below the threshold curve shown in Fig. 1, in absence of the stochastic driving field any initial perturbation at time $t=0$ is damped out and the zero solution is finally reached at $t\rightarrow\infty$. However, the transient dynamics of the decay, which is determined by the degree of normality of the system [4], changes drastically when approaching the transition region, where high levels of amplification are possible before the perturbation is damped out. If we consider the energy of the field stored in the cavity, $u(t)=[\int_0^1 dz F(z,t)F^*(z,t)]=\Sigma_n|F_n(t)|^2$, we may introduce an energy growth rate $G(t)$ defined as the maximum energy that can be stored in the cavity at time t taken over the ensemble of initial field perturbations of unitary energy. $G(t)$ represents therefore the upper boundary to the energy amplification at time t physically realizable for any given initial field perturbation. The energy growth rate $G(t)$ can be calculated in terms of the matrix $\mathcal{A}$ as $G(t)=\|\exp(\mathcal{A}t)\|^2$, where $\|\cdot\|$ denotes the standard 2-norm of a matrix (see [4]). If the system were normal, it would not be possible any energy growth at any time, i.e., $G(t)$ is a decreasing function of time and $G(t)\rightarrow 0$ on a time scale of the order of the inverse of the damping rate of the most unstable normal mode. This also occurs if the system is non-normal provided that the matrix $\mathcal{A}+\mathcal{A}^\dagger$ is asymptotically stable [15]. Conversely, transient energy amplification is possible if the largest eigenvalue of $\mathcal{A}+\mathcal{A}^\dagger$ is positive [3]. An inspection of Eq. (2) reveals that this condition is fullfilled provided that the laser gain exceeds the threshold value for the free-running laser (i.e., $g>l$), a condition that can be fullfilled when the laser is operated close the transition region of Fig. 1. Figure 2 shows a typical behavior of the energy growth rate $G(t)$ for a few values of the frequency detuning γ, indicating the possibility of high transient amplification of energy in the transition region between FM oscillation and FM mode-locking. The high levels of transient amplification can be physically understood by observing that, due to the excess gain, any initial perturbation that is spectrally narrow is amplified by the system in the intial transits inside the cavity. However, in the meanwhile the modulator transfers energy to sidebands modes, resulting in a spectral broadening of the perturbation and, for the laser below threshold, in the final decay of it. The typical temporal scale over which amplification and subsequent decay of the perturbations take place is determined by the number of transits in the cavity needed to spectrally broaden the perturbation, which is in turn inversely proportional to the frequency detuning γ.

In presence of a continuous stochastic forcing, such as that provided by spontaneous emission in the gain medium, the lack of orthogonality of Floquet modes and the capability of the deterministic dynamical system to support transient energy growth allow the Langevin equations (1) to sustain an excess of variance for the field fluctuations larger than that expected if the system were normal [3]. As the system is asymptotically stable and the noise sources are assumed to be δ-correlated Gaussian stationary processes, a stationary statistics for the random amplitudes F_n is reached, which can be determined using standard methods (see, for instance, [16]). In particular the spectral matrix $\mathcal{S}_{l,n}(\omega)$, which is defined as the Fourier transform of the correlation functions $\langle F_n(t)F_l^*(t+\tau)\rangle$, is given by $\mathcal{S}(\omega)=\epsilon(\mathcal{A}+i\omega\mathcal{I})^{-1}(\mathcal{A}^\dagger-i\omega\mathcal{I})^{-1}$, where $\mathcal{I}$ is the identity matrix and ϵ is the variance of the noise. A measure of the non-normality of the system is provided by the ensemble average of the energy stored in the cavity, $\langle u\rangle$, which can be expressed as [3,17]

$$\langle u\rangle=\frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}F(\omega)d\omega, \tag{5}$$

where the spectrum $F(\omega)$ is given by $F(\omega)=\epsilon^{-1}\mathrm{Tr}(\mathcal{S})$. Note that, since $\langle F(z,t)\rangle=0$, the variance of field fluctuations at the plane z in the cavity is given by $\langle|F(z,t)|^2\rangle=\Sigma_{n,l}\langle F_n(t)F_l^*(t)\rangle\exp[2\pi i(n-l)(z-t)]$, so that $\langle u\rangle$ provides a measure of the *spatial* average of the variance of the field fluctuations in the cavity. Differently, for a fixed position z inside the cavity, $\langle u\rangle$ represents the *time* average of the field fluctuations at that position. For detuning values where the system is (nearly) normal, the mean energy stored in the cavity results from a balance between dissipation and forcing for each normal mode of the cavity, and one obtains [17]

$$\langle u\rangle_{normal}=-\sum_{\alpha}\frac{\epsilon}{2\lambda_\alpha}, \tag{6}$$

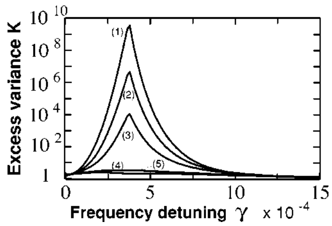

FIG. 3. Excess variance K as a function of frequency detuning γ for a few values of the below-threshold parameter y (see text). Curve (1), $y=0.9$; curve (2), $y=0.7$; curve (3), $y=0.5$; curve (4), $y=0$; curve (5), $y=-0.2$. Parameter values are: $\Delta=0.05$, $\omega_g/\omega_c=100$.

where $\lambda_\alpha=\mathrm{Re}(\mu_\alpha)$. However, for detuning values where the system becomes non-normal, the mean energy stored in the cavity can reach much higher levels due to the continuous transient amplification of the noise. This excess of field fluctuations can be defined by the ratio $K=\langle u\rangle/\langle u\rangle_{normal}$ [see Eqs. (5) and (6)], where $K\geqslant 1$ and $K=1$ for a normal system [3]. Figure 3 shows a typical behavior of the excess variance factor K as a function of the frequency detuning γ for a few values of the below-threshold parameter $y=(g-l)/(g_{th}-l)$ ($y<1$ and $y>0$ if the laser is above the free-running threshold). As can be seen, when y is positive, and thus transient amplification of perturbations is possible in the deterministic dynamics, a large enhancement of field fluctuations by several orders of magnitude is predicted when the laser is operated near the transition region. We note that, although this excess noise has been predicted to occur for the FM laser operated below threshold, we envisage that the strong sensitivity of this system to noise should be also observable when the laser is operated above threshold [18].

In conclusion, we have shown that an intracavity frequency-modulated laser operated close to threshold can sustain a large excess of field fluctuations as a result of the nonorthogonality of Floquet laser modes, which makes possible the continuous transient amplification of noise. Owing to the analogy between spatial and temporal propagation of optical fields [19], we envisage that a similar manifestation of excess noise should occur also when transverse degrees of freedom are considered instead of longitudinal ones. In particular, a plane-plane laser in a stripe geometry with transverse gain guiding and tilted mirrors, capable of producing beam walk-off out of the gain region, is expected to show the same noise features as the detuned FM mode-locking studied in this work, with the tilting angle playing the same role as the frequency detuning parameter.

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[14] A different and perhaps more appropriate way to introduce the notion of modes for a frequency-modulated laser is to rewrite the Langevin equations (1) in the operatorial form $\partial_t F=\mathcal{L}F+\xi(z,t)$, where $\mathcal{L}(z,t)$ is the time-dependent periodic operator given by

$$\mathcal{L}=(g-l)-(\omega_c/\omega_m)\partial_z+\left(\frac{\omega_c}{2\pi\omega_g}\right)^2\partial_z^2+i\Delta\cos[2\pi(t-z)],$$

and $\xi(z,t)$ is a δ-correlated Gaussian noise: $\langle\xi(z,t)\rangle=0$, $\langle\xi^*(z',t')\xi(z'',t'')\rangle=\epsilon\delta(t'-t'')\delta(z'-z'')$. The modes $|\alpha\rangle$ then correspond to the periodic Floquet modes of the time-dependent periodic operator $\mathcal{L}$, μ_α are the corresponding Floquet exponents, and the Langevin equations (4) follow directly by projection of the operatorial equation on the basis of Floquet modes. Notice that, despite the time dependence of Floquet modes, the scalar products $\langle\alpha|\beta\rangle$, $\langle\alpha^\dagger|\beta^\dagger\rangle$, etc., turn out to be time-independent. For a more extended analysis of Floquet modes and of their properties in a frequency-modulated laser, we refer to [13].
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