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Inverse energy cascade in two-dimensional turbulence: Deviations from Gaussian behavior

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High-resolution numerical simulations of stationary inverse energy cascade in two-dimensional turbulence are presented. Deviations from Gaussian behavior of velocity differences statistics are quantitatively investigated. The level of statistical convergence is pushed enough to permit reliable measurement of the asymmetries in the probability distribution functions of longitudinal increments and odd-order moments, which bring the signature of the inverse energy flux. No measurable intermittency corrections could be found in their scaling laws. The seventh order skewness increases by almost two orders of magnitude with respect to the third, thus becoming of order unity.

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The inverse energy cascade in two-dimensional Navier-Stokes turbulence is one of the most important phenomena in fluid dynamics. In agreement with the remarkable prediction by Kraichnan in 1967 [1], the coupled constraints of energy and enstrophy conservation make the energy injected into the system flow toward the large scales. This is a basic difference with respect to 3D turbulence, where energy flows toward small scales in a direct cascade. The dynamical process of structuring and organization of the large scales by the inverse cascade is also of great interest for geophysical fluid dynamics. First numerical and experimental observations of the inverse cascade and the ensuing Kolmogorov energy spectrum were obtained in [2–9]. The important point foreseen in [4] is that the smallness of the skewness suggests that intermittency might be weak. This conjecture was later supported by numerical simulations [10,11] and experiments [12]: scaling laws are compatible with dimensional predictions and both transversal and longitudinal velocity probability distribution functions (PDF's) look not far from Gaussian. The evidence stemming from experiments and simulations is that the inverse transfer takes place via clustering of small-scale equal sign vortices. Strong deviations from Gaussianity appear if the system has a finite size and friction extracting the energy from the large scales is small (or absent). A pile-up of energy akin to the Bose-Einstein condensation then takes place in the gravest mode [1], large scale vortices are formed and energy spectra steeper than the Kolmogorov one are observed [13]. Here we shall not consider the condensation phase, concentrating instead on the inverse cascade statistics. Theoretically, inverse cascade enjoys a great advantage with respect to the direct one: the limit of molecular viscosity $\nu \to 0$ can be taken without any harm in the equations of motion for velocity structure functions. At variance of 3D turbulence, the energy dissipation $\nu\langle(\nabla \boldsymbol{v})^2\rangle$ is indeed vanishing when $\nu \to 0$. The absence of dissipative anomalies is the clue for the analytical solution of inverse cascades in passive scalar advection [15,16]. Intermittency was found to be absent, even though the statistics might be strongly differing from Gaussian. For 2D Navier-Stokes inverse cascade, dissipative terms can again be discarded but the situation is complicated by pressure gradients. They couple indeed the statistics of velocity differences $\delta_r \boldsymbol{v} \equiv \boldsymbol{v}(\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{0})$ at various $\boldsymbol{r}$'s in a nonlocal way. Closures on velocity increments-pressure gradients correlations have been proposed by invoking the quasi-Gaussian behavior of the statistics and quantitative predictions have been derived in this way [17]. The issue of quasi-Gaussian behavior is, however, moot, as deviations are intrinsically entangled to the dynamical process of inverse energy cascade. Standard calculations (see, e.g., [18]) indeed permit one to derive the 3/2 Kolmogorov law for 2D turbulence: $S_L^{(3)}(r) = \langle[\delta_r \boldsymbol{v} \cdot \hat{\boldsymbol{r}}]^3\rangle = 3/2\epsilon r$, where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. The energy flux is denoted by ϵ and the fact that it goes upscale reflects into the positive sign of the moment. Precise quantitative informations on the deviations from Gaussian behavior are, however, difficult to obtain. Odd-order structure functions involve for example strong cancellations between negative and positive contributions and the 3/2 law itself could not be observed in previous studies, due to lack of resolution and/or statistical convergence. It is our purpose here to present the results of high-resolution numerical simulations aimed at quantitatively analyzing deviations from Gaussian behavior in the inverse energy cascade.

Specifically, the 2D Navier-Stokes equation for the vorticity $\omega(\boldsymbol{r},t) = -\Delta\psi(\boldsymbol{r},t)$ is

$$\partial_t \omega + J(\omega,\psi) = \nu\Delta\omega - \alpha\omega - \Delta f, \tag{1}$$

where ψ is the stream function, the velocity $\boldsymbol{v} = \nabla^\perp \psi = (\nabla_y \psi, -\nabla_x \psi)$, and J denotes the Jacobian. The friction linear term $-\alpha\omega$ extracts energy from the system at scales comparable to the friction scale $\eta_{\rm fr} \sim \epsilon^{1/2}\alpha^{-3/2}$, assuming a Kolmogorov scaling law for the velocity. To avoid Bose-Einstein condensation in the gravest mode we choose α to make $\eta_{\rm fr}$ sufficiently smaller than the box size. The other relevant length in the problem is the small-scale forcing correlation length l_f, bounding the inertial range for the inverse cascade as $l_f \ll r \ll \eta_{\rm fr}$. We use a Gaussian forcing with correlation function $\langle f(\boldsymbol{r},t)\, f(\boldsymbol{0},t')\rangle = \delta(t-t')\, F(r/l_f)$. The δ-correlation in time ensures the exact control of the energy injection rate. The forcing space correlation should decay rapidly for $r \gg l_f$ and we choose $F(x) = F_0 l_f^2 \exp(-x^2/2)$, where F_0 is the energy input. The numerical integration of Eq. (1) is performed by a standard 2/3-dealiased pseudospec-

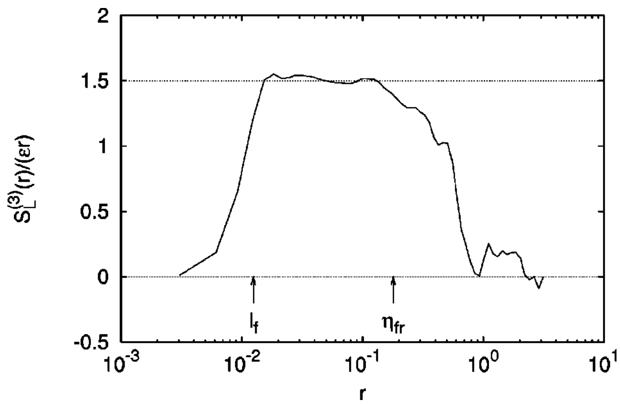

FIG. 1. Compensated third order longitudinal structure function $S_L^{(3)}(r)/(\epsilon r)$. The dotted line is the value 3/2. Note the linear vertical scale. The labels l_f and η_{fr} indicate the forcing and the friction length scales, respectively.

tral method on a doubly periodic square domain of $N^2 = 2048^2$ grid points. The viscous term in Eq. (1) has the role of removing enstrophy at scales smaller than l_f and, as customary, it is numerically more convenient to substitute it by a hyperviscous term (of order eight in our simulations). Time evolution is obtained by a standard second-order Adams-Bashforth scheme. After the system has reached stationarity, analysis is performed over eighty snapshots of the velocity field equally spaced by one large-eddy turnover time.

Let us now discuss the results. In Fig. 1 we present the third-order longitudinal structure function $S_L^{(3)}(r)$ compensated by the factor $1/(\epsilon r)$, showing a neat plateau at the value 3/2 (in agreement with the Kolmogorov law) over a range of almost one decade of scales. In Fig. 2 the energy spectrum $E(k)$ is presented, which displays Kolmogorov scaling $k^{-5/3}$, and the energy flux $\Pi(k)$. Although at small wave numbers it is visible the effect of large-scale friction on the energy flux, nevertheless $\Pi(k) \simeq \epsilon$ for almost one decade. A careful inspection of the spectrum (see upper inset of Fig. 2) shows that at low wave numbers there is a slight deviation from the Kolmogorov slope. This can be recognized as a bottleneck effect (see Ref. [19] in the context of the direct energy cascade), which can be very marked when

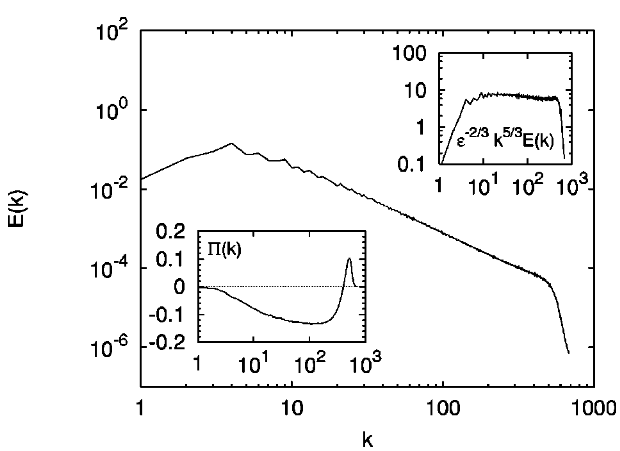

FIG. 2. Energy spectrum $E(k)$. In the lower inset the energy flux $\Pi(k)$ is shown. In the upper inset is the compensated spectrum $\epsilon^{-2/3}k^{5/3}E(k)$.

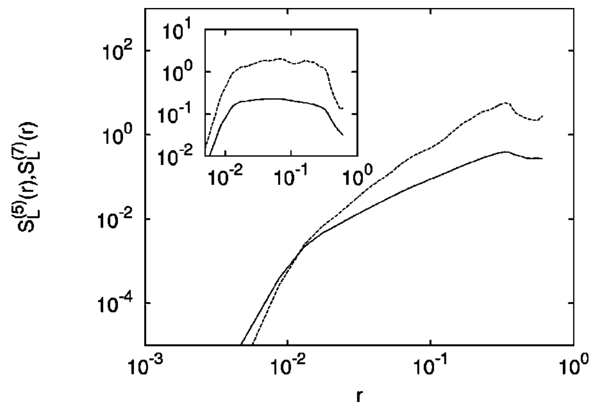

FIG. 3. Structure functions of order 5 (lower line) and 7 (upper line). The compensated curves $S_L^{(5)}(r)/(C_L^{(2)}\epsilon^{2/3}r^{2/3})^{5/2}$ (lower line) and $S_L^{(7)}(r)/(C_L^{(2)}\epsilon^{2/3}r^{2/3})^{7/2}$ (upper line) are shown in the inset.

hypodissipative terms $-\alpha_p(-\nabla^2)^{-p}\omega$ replace friction, as in Ref. [13].

We also have performed simulations with several hypoviscosity terms, which show that the magnitude of the hump in the compensated spectrum increases with the order p of the large-scale dissipation. This effect is related to the presence of large-scale vortical structures, which do not appear if the order of hypodissipation is taken small enough, or if damping is properly parametrized [14]. The Kolmogorov constant in

$$E(k) = C\,\epsilon^{2/3}\,k^{-5/3}, \tag{2}$$

is found to be $C = 6.0 \pm 0.4$. Previous numerical simulations and experiments report values of the Kolmogorov constant C ranging from 5.8 to 7.0 [6,8–13]. The structure function constants corresponding to Eq. (2) are $C_L^{(2)} = 3C_T^{(2)}/5 = [\sqrt{3}\pi/2^{5/3}\Gamma(4/3)^2]C = 12.9 \pm 0.8$, where the first two equalities follow from isotropy and incompressibility and

$$S_L^{(n)}(r) = \langle[\delta_r \boldsymbol{v}\cdot\hat{\boldsymbol{r}}]^n\rangle = C_L^{(n)}(\epsilon r)^{n/3}. \tag{3}$$

For transverse moments, $\hat{\boldsymbol{r}}$ is substituted in Eq. (3) by $\hat{\boldsymbol{r}}_\perp$, perpendicular to it, and C_L by C_T. It is of interest to remark that longitudinal and transverse velocity increments are uncorrelated, i.e., $\langle[(\delta_r\boldsymbol{v}\cdot\hat{\boldsymbol{r}})(\delta_r\boldsymbol{v}\cdot\hat{\boldsymbol{r}}_\perp)]\rangle = 0$. The relatively large value of $C_L^{(2)}$ implies a small skewness of the longitudinal velocity differences $(3/2)/(C_L^{(2)})^{3/2} = 0.03$. Albeit the longitudinal PDF looks close to Gaussian and quite symmetric, nevertheless on a more quantitative ground asymmetries turn out to be quite strong as shown by the two curves of $S_L^{(5)}(r)$ and $S_L^{(7)}(r)$ in Fig. 3. First, we can observe that their scaling behavior is in agreement with Kolmogorov predictions. The existence of fluctuations do not permit one to fully rule out nonvanishing intermittency corrections, but they are bounded to be minute and within the error bars with the present statistics. Second, the constants are $C_L^{(5)} \simeq 130$ and $C_L^{(7)} \simeq 14\,000$, giving for the hyperskewness $C_L^{(5)}/(C_L^{(2)})^{5/2} \simeq 0.22$ and $C_L^{(7)}/(C_L^{(2)})^{7/2} \simeq 1.8$. The error bars can be estimated from rms fluctuations of compensated plots and for the seventh order (which is of course the most delicate) they amount to 20%.

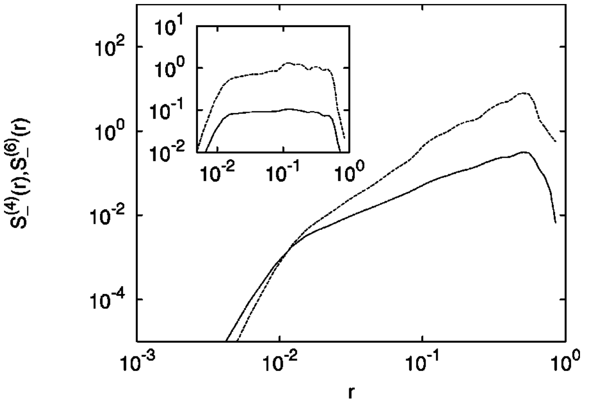

FIG. 4. Antisymmetric structure functions $S_-^{(n)}(r)$ of order $n=4,6$. In the inset $S_-^{(4)}(r)/(C_L^{(2)}\epsilon^{2/3}r^{2/3})^2$ (lower line) and $S_-^{(6)}(r)/(C_L^{(2)}\epsilon^{2/3}r^{2/3})^3$ (upper line).

Another striking evidence for the importance of the longitudinal PDF asymmetries is provided in Fig. 4. We consider here the antisymmetric part of the PDF $\mathcal{P}(\delta v_L(r)) - \mathcal{P}(-\delta v_L(r))$ (shown in Fig. 5) and calculate "antisymmetric structure functions" such as $S_-^{(4)} = \int_0^\infty u^4(\mathcal{P}(u) - \mathcal{P}(-u))\,du$. Both the fourth and the sixth moment show a scaling compatible with Kolmogorov prediction $S_-^{(n)}(r) = C_-^{(n)}(\epsilon r)^{n/3}$, with $C_-^{(4)}/(C_L^{(2)})^2 \simeq 0.08$ and $C_-^{(6)}/(C_L^{(2)})^3 \simeq 0.6$. This indicates that the non-Gaussian antisymmetric part, although visually small, has imprinted all the relevant scaling informations on the inverse cascade.

The increase of the skewness by almost two orders of magnitude from the third to the seventh order is particularly informative. Indeed, whereas (hyper)flatness necessarily increases with the order (a consequence of Hölder inequalities), (hyper)skewness might *a priori* reduce. Our main motivation was precisely to find out whether the skewness was decreasing or increasing with the order and the answer to this question shows that hyperskewness is definitely not a "small parameter" to be used in perturbative schemes for the statistical properties of the inverse energy cascade. Different adimensionalizations might of course be considered, as, e.g.,

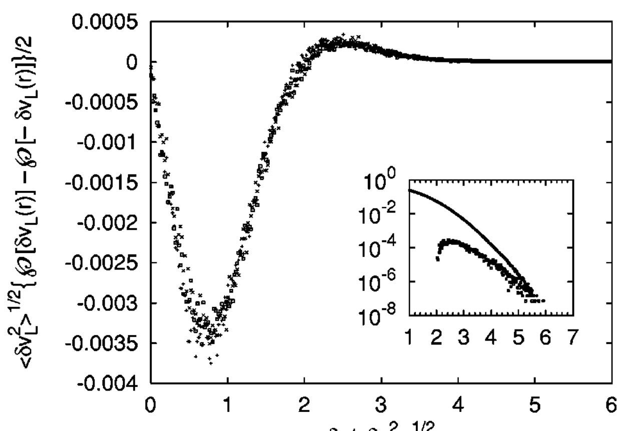

FIG. 5. Antisymmetric part of the longitudinal velocity increments PDF, at three separations ranging from $r=0.05$ to $r=0.1$, within the inertial range of scales. In the inset, the antisymmetric part of the PDF at $r=0.1$ (lower points) compared with the symmetric part (upper points).

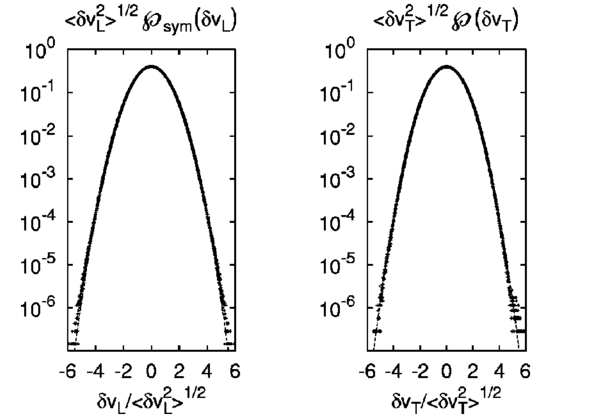

FIG. 6. Left: symmetric part of the longitudinal velocity difference PDF. Right: PDF of transverse velocity differences. The forcing is restricted to a band of wave numbers. Gaussian distributions are shown as solid lines.

$S_L^{(2n+1)}/(S_L^{(2n)})^{(2n+1)/2n}$, which are guaranteed to give smaller numerical values for $n=2,3$. We prefer then to directly compare in the inset of Fig. 5 the tails of the antisymmetric and the symmetric part of the PDF (the latter being very close to Gaussian, see Fig. 6). The figure unambiguously shows that the antisymmetric part, although much smaller than the symmetric one for moderate fluctuations, tends to become comparable to it (remaining of course always below it) for large fluctuations. The predictions in [17], although based on a closure explictly invoking small deviations from Gaussian behavior, turn out to be compatible with the numerical results. This indicates that the closure is likely to be more robust and "nonperturbative" than its derivation might suggest.

To address the issue of the expected universality of the present results with respect to the type of forcing we also performed numerical simulations with an injection rate characterized by the spectral correlation function $\langle f(\boldsymbol{k},t)\,f(\boldsymbol{k}',t')\rangle = \delta(t-t')\,\delta(\boldsymbol{k}+\boldsymbol{k}')\,\delta(1-kl_f)$ [6]. At variance with the former choice, this forcing is limited to a narrow bandwidth in Fourier space but its spatial correlations decay rather slowly. Averages have been taken over 20 snapshots of the velocity field.

Odd-order structure functions and the antisymmetric part of the PDF do not show any visible dependence on the details of the energy input. Conversely, the symmetric part of the PDF of velocity differences and even order moments are more sensitive. For the forcing limited to a shell of wavenumbers, both the (symmetrized) longitudinal and the transverse pdfs are visually indistinguishable from Gaussian, as shown in Fig. 6.

Deviations of kurtosis and hyperkurtosis from their Gaussian values are small and compatible with those presented in [12]. For the forcing localized in physical space, the far tails of the PDF at scales $O(l_f)$ tend to be (symmetrically) broader. This tendency is due to the formation of small vortices of size comparable to l_f, which generate large velocity differences (especially transverse ones) across a distance of the order of their size. The effect becomes, of course, negligible at scales larger than l_f but it might affect the quality and the extension of the scaling region for even order structure functions. No coherent structure of size larger than l_f has ever been detected in our simulations. This confirms that the inverse cascade does not proceed by vortex merging, as also observed in experiments [12]. Note that the vortices formed at the forcing scale (or smaller) do not affect odd-order structure functions [20].

In conclusion, we have presented quantitative evidences for deviations from Gaussian behavior of the velocity increment statistics in the inverse energy cascade. Odd-order structure functions display a power-law scaling compatible with classical Kolmogorov predictions. Numerical prefactors in adimensionalized structure functions are expected to be universal with respect to the forcing statistics and have been measured up to the seventh order. Despite the small value of the skewness, asymmetries in longitudinal velocity statistics have been shown to be important and should therefore be incorporated and treated systematically in theoretical models for the inverse energy cascade.

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