Hysteresis studies in a noisy autoassociative neural network

Renuka Rai and Harjinder Singh*
Department of Chemistry, Panjab University, Chandigarh-160014, India
(Received 19 August 1999)

We define magnetization for a noisy autoassociative neural network driven by an external periodic field. Numerical simulations are carried out to investigate the effect of drive amplitude and frequency and noise strength on the area of the hysteresis loop. We observe that in the presence of weak periodic signal, the network exhibits a maximum in the hysteresis loop area at a nonzero noise intensity indicating maximum synchronization between the periodic signal and the response. It also goes through a maximum as a function of signal frequency.

PACS number(s): 87.10.+e, 05.40.−a, 05.45.−a, 75.60.Ej

The role of noise and nonlinearity in periodically driven multistable systems has been a subject of recent interest [1]. A commonly observed nonequilibrium phenomenon is hysteresis which reflects how the system responds to the external field sweep [2,3]. There have been many attempts to study the hysteresis phenomenon, both experimentally [4–6] as well as theoretically and numerically [7–16]. It is often modeled using a system of differential equations that display discontinuous bifurcations [7]. Rao *et al.* [9] have studied hysteresis in $(\phi^2)^2$ and $(\phi^2)^3$ model and lattice spin systems. They observed a power law dependence of the area of the hysteresis loop on the amplitude and frequency of the external field. Their results are consistent with the experimental studies on ferroelectrics [4] and charge density waves [5]. Mahato and Shenoy [2] have explained hysteresis using a first-passage time formalism. They observed that the hysteresis loop area shows a stochastic resonance behavior with respect to the noise strength. Mahato and Jayannawar [11] have used the master equation approach to obtain magnetization and obtained similar results. Apart from these models, theoretical studies of hysteresis have been performed on several other models. These include mean field calculation of a kinetic Ising model [12], Monte Carlo simulations of spin-$\frac{1}{2}$ Ising model [9,10], dissipative quantum systems [14], bistable maps [15], and neural networks [16].

In this Brief Report, we study hysteresis in a noisy autoassociative neural network which has previously been shown to be an accurate model of the bistable perceptual process involved in the interpretation of ambiguous figures [17]. Studies of the perception of ambiguous figures is characterized by noisy bistable dynamics [18] and it has been established that noise can improve the performance of certain neural networks [19]. In the presence of a subthreshold periodic drive, it has been shown by Riani and Simmonotto [20] that the signal to noise ratio of the output goes through a maximum as a function of noise intensity, which is a signature of the phenomenon of stochastic resonance. In order to understand the kinetics aspects of the phenonmenon, i.e., how the system responds to the external periodic drive, we investigate the hysteresis behavior of this network. The model consists of a network made of globally coupled binary neurons with activation level 0 and 1. There are two groups of neurons n_X and n_Y, each of which is associated with one of the two stable states. The two groups of neurons are mutually exclusive, i.e., the excitation of one group must inhibit the other group. The energy function for this network is

$$E=-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} S_i W_{ij} S_j, \tag{1}$$

where S_i is the activation of the i^{th} neuron and W_{ij} is an element of synaptic connection matrix. The connection matrix is made up of four parts: two square blocks representing the positive (excitatory) connections within n_X and n_Y and two rectangular blocks representing the negative (inhibitory) interconnections linking the two groups. We will mimic a symmetric bistable system using this model, so the connections among the neurons within each group are taken to be of equal strength (m). The interconnections between the two groups are taken to be of negative strength $(-r)$ because it is inhibitory. It has been shown that this value of energy can be simplified to

$$E=-\frac{1}{2}x(x-1)m-\frac{1}{2}y(y-1)m+xyr, \tag{2}$$

where the network has x active neurons in the population n_X and y active neurons in the population n_Y.

The system evolves according to a modified hopfield dynamical rule [21]. Initially the system is present in one of a randomly chosen state. An external perturbation is introduced in the form of a small additive perturbation to the energy

$$E=-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} S_i W_{ij} S_j + a(t)\left(\sum_{i\in A} S_i - \sum_{i\in B} S_i\right), \tag{3}$$

where $a(t)$ is an external sinusoidal drive with amplitude A and frequency Ω and the summation is extended over all the neurons in the populations n_X and n_Y. During alternate half cycle, the field favors the activation of one population and

*Electronic address: laltu@panjabuniv.chd.nic.in

the inhibition of the other. Trajectories generated during one half cycle then preferentially migrates toward one stable state and toward the other state on alternate half cycles. Thus the possibility to realize one of the states is periodically modulated.

The network is asynchronously updated at each time step by changing the activation levels of N_0 randomly chosen neurons. After each update, a new value of the energy is calculated, and the resulting state is either accepted or rejected according to the following rule: accept with probability one if $\Delta E \leqslant 0$, accept with probability $p = \exp(-\Delta E/k_B D)$ if $\Delta E \geqslant 0$, where D is the temparature (or noise). This procedure is the Metropolis algorithm [22]. In this way the effect of noise has been accounted for by the simulation.

To begin with, at $t=0$, the system is present in a randomly chosen state, i.e., n_{X0} neurons of kind X and n_{Y0} neurons of kind Y. For a given A, Ω, and D, the trajectory evolves according to the rules explained earlier and the residence time distributions (RTD), $\rho_1(T)$ and $\rho_2(T)$ in well one and two, respectively, have been obtained. If t_i denotes the sequence of escape times, then the normalized distribution of quantities $T(i) = t_i - t_{i-1}$, represents the RTD [Fig. 1(a)]. The escape events directly give escape field distributions $\rho_{12}(a(t))$ and $\rho_{21}(a(t))$. The quantity $\rho_{12}(a(t))$ is the distribution of the field values $a(t)$ at which the passage takes place from well 1 to well 2 [Fig. 1(b)]. The distributions $\rho_{12}(a(t))$ and $\rho_{21}(a(t))$ determine the evolution of the fraction of population in the ith well, $m_i(t)$, from the $(t-1)$th step as

$$m_i(t) = m_i(t-1) - \rho_{ij}(a(t-1))m_i(t-1) + \rho_{ji}(a(t-1))m_j(t-1), \tag{4}$$

where $i=1,2$; $j=1,2$ and $i \neq j$.

It is observed that $m_i(t)$ is periodic, i.e., $m_i(t) = m_i(t+T_0)$, where T_0 is the time period of the drive. Therefore we can consider m_1 and m_2 to be functions of field. We have defined magnetization as the difference between the fractions of the population in the two wells, i.e.,

$$m(a) = m_2(a) - m_1(a). \tag{5}$$

As shown in Fig. 1(c), the plot of $m(a)$ vs $a(t)$ is a closed loop. This hysteresis loop has been obtained using escape field distributions which contain the information of all the peaks in the RTD. When the RTD show sharp peaks at $t = (n-1/2)T_0$ where $n=1,2,3,\ldots$ [Fig. 1(a)], the escape field distribution will show one sharp peak at $a=-A$ [Fig. 1(b)]. Thus when there is maximum synchronization, we get a rectangular hysteresis loop with maximum area. Similarly, when the passage takes place all over the period, there is a continuous variation of $m(a)$, instead of a sudden jump. So the hysteresis loop has a very small area. Thus hysteresis loop area reflects the degree of synchronization of the passage across the barrier with the input signal. Instead of calculating the magnetization as the difference between the populations of the two wells, we use this procedure because it involves observing the system over many cycles of the drive. In contrast, in the other procedures which do not involve the master equation approach, hysteresis loop is obtained in one cycle and averaging is done on many cycles just to remove the fluctuations.

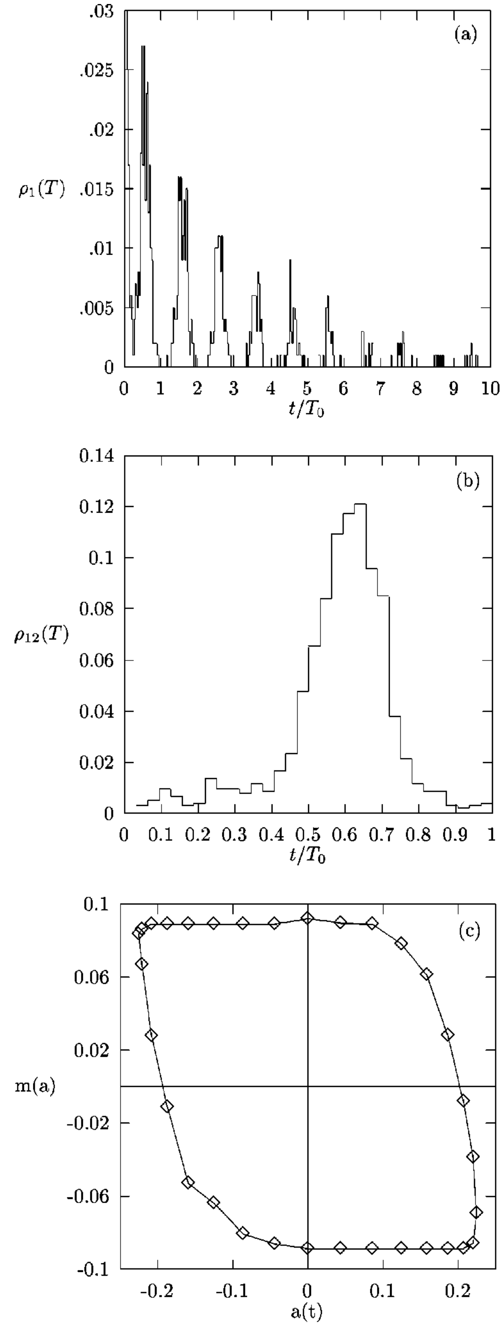

FIG. 1. (a) The residence time distribution $\rho_1(T)$ in well one. (b) The distribution $\rho_{12}(a)$ of the field value $a(t)$ at which the passage from well 1 to 2 takes place. (c) The corresponding hysteresis loop $m(a)$ for $A=0.225$ (arbitrary units), $D=0.4$ (scaled for $k_B=1.0$), and $\Omega=1/32$ (arbitrary units).

We have carried out numerical simulations for various values of ω and D and obtained residence time distributions, escape time distributions, and magnetization $m(a)$. The dependence of the hysteresis loop on noise strength and drive frequency is studied. In all our calculations we take the same values as used by Riani *et al.* in Ref. [20], i.e., $m=0.02$, $r=0.04$, and the total number of neurons (n_X+n_Y) are 40.

The hysteresis loop area is found to show a maximum as a function of noise strength D (Fig. 2). Since the signal is

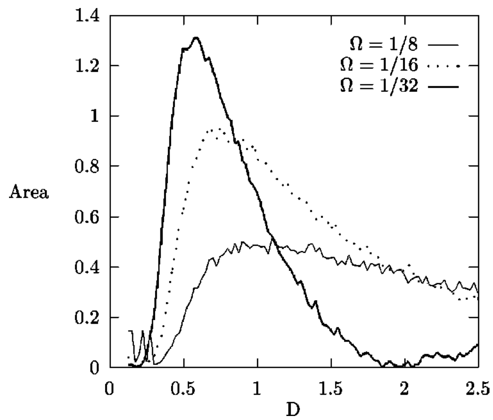

FIG. 2. Area of the hysteresis loop (arbitrary units) as a function of noise intensity D (scaled for $k_B = 1.0$) for $A = 0.25$ (arbitrary units) for different frequencies (arbitrary units).

subthreshold, for very low D the system may not switch to the other well. But as the noise is increased, it aids the system to flip to the other well. This is most likely when the signal is at its peak. For larger noise, since the flips can occur almost all the time, the regularity is reduced. Therefore, the passage between the wells is most synchronized for some optimum value of D where the maximum in hysteresis loop area is observed.

In Fig. 3 we show the frequency dependence of hystersis loop area. It goes through a maximum as a function of the drive frequency. For a very high frequency of the drive, area approaches zero. A possible explanation for this observation is given below.

As shown in Fig. 4(a), for very low frequencies, most of the jumps take place even before the external drive has its peak value. Consequently, the escape time distribution shows a peak before $t = T_0/2$ [Fig. 4(b)]. As we increase the frequency, the RTD spreads to many cycles, but most of the jumps take place when the barrier height is minimum. If the particle fails to cross the barrier, then it has to wait for the next cycle. Thus the peaks around $t = (n - 1/2)T_0/2$ become narrower [Fig. 4(a)] and we get a sharp peak at $T_0/2$ in the $\rho_{12}(a)$ vs t/T_0 plot [Fig. 4(b)]. As a result, the hysteresis loop area increases. If we still increase the frequency, the

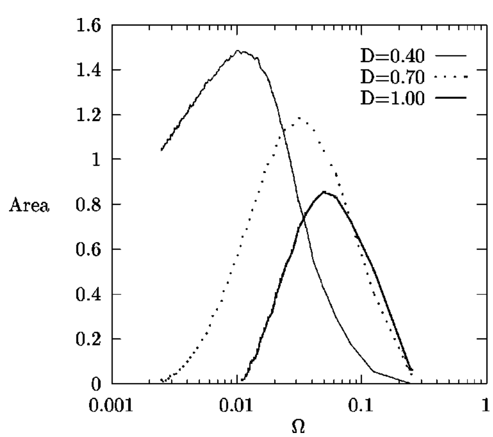

FIG. 3. Area of the hysteresis loop (arbitrary units) as a function of drive frequency Ω (arbitrary units) for $A = 0.25$ (arbitrary units).

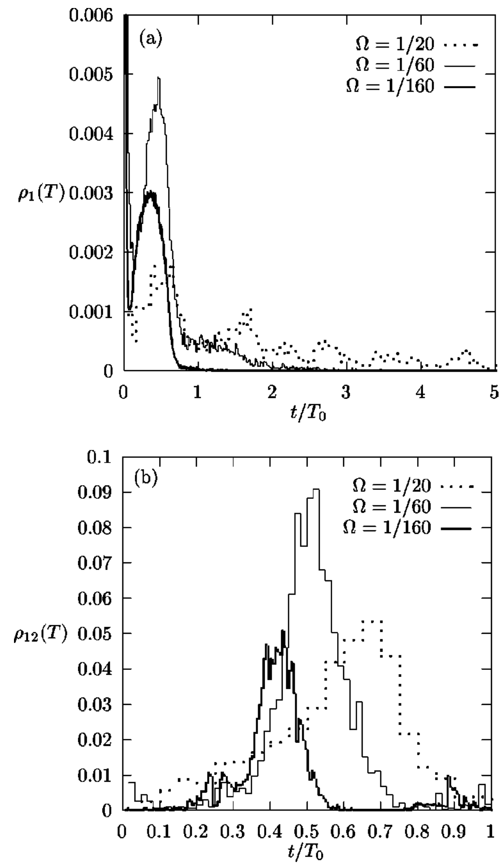

FIG. 4. (a) Residence time distribution $\rho_1(T)$ and (b) escape field distribution $\rho_{12}(a)$ as a function of scaled time t/T_0 for $A = 0.25$ (arbitrary units) and $D = 0.5$ (scaled for $k_B = 1.0$).

average number of cycles the particle takes to cross the barrier increases. Also the peaks become broader. This is because the particle takes some time to go from the metastable point to the other fixed point and by that time $a(t)$ moves away from $-A$. Due to this the $\rho_{12}(a)$ peak becomes broader [Fig. 4(b)] and its position is displaced from $T_0/2$. This results in a decrease in the hysteresis loop area.

The variation of hysteresis loop area as a function of noise intensity is similar to that of signal to noise ratio vs D profile [20] and indicates the occurrence of stochastic resonance. Since the introduction of the phenomenon to explain the glaciation cycle of earth [23], stochastic resonance has been employed in explaining various phenomena [24] and has been studied extensively [25,26]. Recently, quantifiers other than signal to noise ratio, e.g., residence time distribution, signal amplitude, etc., have been proposed as charactreizers of stochastic resonance. It has been shown that hysteresis loop area is also an equally important quantifier [2,11,15] and it reflects the degree of synchronization between the input signal and the response. In this study also similar results are obtained and we observe that the maximum in area of the hysteresis loop appears not only as a function of noise intensity but also as a function of drive frequency. Thus, the synchronization between the response and the input signal can be achieved by varying either drive frequency or noise strength.

In conclusion, we have studied the phenomenon of hysteresis in a noisy autoassociative neural network. It is interesting to observe that the dependence of hysteresis loop on noise strength and drive frequency is qualitatively similar to that obtained for a wide variety of systems such as continuous bistable systems [9,2,11], bistable maps [15], Ising system [10], etc. Thus, apart from stochastic resonance other features of noisy bistable dynamics can also be demonstrated in these systems. The study will be useful in understanding history dependent effects in a variety of systems, e.g., spin glasses [16,27], that can be modeled by an autoassociative neural network.

ACKNOWLEDGMENTS

R.R. would like to thank UGC. Financial support for this work came from DST, Government of India.

[1] P. Jung, Phys. Rep. **234**, 175 (1993).

[2] M.C. Mahato and S.R. Shenoy, Phys. Rev. E **50**, 2503 (1994).

[3] P.M. Gade, R. Rai, and H. Singh, Phys. Rev. E **56**, 2518 (1997).

[4] M. Lederman, S. Schultz, and M. Ozaki, Phys. Rev. Lett. **73**, 1986 (1994); Y.L. He and G.C. Wang, *ibid.* **70**, 2336 (1993); W.J. Merz, Phys. Rev. **91**, 513 (1953); W.J. Merz, *ibid.* **91**, 690 (1954); H.H. Wieder, J. Appl. Phys. **26**, 1479 (1955).

[5] Gy. Hutiray and G. Mihaly, in *Charge Density Waves in Solids*, Vol. 217 of *Lecture Notes in Physics*, edited by Gy. Hutiray and J. Solyom (Springer-Verlag, Berlin, 1985).

[6] E.C. Gage and L. Mandel, J. Opt. Soc. Am. **6**, 287 (1989).

[7] A. Visitin, *Differential Models of Hysteresis* (Springer, Berlin, 1994); M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions* (Springer, New York, 1996).

[8] R. Gilmore, Phys. Rev. A **20**, 2510 (1979); G.S. Agarwal and S.R. Shenoy, *ibid.* **23**, 2719 (1981); S.R. Shenoy and G.S. Agarwal, *ibid.* **29**, 1315 (1984).

[9] M. Rao, H.R. Krishnamurthy, and R. Pandit, Phys. Rev. B **42**, 856 (1990); M. Rao and R. Pandit, *ibid.* **43**, 3373 (1991); M. Rao, H.R. Krishanamurthy, and R. Pandit, J. Phys. C **1**, 9061 (1989); D. Dhar and P.B. Thomas, J. Phys. A **25**, 4967 (1992).

[10] S.W. Sides, P.A. Rikvold, and M.A. Novotny, Phys. Rev. E **59**, 2710 (1999); S.W. Sides, P.A. Rikvold, and M.A. Novotny, *ibid.* **57**, 6512 (1998).

[11] M.C. Mahato and A.M. Jayannavar, Phys. Rev. E **55**, 3716 (1997); **55**, 6266 (1997).

[12] P. Jung, G. Gray, R. Roy, and P. Mandel, Phys. Rev. Lett. **65**, 1873 (1990); T. Tomé and M.J. de Oliveira, Phys. Rev. A **41**, 4251 (1990); C.N. Luse and A. Zangwill, Phys. Rev. E **50**, 224 (1994); M. Locher, G.A. Johnson, and E.R. Hunt, Phys. Rev. Lett. **77**, 4698 (1996).

[13] M. Acharyya and B.K. Chakarbarti, J. Phys. A **27**, 1533 (1994); W.S. Lo and R.A. Pelcovits, Phys. Rev. A **42**, 7471 (1990); M. Acharyya and B.K. Chakrabarti, J. Phys. A **52**, 6550 (1995).

[14] M. Thorwart and P. Jung, Phys. Rev. Lett. **78**, 2503 (1997); T.P. Pareek, M.C. Mahato, and A.M. Jayannawar, Mod. Phys. Lett. B **11**, 861 (1997).

[15] R. Rai and H. Singh, Mod. Phys. Lett. B **13**, 181 (1999).

[16] P. Shukla and T.K. Sinha, Phys. Rev. E **49**, 4811 (1994); P. Shukla, *ibid.* **56**, 2265 (1997).

[17] F. Masulli, M. Riani, and E. Simonotto, Proc. SPIE **1710**, 267 (1992).

[18] L.A. Necker, Philos. Mag. **3**, 329 (1832); H. Goldhamer, Am. J. Psychol. **46**, 189 (1934), and references therein.

[19] J. Buhmann and K. Schulten, Biol. Cybern. **56**, 313 (1987).

[20] M. Riani and E. Simonotto, Phys. Rev. Lett. **72**, 3120 (1994).

[21] J.J. Hopfield, Proc. Natl. Acad. Sci. USA **79**, 2554 (1982); J. J. Hopfield, **81**, 3088 (1994).

[22] N. Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller, J. Chem. Phys. **21**, 1087 (1953).

[23] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981).

[24] See, for example, *Proceedings of the NATO ARW Stochastic Resonance in Physics and Biology*, edited by F. Moss, A. Bulsara and M.F. Shlesinger [J. Stat. Phys. **70**, 1 (1993)].

[25] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).

[26] B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988); S. Fauve and F. Heslot, Phys. Lett. **97A**, 5 (1983); T. Zhou, F. Moss, and P. Jung, Phys. Rev. A **42**, 3161 (1990).

[27] K. Binder and A.P. Young, Rev. Mod. Phys. **58**, 801 (1986).