Stochastic resonance at phase noise in signal transmission

François Chapeau-Blondeau
Laboratoire d'Ingénierie des Systèmes Automatisés (LISA), Université d'Angers, 62 avenue Notre Dame du Lac, 49000 Angers, France

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A model is developed for a periodic signal corrupted by an arbitrarily distributed phase noise and transmitted by an arbitrary memoryless system. The model establishes a new form of the phenomenon of stochastic resonance, whereby signal transmission can be enhanced by addition of noise. This is revealed by the standard signal-to-noise ratio of stochastic resonance, which here receives an explicit theoretical expression, and which is shown improvable via noise addition. This model is the first to propose a theory of stochastic resonance with phase noise. It represents a unique framework for further investigations on stochastic resonance and its applications.

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Stochastic resonance (SR) is a phenomenon of noise-enhanced signal transmission which can occur under various forms in different types of nonlinear systems (see Refs. [1,2] for recent reviews). SR has been observed for instance in electronic circuits [3,4], neurons [5–7], optical devices [8–10], and its applicability is gradually extending to wider classes of nonlinear systems and signals. Thus far, SR has essentially been reported and analyzed in the response of nonlinear systems driven by the additive mixture of a coherent signal (usually periodic) and a noise. Some studies have also considered a multiplicative noise, or more precisely a periodically modulated noise [11] or a state-dependent noise [12]. Here, we address a different type of coupling between the signal and the noise, i.e., the noise occurs as a random perturbation in the phase of a periodic signal. Such a situation arises, for instance, when a periodic wave travels through a fluctuating medium or interface. SR with phase noise has been considered only once in the literature, in Ref. [13], where SR is shown by means of a numerical simulation in a specific nonlinear system. By contrast here, we shall develop a complete theoretical analysis for SR at phase noise in a class of nonlinear systems.

We consider a sinusoidal signal of period T corrupted by a phase noise $\eta(t)$ according to

$$x(t)=\cos\left[\frac{2\pi}{T}t+\eta(t)\right]. \tag{1}$$

A signal comparable to $x(t)$ of Eq. (1) was used in Ref. [13], to drive a level-crossing detector firing output pulses. This nonlinear transmission system was complicated enough to hinder an analytical analysis, and SR was shown in Ref. [13] through simulation. We consider here another type of transmission, described by

$$y(t)=g[x(t)], \tag{2}$$

where $g(\cdots)$ is an arbitrary function operating on real numbers.

The noise $\eta(t)$ is a stationary white noise with cumulative distribution function $F_\eta(u)$ and probability density function $f_\eta(u)=dF_\eta(u)/du$. The signal $x(t)$ of Eq. (1) results as a cyclostationary random signal of period T, with its cumulative distribution function $F_x(u,t)$ and its probability density function $f_x(u,t)=dF_x(u,t)/du$ which are both T-periodic functions of t. The output signal $y(t)$ of Eq. (2) will also result as a cyclostationary random signal of period T, in general, provided the function $g(\cdots)$ is not too peculiar.

Information about the coherent sinusoid of period T can be extracted from the random output signal $y(t)$, in the frequency domain, in the following way, which conforms to the standard setting of SR [1,2,14]. In general, due to the cyclostationarity of $y(t)$, the power spectral density of $y(t)$ will display a coherent part, constituted by spectral lines at integer multiples of the coherent frequency $1/T$. The power contained in the coherent spectral line at frequency n/T is given [14,15] by $|\overline{Y_n}|^2$, where $\overline{Y_n}$ is the order n Fourier coefficient of the T-periodic nonstationary output expectation $E[y(t)]$:

$$\overline{Y_n}=\frac{1}{T}\int_0^T E[y(t)]\exp\left(-in\frac{2\pi}{T}t\right)dt, \tag{3}$$

where the expectation $E[y(t)]$ at a fixed time t is expressable as

$$E[y(t)]=\int_{-\infty}^{+\infty} g(u)f_x(u,t)du. \tag{4}$$

Incoherent noisy fluctuations in the output signal $y(t)$, will show up in its power spectral density as a continuous broad-band noise background whose constant amplitude is measured by the stationarized output variance [14,15]

$$\overline{\mathrm{var}(y)}=\frac{1}{T}\int_0^T \mathrm{var}[y(t)]dt, \tag{5}$$

where the nonstationary variance $\mathrm{var}[y(t)]$ at a fixed time t is computable as

$$\mathrm{var}[y(t)]=\int_{-\infty}^{+\infty} g^2(u)f_x(u,t)du-E^2[y(t)]. \tag{6}$$

In the output power spectral density, it is possible to quantify the degree the coherent spectral lines emerge out of the broadband noise background by means of a signal-to-noise ratio (SNR) defined as

$$\mathcal{R}\left(\frac{n}{T}\right)=\frac{|\overline{Y_n}|^2}{\overline{\text{var}(y)}\Delta t \Delta B}, \tag{7}$$

which represents (as in Ref. [14], the expression of the SNR of Eq. (7) is obtained in a discrete-time framework where the signals are sampled at $\Delta t \ll T$, which provides an easy way to circumvent nonphysical pathologies of the continuous-time white noise such as an infinite variance associated to a zero correlation duration) the ratio of the power $|\overline{Y_n}|^2$ contained in the coherent spectral line at frequency n/T, to the power contained in the continuous noise background in a small frequency band ΔB around n/T.

Such a SNR in the frequency domain, is the standard measure most frequently used to characterize SR with a periodic signal and additive or multiplicative noise [1,2]. Here, we naturally turn to this measure for a characterization of this new form of SR at phase noise. A specificity here is that the present model allows an explicit computation of the SNR of Eq. (7), for any transmission function $g(\cdots)$, by means of Eqs. (3)–(6), provided we can relate the probability density $f_x(u,t)$ to the known statistical properties of the phase noise $\eta(t)$.

If we introduce the T-periodic cyclostationary random signal

$$\xi(t)=[2\pi t/T+\eta(t)] \text{ modulo } [-\pi,\pi), \tag{8}$$

we have, for the cumulative distribution function $F_x(u,t) = \Pr\{x(t)\leq u\}$, the relations

$$\begin{aligned}1-F_x(u,t)&=\Pr\{x(t)=\cos[2\pi t/T+\eta(t)]>u\}\\ &=\Pr\{-\text{acos}(u)<\xi(t)<\text{acos}(u)\}\\ &=F_\xi[\text{acos}(u),t]-F_\xi[-\text{acos}(u),t],\\ &\quad \text{for } u\in[-1,1),\end{aligned} \tag{9}$$

leading to

$$F_x(u,t)=\begin{cases}0 & \text{for } u<-1,\\ 1-F_\xi[\text{acos}(u),t]+F_\xi[-\text{acos}(u),t] & \text{for } -1\leq u<1,\\ 1 & \text{for } u\geq 1,\end{cases} \tag{10}$$

with the cumulative distribution function of $\xi(t)$,

$$F_\xi(u,t)=\Pr\{\xi(t)\leq u\}=\sum_{k=-\infty}^{+\infty}\Pr\{\eta(t)\in[-\pi+2k\pi-2\pi t/T,u+2k\pi-2\pi t/T]\} \quad \text{for } u\in[-\pi,\pi), \tag{11}$$

k integer. According to Eq. (11) we thus have

$$F_\xi(u,t)=\begin{cases}0 & \text{for } u<-\pi,\\ \displaystyle\sum_{k=-\infty}^{+\infty}[F_\eta(u+2k\pi-2\pi t/T)-F_\eta(-\pi+2k\pi-2\pi t/T)] & \text{for } -\pi\leq u<\pi,\\ 1 & \text{for } u\geq\pi.\end{cases} \tag{12}$$

The probability density of $x(t)$ is $f_x(u,t)=dF_x(u,t)/du$; i.e., acording to Eq. (10):

$$f_x(u,t)=\begin{cases}0 & \text{for } u<-1,\\ (f_\xi[\text{acos}(u),t]+f_\xi[-\text{acos}(u),t])\dfrac{1}{\sqrt{1-u^2}} & \text{for } -1\leq u<1,\\ 0 & \text{for } u\geq 1,\end{cases} \tag{13}$$

where $f_\xi(u,t)=dF_\xi(u,t)/du$ is the probability density function of $\xi(t)$, given through Eq. (12) by

$$f_\xi(u,t)=\begin{cases}0 & \text{for } u<-\pi,\\ \displaystyle\sum_{k=-\infty}^{+\infty} f_\eta(u+2k\pi-2\pi t/T) & \text{for } -\pi\leq u<\pi,\\ 0 & \text{for } u\geq\pi.\end{cases} \tag{14}$$

Collecting the above equations allows one to explicitly compute the SNR of Eq. (7), for any transmission function $g(\cdots)$, as a function of the properties of the phase noise $\eta(t)$ conveyed for instance by $f_\eta(u)$, and then to check the possibility of a nonmonotonic resonant evolution of the SNR when the noise level is raised, which represents the signature of SR.

For a simple illustration, we consider the case (often tested for SR with additive noise) of a hard threshold $g(u)=1$ when $u>\theta$ and $g(u)=0$ otherwise. In this case, Eqs. (4)

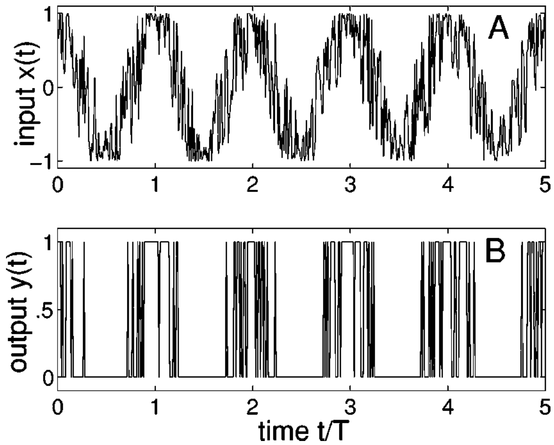

FIG. 1. (a) Input signal $x(t)$ of Eq. (1) when the phase noise $\eta(t)$ is uniform over $[-A,A]$ with $A=1$ rad, the step of the time discretization is $\Delta t = T/100$. (b) Ouput signal $y(t)=g[x(t)]$ of Eq. (2) when $g(\cdots)$ is a hard threshold $\theta=0.5$.

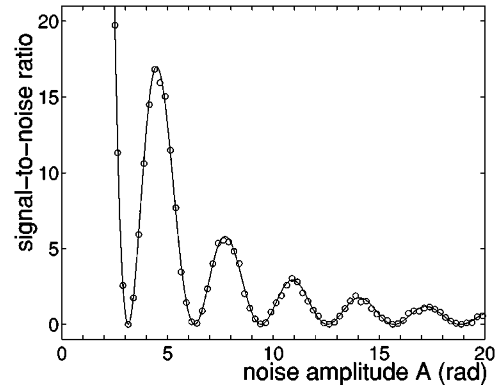

FIG. 2. Signal-to-noise ratio $\mathcal{R}(1/T)$ as a function of the amplitude A of the phase noise $\eta(t)$ uniform over $[-A,A]$, when $g(\cdots)$ of Eq. (2) is a hard threshold $\theta=0.5$. The solid line is the theoretical evaluation from Eq. (7), and the open circles result from a numerical simulation (we arbitrarily chose throughout $\Delta B \Delta t = 10^{-3}$).

and (6) lead, respectively, to

$$E[y(t)] = 1 - F_x(\theta,t) \tag{15}$$

and

$$\text{var}[y(t)] = F_x(\theta,t)[1 - F_x(\theta,t)]. \tag{16}$$

A simple choice for the noise is $\eta(t)$ uniform over $[-A,A]$, giving $f_\eta(u)=1/(2A)$ for $u\in[-A,A]$ and $f_\eta(u)=0$ elsewhere. Since this density $f_\eta(u)$ of $\eta(t)$ is an even function of u, the density $f_x(u,t)$ of $x(t)=\cos[2\pi t/T+\eta(t)]$, and also $F_x(u,t)$, are both even functions of t, thanks to the symmetry of the cosine. It is thus enough in Eqs. (13) or (10), to evaluate $f_x(u,t)$ or $F_x(u,t)$, two T-periodic functions of t, over $t\in[0,T/2]$; and thus enough also to evaluate $f_\xi(u,t)$ of Eq. (14), or $F_\xi(u,t)$ of Eq. (12), over $t\in[0,T/2]$. For this purpose, in Eqs. (14) and (12), the nonvanishing contributions to the sum are restricted to $k\in[k_{\min},k_{\max}]$, with $k_{\min}=-1-\text{int}[0.5+A/(2\pi)]$ and $k_{\max}=2+\text{int}[A/(2\pi)]$, where $\text{int}(\cdots)$ returns the closest integer towards zero.

In these conditions, Fig. 1 represents typical time evolutions of the noisy input $x(t)$ and output $y(t)$. The spectral analysis is then performed on the output signal $y(t)$. For the SNR in the frequency domain of Eq. (7), Fig. 2 shows the SNR theoretically computed with the present model, compared to the SNR numerically estimated on a simulation of $y(t)$ as in Fig. 1(b). The agreement is very good within the accuracy of the numerical estimation, since the present theoretical derivation of the SNR is exact, in contrast to many models for SR with additive or multiplicative noise which often have to resort to approximations.

The interesting point, revealed in Fig. 2, is that the SNR undergoes a nonmonotonic evolution when the noise level is raised. At zero noise, the SNR of Fig. 2 goes to infinity. This is because with this type of threshold detector and phase noise, the input signal $x(t)$ of Eq. (1) has a maximum amplitude always above threshold and unaffected by the noise, and thus in the absence of noise $x(t)$ is perfectly recovered by the spectral analysis at the output whence the infinite SNR. By contrast, for SR with additive noise, the coherent input is usually subthreshold at zero noise and requires noise addition to overcome the threshold; thus no transmission takes place in the absence of noise whence a vanishing output SNR at zero noise. Then in Fig. 2, when the noise is gradually raised above zero, the output SNR first rapidly drops to very low values, and next, larger noise levels can in turn increase the SNR. The multipeaked structure of the SNR as revealed by the model, has its origin in the specific action of the noise on the phase of a periodic signal as in Eq. (1). The nonmonotonic evolution of the SNR in Fig. 2 characterizes a form of SR, with ranges where the SNR gets improved when the noise level is raised. This is the common feature in SR: an increase of the noise can be beneficial to the transmission of a coherent signal, yet with specificities here contained in the present theory for a phase noise.

Figure 3 shows that the SR effect is preserved for any value of the threshold θ over the interesting range $[0,1)$. The exploitation of the model shows that many other conditions also lead to SR at phase noise. Figure 4 compares SR with a uniform and a dichotomous distribution for the phase noise

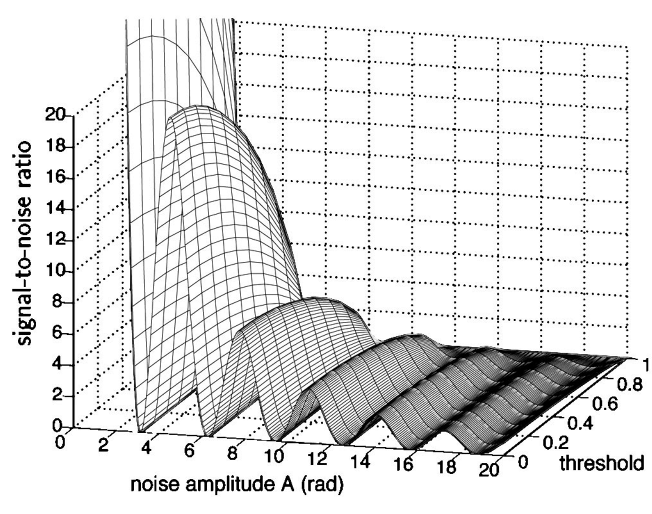

FIG. 3. Signal-to-noise ratio $\mathcal{R}(1/T)$ from Eq. (7) as a function of the threshold θ and of the amplitude A of the phase noise $\eta(t)$ uniform over $[-A,A]$, when $g(\cdots)$ of Eq. (2) is a hard threshold θ.

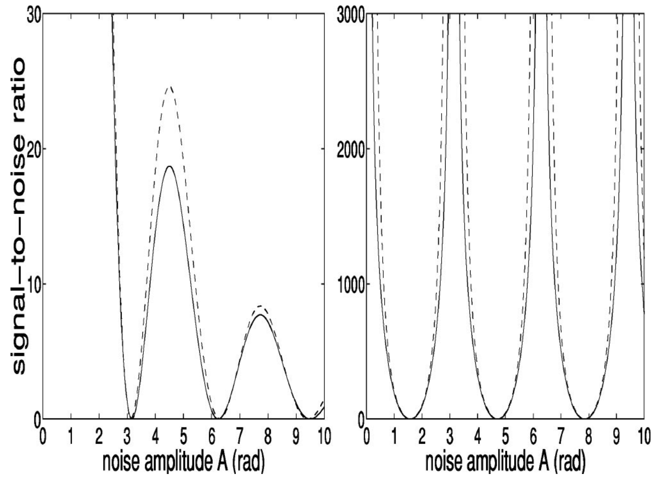

FIG. 4. Signal-to-noise ratio $\mathcal{R}(1/T)$ from Eq. (7) as a function of the amplitude A of the phase noise $\eta(t)$, when $g(\cdots)$ of Eq. (2) is a hard threshold $\theta=0$ (solid line) or a linear detector $g(u)=u$ (dashed line), with $\eta(t)$ uniform over $[-A,A]$ (left) and $\eta(t)=\pm A$ dichotomous (right).

$\eta(t)$. The multipeaked structure of the SNR is still observed in Fig. 4, and with the dichotomous noise the periodic alternation of zero and infinite values of the SNR can be understood directly from the behavior of $x(t)$ of Eq. (1) when the noise $\eta=\pm A$ assumes values at integer multiples of $\pi/2$. Our model shows that SR at phase noise can be obtained with a diodelike nonlinearity $g(\cdots)$, as the one used in Ref. [4] for SR with additive noise. Figure 4 also shows that SR at phase noise can even be obtained with the purely linear detector $g(u)=u$. This is a property usually not present in conventional periodic SR with additive noise and measured by the standard SNR, which requires transmission by a nonlinear system in order to induce a coupling between signal and noise capable of SR. Here, the noise coupled to the signal through its phase constitutes a sufficient nonlinear coupling, that in itself can generate SR. The SNR delivered by the linear detector $g(u)=u$ can also be interpreted as the input SNR, resulting from direct observation of $x(t)$ of Eq. (1), and prior to transmission by a nonlinear detector outputting $y(t)\not\equiv x(t)$. Then, as illustrated in Fig. 4, we were not able to find a simple nonlinear detector capable of delivering an output SNR larger than this input SNR for SR at phase noise. This was so, even with a dichotomous noise which was shown the most favorable noise to obtain an output SNR larger than the input SNR in SR with additive noise in static detectors [15]. This again contrasts SR with phase or with additive noise: possibility of SR with a linear detector but no input-output SNR gain with the former, the opposite with the latter.

The present theory provides a characterization of an alternative form of SR, at phase noise. The model can handle an arbitrary distribution for the noise associated to an arbitrary memoryless detector $g(\cdots)$. Illustrations were provided with a uniform noise (this type of noise was used in the numerical simulation of Ref. [13] reporting the first observation of SR at phase noise, though in a different system) and with a dichotomous noise, associated with a treshold detector (the simplest detector most often used to obtain SR with additive noise) and with a linear detector which was shown here capable of a form of SR. Additionally, the expression of the SNR of Eq. (7) allows the examination of SR at higher-order harmonics. The model can also be used to characterize SR when the static detector $g(\cdots)$ is followed by any dynamic linear system. In this case the SR property measured by the SNR of Eq. (7) will not be affected, since both the numerator and the denominator of Eq. (7) will be multiplied by the same factor given by the squared modulus of the transfer function of the linear system. We also note that the present treatment can easily be transposed to a coherent signal which is not a sinusoid, for instance to a periodic train of pulses (neural action potentials, solitons) that would be corrupted by phase noise, and could possibly lead to SR. The present model is the first of its kind to propose a theory of SR at phase noise, in a broad class of conditions or systems. As such it constitutes a unique framework for further investigations on SR and its applications.

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