Quasiperiodicity and transition to chaos

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In coupled Lorenz systems, we find that the three-frequency quasiperiodcity exists in the finite parameter range robustly. We also find the period-doubling bifurcation of the torus and the quasiperiodic windows in superchaos. Based on the separation of the dynamics, corresponding explanations are given by making use of the synchronous dynamics.

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Transition to chaos is a fundamental problem in the nonlinear dynamics. Several routes [1–4] have been found in low-dimensional chaotic systems (we define a low-dimensional chaotic system as the system with a chaotic attractor whose dimension is less than 3). Recently, the transition to high-dimensional chaos also attracts a great deal of attention. Harrison and Lai [5] presented a route to high-dimensional chaos where the system first comes to low-dimensional chaos, then to a high-dimensional one (superchaos) with the second Lyapunov exponent passing through zero continuously. However, there exists another type of high-dimensional chaos that has only one positive Lyapunov exponent and more than one zero Lyapunov exponent. Generally speaking, this type of chaos cannot be attained only through the transition form of periodic behavior or a fixed point. To investigate the transition to this type of high-dimensional chaos, the quasiperiodicity route will be a good candidate.

Evidence of the route to chaos from quasiperiodicity was first known in late 1970s [6,7], our understanding of the transition still appears quite limited. Most of investigations in the 1980s was concentrated on two-dimensional (2D) maps and three-dimensional (3D) flows, [7–11] where the transition involves with the frequency locking or wrinkles or corrugation on the torus. However, one new type of transition to chaos with dimension larger than 3 was recently proposed where the torus is directly destroyed without frequency locking and wrinkle of the torus [12]. The mechanism of the transition is believed to be heteroclinic bifurcation occurring in a four-dimensional (4D) system. It raises a question: can we find another situation where torus is destroyed without wrinkle and frequency lock in a different way?

It has been controversial regarding whether there exists quasiperiodicity with more than two incommensurate frequencies. Ruelle and Takens [4] showed that three-frequency quasiperiodicity is unlikely since it can be destroyed by small perturbations. The experiments that followed were inconclusive [13–16]. Grebogi *et al.* [17] reported the results of some numerical experiments on a model system. They found that chaos in their system appeared to have zero measure until coupling was almost 3/4 of the critical coupling. Three-frequency quasiperiodicity is also found in other numerical simulations and electronic circuits [18]. The common point in these investigations is that the systems behave in the way of three-frequency quasiperiodcity in the absence of coupling or for small driving. The interesting points are whether we can find three-frequency quasiperiodicity in systems with more complex dynamics and whether we can find the transition involved with three-frequency quasiperiodicity.

In this paper, we address the transition to high-dimensional chaos via quasiperiodicity and the robustness of three-frequency quasiperiodicity. Furthermore, we will show the existence of interesting quasiperiodicity windows in superchaos.

The model we used is the coupled N Lorenz oscillators [19,20]

$$\begin{aligned} \dot{x}_i &= \sigma(y_i - x_i), \\ \dot{y}_i &= \rho x_i - y_i - x_i z_i + (\epsilon - r)(x_{i+1} - x_i) + (\epsilon + r)(x_{i-1} - x_i), \\ \dot{z}_i &= x_i y_i - z_i, \\ i &= 1,2,\ldots,N, \end{aligned} \tag{1}$$

where ϵ is the diffusion coupling and r is the flow coupling. When we fix $\sigma=10$ and $\rho=28$, the single Lorenz system will be in chaos. $N=4$ and $\epsilon=14$ throughout the paper unless we mention it specifically. The Lorenz oscillators are in synchronization for $r=0$. When r increases beyond a critical value, a Hopf bifurcation related to the synchronous chaos will occur (for details see Ref. [17]). In this paper, we only survey a small range of parameter r beyond the Hopf bifurcation.

We linearize Eq. (1) along the typical orbit of the system (x_i, y_i, z_i)

$$\begin{aligned} \dot{\delta x}_i &= \sigma(\delta y_i - \delta x_i), \\ \dot{\delta y}_i &= \rho\,\delta x_i - \delta y_i - \delta x_i z_i - x_i \delta z_i + (\epsilon - r)(\delta x_{i+1} - \delta x_i) \\ &\quad + (\epsilon + r)(\delta x_{i-1} - \delta x_i), \\ \dot{\delta z}_i &= \delta x_i y_i + x_i \delta y_i - \delta z_i, \\ i &= 1,2,\ldots,N, \end{aligned} \tag{2}$$

where $(\delta x_i, \delta y_i, \delta z_i)$ is the perturbation around the typical orbit of the system. We integrate Eqs. (1) and (2) by the use of Runge-Kutta method. Based on Eqs. (1) and (2), we can

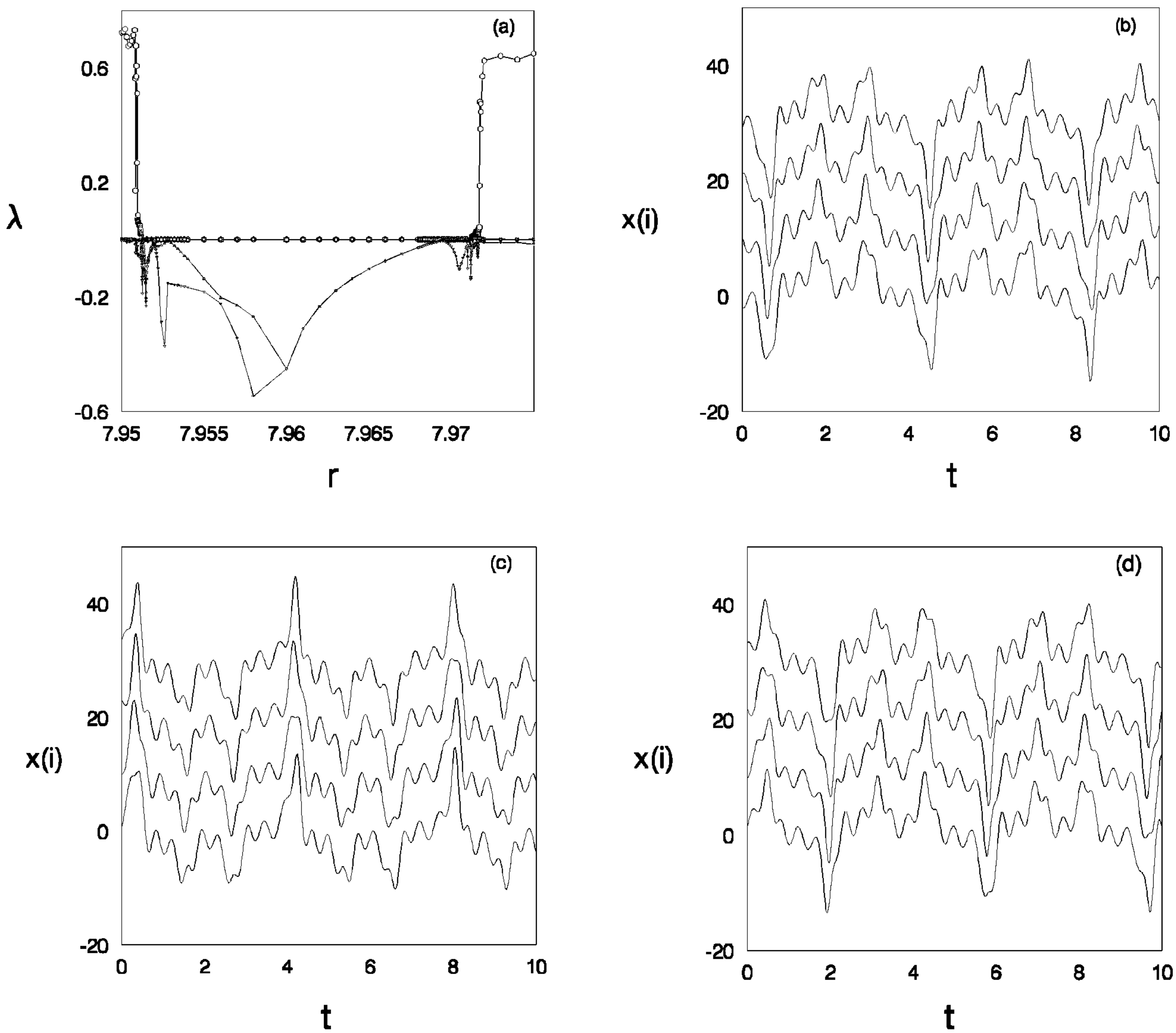

FIG. 1. (a) The first four Lyapunov exponents in coupled Lorenz systems. $N=4$, $\epsilon=14$, $\sigma=10$, and $\rho=28$. These parameters remain the same for all the following figures unless we mention it specifically. The time sequences of variable $x(i)(i=1,2,3,4)$ for four sites when (b) $r=7.954$; (c) $r=7.97$; (d) $r=7.9714$.

calculate the Lyapunov exponents spectrum with the repeated use of the Gram-Schmidt reorthonormalization procedure on the vector frame $(\delta x_i, \delta y_i, \delta z_i)$. (The detailed algorithm of the Lyapunov exponents spectrum may be found in Ref. [21]) The first four Lyapunov exponents in the range of $r\in[7.95,7.975]$ have been shown in Fig. 1(a). It is obvious that quasiperiodic motion exists in this region and the transitions to chaos can be seen at the large and small r ends. In Figs. 1(b), 1(c), and 1(d), we show the time sequence of variable x of the four sites for different parameters for which the system has different dynamics. The common aspect is that the dynamics of the system may be divided into two components, for example, $x(i)=x_T(i)+x_S(i)$. One is the synchronous component (SC), (x_S, y_S, z_S), where all sites behave in the same way; the other is the traveling wave component (TWC), $(x_T(i), y_T(i), z_T(i))$, where the different sites behave in the same periodic way but with a constant phase difference of $\pi/2$ between two adjacent sites. That is, $x_T(i,t)=x_T[i+1,t-(\tau/4)]$ where τ is the temporal period of the TWC. To single out the SC, we may consider the quantities X,Y, and Z, which are the sums of the variables x,y, and z for all sites. We take X, for example,

$$X=\frac{1}{4}\sum x_i=\frac{1}{4}\sum (x_T+x_S)(i)$$

$$=x_S+\frac{1}{4}\sum\left(\sum_k a_k e^{i(2\pi k/\tau)t+(i\pi/2)}\right)=x_S+o(a_4),$$

where a_k is the Fourier coefficient of the TWC. By numerical simulation, we know that $a_k(k>1)$ is much smaller than a_1 and $o(a_4)$ is a small amount corresponding to x_S. Thus when we take X, Y, and Z into consideration, we eliminate the TWC up to the small amount of $o(a_4)$. As a result, if SC is periodic (or two-frequency quasiperiodicity) the system will behave in the way of two-frequency (or three-frequency) quasiperiodicity. Another important thing is that both the frequency and the strength of the TWC will increase with the increase of r in this region.

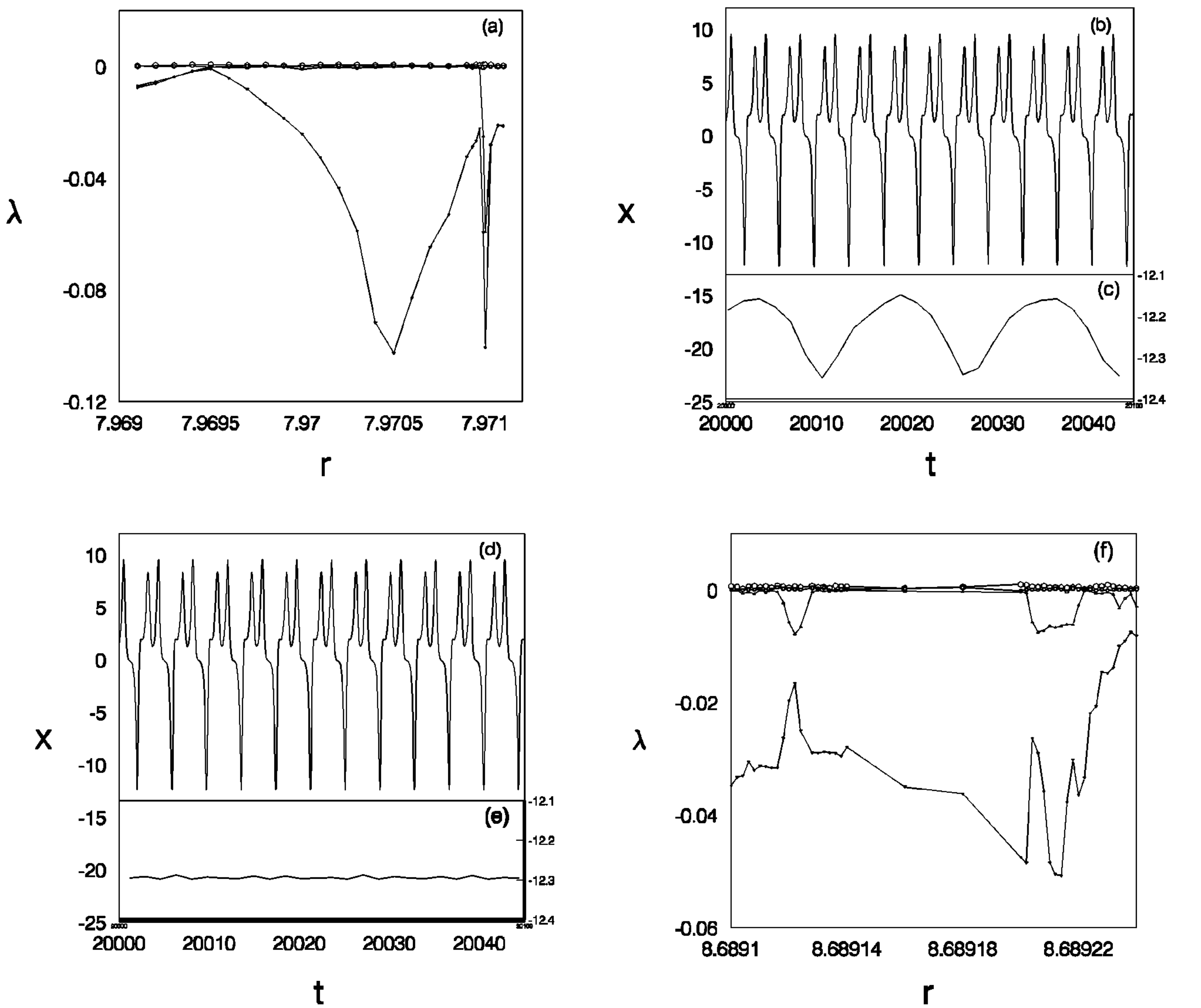

FIG. 2. The Lyapunov exponents in the range of $r\in(7.969,7.9712)$. (b) and (d) The time sequences of variable X for the synchronous motion for different r, (b) $r=7.9705$ and (d) $r=7.965$. (c) and (e) The profile of the minimum of X with the same parameters as (b) and (d), respectively. (f) The Lyapunov exponents for $N=4$, $\epsilon=16$.

We zoom in the Lyapunov exponents spectrum in the region of $r\in(7.969,7.9712)$ in Fig. 2(a). Three zero Lyapunov exponents are found, which means that three-frequency quasiperiodicity exists in this region. To further confirm our findings, we plot the time sequence of quantity X for $r=7.9705$ in Fig. 2(b). To magnify the weak periodic modulation, we show the profile of the minimum of X in Fig. 3(c). In these plots we may find the slow periodic modulation whose amplitude is very small on SC. In Fig. 2(d), we show the results for $r=7.96$ where the system is in two-frequency quasiperiodicity. The profile of the minimum of X is also shown in Fig. 3(e). In these plots, no periodic modulation is found and the SC is periodic. The fluctuations in Figs. 3(c) and 3(e) are caused by $o(a_4)$. All of these results show that there exists three-frequency quasiperiodicity, and the three-frequency quasiperiodicity exists in a finite parameter range robustly. The three-frequency quasiperiodicity is developed from the two-frequency quasiperiodicity via Hopf bifurcation at about $r=7.9695$ and then loses its stability and changes to a different two-frequency quasiperiodicity via a saddle-node bifurcation at around $r=7.9709$. Actually, we can also find various saddle-node bifurcations between two- and three-frequency quasiperiodicities. One section of the Lyapunov exponents spectrum for $\epsilon=16$ is shown in Fig. 2(f). We can find that the three-frequency quasiperiodicity finally settles down to two-frequency after a series of transitions between three-frequency and two-frequency quasiperiodcity. It is noted that all of the transitions are saddle-node bifurcation. In this system, we do not find the direct transition to chaos from three-frequency quasiperiodicity.

Now we are interested in the transition to chaos around $r=7.951$. In Fig. 3(a), we may notice that there exist several instabilities during the transition from two-frequency quasiperiodicity to chaos such as those at $r=7.953$ and $r=7.952$. These instabilities do not change the nature of the quasiperiodicity. To investigate how the torus changes during these instabilities, we study SC on the X-Z plane. In Fig. 3(b) $r=7.9532$ where SC rotates two cycles in the right half-plane then rotates one cycle in the left-half plane [22]. When we increase r to 7.9524 in Fig. 3(c), we find that the rotation of the torus doubles, namely, it rotates four cycles in one half-plane and two cycles in the other half-plane. If we only trace

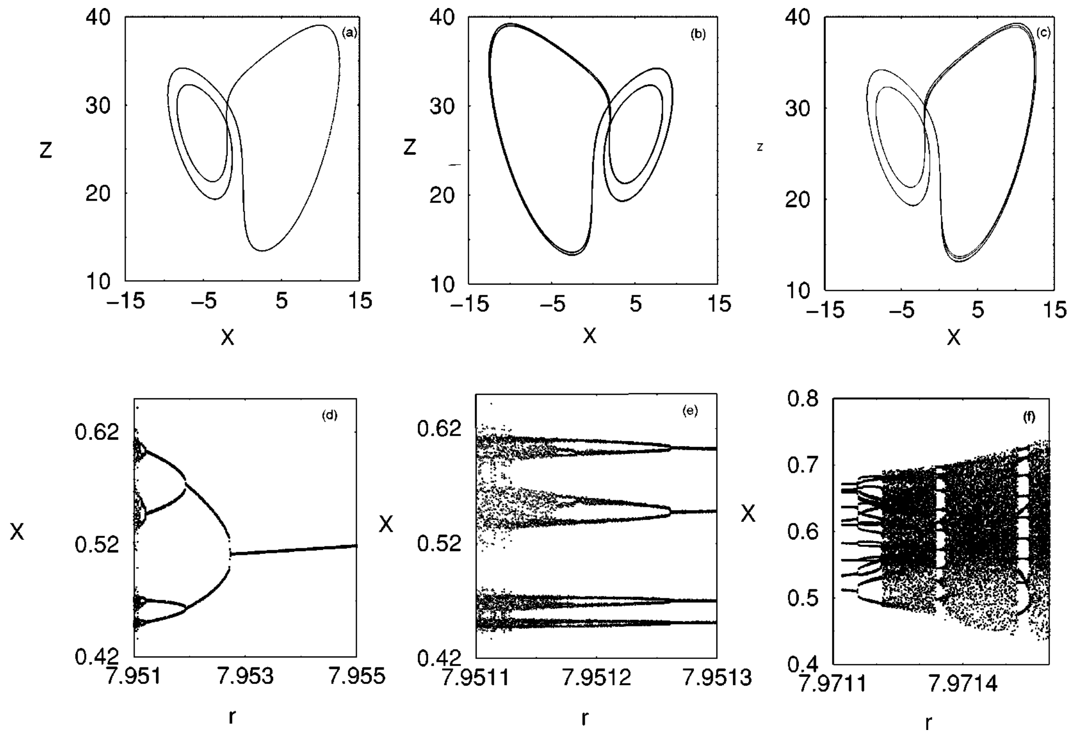

FIG. 3. (a) The Lyapunov exponents versus r; (b), (c) and (d) the trajectories of the synchronous motion for different parameters r, (b) $r=7.9532$, (c) $r=7.9524$, and (d) $r=7.9513$. (e), (f), and (g) The bifurcation diagrams for the synchronous motion.

a section of the trajectory in the plane X-Z, for example, $X \in (-12,-10)$ [or $X \in (10,12)$] and $Z \in (34,38)$, we may find two curves while there is only one curve for $r=7.9532$. If we further decrease r, the period of SC will double again and so will the number of the curves [Fig. 3(d)].

It will be more convenient to investigate the transition if the bifurcation diagram is built up. We define a Poincaré section for SC when the trajectory passes through the X-Z plane from negative Y to positive Y. The results are shown in Fig. 3(e). The period-doubling bifurcation to chaos is observed. In Ref. [23], the authors suggested a model to simulate the period-doubling bifurcation of a torus by applying a periodic driving to a system that manifests period-doubling bifurcation to chaos. Differing from their model, our period-doubling bifurcation of the torus is not artificial. The periodic driving and the dynamics, which manifests period-doubling bifurcation, occur spontaneously, and they cannot exist in a single Lorenz system of the same parameters. This invites a question: will the period-doubling bifurcation undergo infinite time? The authors of [23] stated that the bifurcation only proceeds several times or even transforms to chaos without period-doubling bifurcation when the coupling between the driving and the system of period-doubling bifurcation is strong. We also find that the period-doubling bifurcation only occurs for a few times in Fig. 3(f). After the bifurcation from period 8 to period 16, we cannot observe period-doubling bifurcation any more. For the interruption of the period-doubling bifurcation sequence, we can have a good understanding based on the separation of the dynamics of the system. We know from Eq. (2) that the effect of the TWC cannot be eliminated completely.

There still exists a small quantity $o(a_4)$. The main effect of $o(a_4)$ is to thicken all the branches of the solution on the synchronous manifold. The thickening phenomenon can be observed in Figs. 3(d) and 3(e). When the distance between any pair of adjacent branches of the solution of SC becomes smaller than the thickness of the corresponding solution branches, SC will no longer obey the original periodic path. It will jump between the two branches under the control of TWC. With further decreasing r, SC will jump between two more distant branches. It is the jump between the different branches that interrupts the bifurcation sequence. Though we do not know the exact form of the coupling between the TWC and SC, we know that the strength of the coupling between the TWC and SC is determined by the quantity $o(a_4)$. Therefore, it is interesting to notice that the coupling between the SC and TWC in our case is not a constant. This conclusion is different from that of Ref. [23]. For the transition near $r=7.951$, the bifurcation sequence terminates at a weak coupling. However, we can also observe the period-doubling bifurcation, which terminates at a strong coupling, for example, the transition to chaos at the higher r end [see Fig. 3(g)]. In the case shown in Fig. 3(g), the two-frequency quasiperiodicity is developed from the three-frequency quasiperiodicity and its SC has a high period. As a result, its SC has a more branches than those in Figs. 3(e) and 3(f), and the distances between any adjacent branches are much smaller. It is also the reason why fewer period-doubling bifurcations are observed.

Some remarks are necessary to be made. First, the period-doubling bifurcation of the torus discussed above is one example that shows the two-frequency quasiperiodicity yields to chaos without frequency locking and wrinkle of the torus. The torus keeps its smoothness until becoming chaos (in Fig. 3). Though our results are obtained from SC, we can draw the same conclusion even if we construct other Poincaré sections such as an $x(1)-y(1)$ plane with $z(1)$ taking its maximum. Second, it is important to notice that there is only one

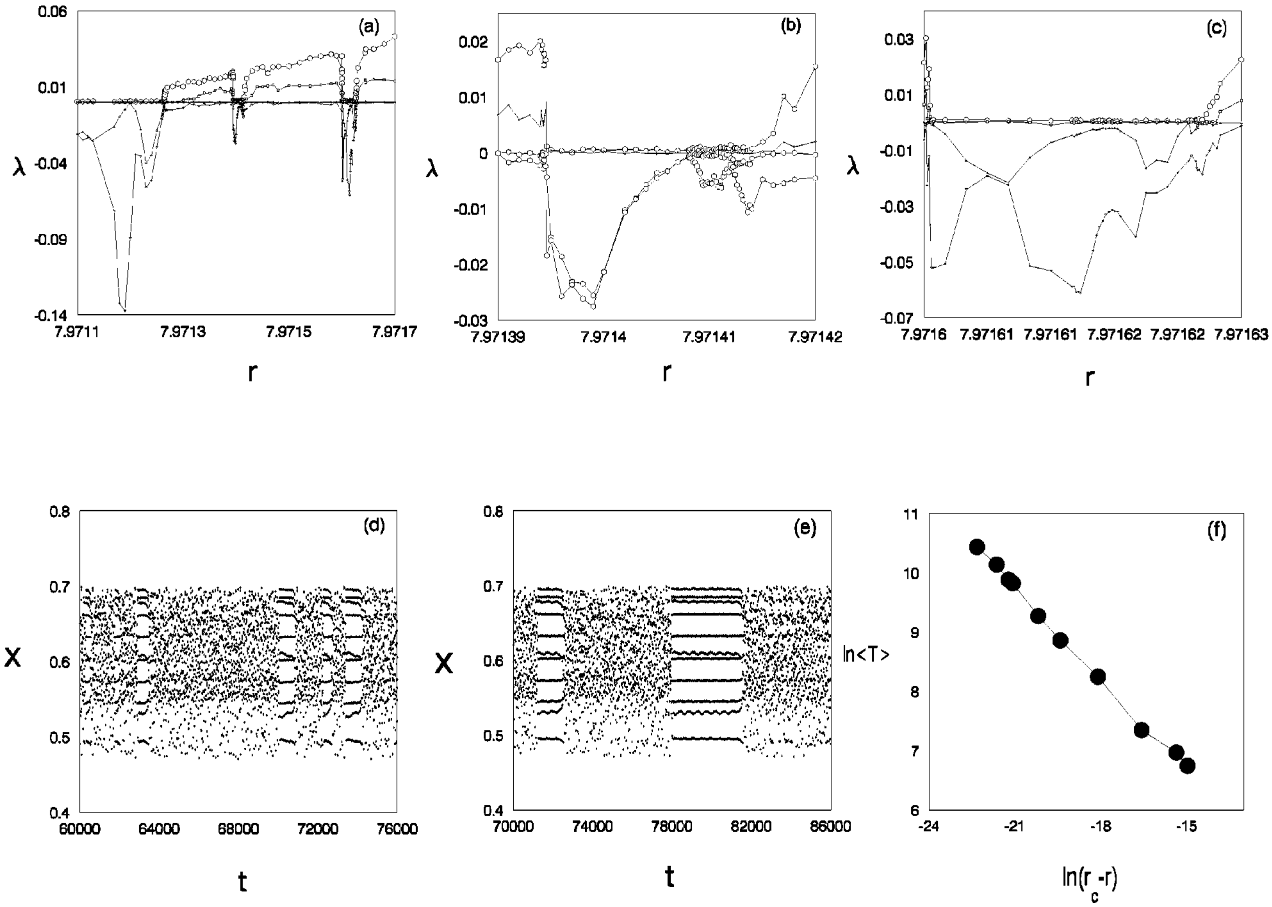

FIG. 4. (a) The Lyapunov exponents in the range of $r \in [7.9711,7.9717]$. (b) and (c) The magnified plots of the Lyapunov exponents of (a) around (b) $r \in [7.97139,7.97142]$ and (c) $r \in [7.9716,7.97163]$. (d) and (e) The evolutions of X in the Poincaré section for $r = 7.971\,394\,3$ and $r = 7.971\,394\,612$. (f) $\ln\langle T\rangle$ vs $\ln|r-r_c|$.

positive Lyapunov exponent and two zero Lyapunov exponents after the onset of the chaos. According to the Kaplan-Yorke conjecture [24], we know the dimension of the attractor is larger than 3. That is, a high-dimension chaos via period-doubling bifurcation of the two-frequency torus is attained. Finally, as shown in Fig. 4(a), we can find that superchaos will appear after the birth of the high-dimensional chaos. Here the second Lyapunov exponent gradually becomes positive at a parameter with a finite distance from the onset of the chaos, which is in agreement with the statement provided by Harrison and Lai [5] that the superchaos is induced gradually. Nonetheless, the difference is obvious: we first encounter the high-dimensional chaos just when the regular motion loses stability, while Harrison *et al.* first encounter a low-dimensional chaos.

In Fig. 4(a), we may notice an interesting phenomenon. In the super-chaotic region, we find that there exists regular motion windows located at $r \in [7.971\,39,7.971\,42]$ and $r \in [7.9716,7.971\,63]$. The regular motion windows can be observed in Fig. 3(g) also. We plot in Figs. 4(b) and 4(c) the detailed Lyapunov exponents of the spectrum. To our surprise, we find that they are both two-frequency quasiperiodicity in these two windows (to our knowledge, this is first time that the quasiperiodicitiy window is observed) and the quasiperiodic windows come from superchaos abruptly. In the first window, the two-frequency quasiperiodicity first undergoes the Hopf bifurcation to three-frequency quasiperiodicity followed by frequency locking, then undergoes the period-doubling bifurcation to chaos. In the second window, there exists no three-frequency quasiperiodicity, and the two-frequency quasiperiodicity moves to chaos directly via period doubling of the torus. In both windows, the superchaos is restored continuously from the high-dimension chaos with one positive and two zero Lyapunov exponents. The existence of the quasiperiodic windows may be explained based on the separation of the dynamics of the system. We know that all the qualitative changes of the system are mainly related to the SC. The existence of the quasiperiodic windows is due to the existence of the periodic windows for the SC. The sudden appearance of the quasiperiodic windows is because the periodic windows appear via tangent bifurcations for the SC. The evolutions of X in the Poincaré section defined above for $r = 7.971\,394\,3$ and $r = 7.971\,394\,612$ are recorded in Figs. 4(d) and 4(e). The intermittency is clear in these two plots. The closer to the critical $r_c \simeq 7.971\,394\,613\,7$, the longer is the duration of the periodic phase. The averaged life for periodic phase $\langle T\rangle$ is plotted vs $|r-r_c|$. The power law is clear and the exponent is $-\frac{1}{2}$, that is, $\langle T\rangle \sim |r-r_c|^{-1/2}$. It agrees with the result for the intermittency of saddle-node type [25]. The same conclusion can be drawn for the second quasiperiodic window.

For different parameter sets, we can observe similar pictures as those discussed above. The effects of different parameters are changing the location of the quaiperiodic

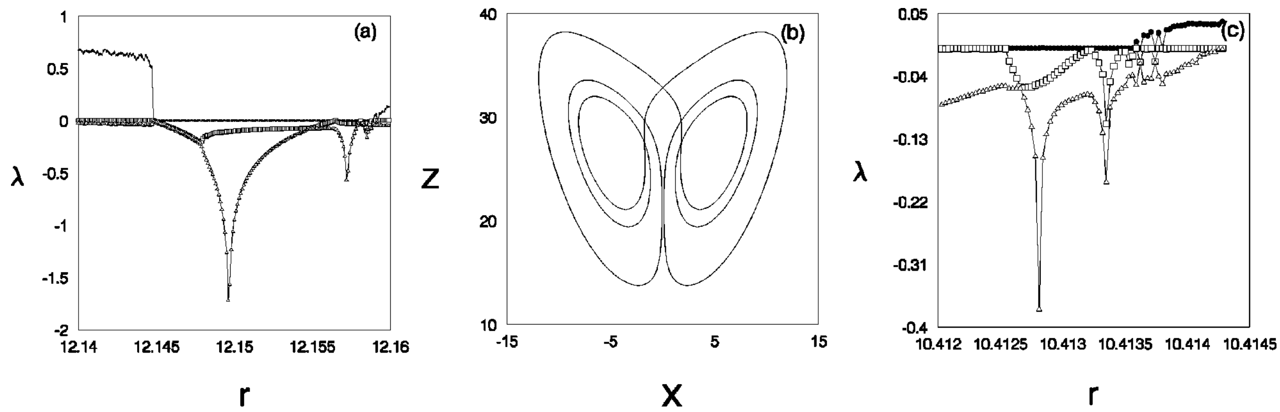

FIG. 5. The Lyapunov exponents when $\epsilon=28$. (b) The symmetric periodic synchronous orbit for $\epsilon=28.$ and $r=12.15$. (c) The Lyapunov exponents when $N=10$ and $\epsilon=50$.

''dent'' and selecting the periodic synchronous orbit. In Fig. 5(a), we show the first four Lyapunov exponents when $\epsilon=28$. A similar structure to the Lyapunov exponents is found in Fig. 3(a) and shows the period-doubling bifurcation of a torus to chaos. It is worth mentioning that the periodic synchronous orbit is different from the one in Fig. 3. In this case, the synchronous orbit is invariant under the transformation $(X,Y,Z)\rightarrow(-X,-Y,Z)$ [Fig. 5(b)]. Furthermore, the phenomena observed here are independent of the dimension of the system. We show the first four Lyapunov exponents in a section of the quasiperiodic ''dent'' in Fig. 5(c) where three-frequency quasiperiodicity and the period-doubling bifurcation are seen. The quasiperiodic windows in superchaos can be found also if we zoom in on this plot.

It is necessary to note that the results obtained here are not limited to the coupled Lorenz systems. One possible experimental setup to realize those phenomena is the coupled Chua's circuits, which are used in Ref. [20].

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