Amplification of near-resonant signals via stochastic resonance in a chaotic CO_2 laser

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We show experimentally and numerically that a chaotic CO_2 laser with modulated losses operating in the region of an intermittency resulting from the band-merging crisis can serve as an amplifier of near-resonant signals, i.e., signals with a frequency close to the first subharmonic frequency, via deterministic stochastic resonance. The mechanism underlying stochastic resonance in this case is a synchronization of the random switching events between two chaotic repellers after the band-merging crisis with near-resonant signals at the detuning frequency. We demonstrate experimentally that the gain factor in chaos is larger than near the first period-doubling bifurcation by a factor of 2. Numerical results obtained in a two-level rate-equation model are in good agreement with the experimental ones.

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I. INTRODUCTION

The amplification of periodic signals near the onset of instabilities is known to be inherent in bifurcating dynamical systems, as shown theoretically in [1] for the case of simple critical points (such as period doubling, symmetry breaking, and saddle-node bifurcations). Experimentally, investigations of the amplification of near-resonant signals, i.e., signals with a frequency close to one of the subharmonic frequencies, near period-doubling bifurcations have been carried out in different nonlinear systems [2–5] including a CO_2 laser with modulated losses [6,7]. They have shown a high sensitivity to the effect of near-resonant signals that allows one to hope for practical implementations. In a chaotic state, nonlinear dissipative systems may display another class of bifurcation events, such as crises, which manifest themselves as sudden discontinuous changes in the chaotic dynamics as a control parameter is varied [8]. For the case of the band-merging crisis, Anishchenko *et al.* [9,10] on the basis of a simple map have shown theoretically that chaotic systems may enhance periodic signals via the mechanism of stochastic resonance (SR). This so-called *deterministic* SR [9,10] takes place in the region of crisis-induced intermittency past its critical point. In contrast with conventional SR, they have proposed making use of the intrinsic chaotic dynamics as an internal source of noise for SR instead of an external one. In this case, increase of the bifurcation parameter above a critical point corresponding to the onset of the crisis leads to increase of the stochasticity in the system. A similar chaotic dynamics approach to SR was considered in [11]. Recently, amplification of weak signals and stochastic resonance in a general class of dynamical systems with reflection symmetry has been theoretically studied in [12]. In these cases, the signature of SR is that the response to a deterministic periodic signal, the signal-to-noise ratio (SNR), or the area under the first peak in the residence-time distribution passes through a maximum at the optimal value of the bifurcation parameter. Such a type of deterministic SR has been experimentally observed in the region of type-III intermittency between chaotic and regular phases in ferromagnetic resonance experiments [13]. Later, deterministic SR was also observed in a loss-modulated CO_2 laser in the bistability domain between two chaotic attractors of period-2 and period-3 branches [14]. *Though deterministic SR has already been demonstrated, one question still remains to be answered, which is very important from both theoretical and practical standpoints: how large can the gain factor be when using chaos as an amplifier of periodic signals in real physical systems?*

It should be noted that SR is by now a well-established phenomenon, which takes place in bistable, excitable, and threshold systems, where adding a certain level of noise may enhance the system response to weak periodic signals. Originally, the concept of SR was proposed in [15,16] to explain some specific phenomenology in climatology. At first, SR was experimentally observed in such systems as an electric circuit [17], a bidirectional ring dye laser [18], and then in many other devices in different fields [19].

In this paper, in contrast with previous studies of deterministic SR with low-frequency signals, we used near-resonant signals (NRS's), i.e., signals with a frequency close to the first subharmonic frequency. Though near-resonant signals have been used earlier for investigations of SR in nonlinear electrical circuits [20], most attention has been focused on the study of SR in the region of a period-doubled response with an external source of noise or chaos. Moreover, SR has been investigated mainly on the beat frequency (i.e., at a frequency equal to the detuning of NRS's from half the driving frequency). Here we shall focus on the system response at exactly the frequency of the NRS.

According to Dykman *et al.* [21], along with conventional or classical bistable SR with a weak periodic signal and a relatively low frequency with respect to the fundamental frequency in the system, there is also a high-frequency stochastic resonance. In the latter case, SR can be observed in a bistable system with a signal frequency close to the main driving frequency in the narrow range of system parameters

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where a kinetic phase transition takes place. In this context, SR with NRS's occupies an intemediate range of frequencies between conventional SR and high-frequency SR.

The second feature of our experiments is that we explore SR with NRS's in the region of an intermittency that results from the band-merging crisis. Such a temporal behavior is inherent in nonautonomous systems which display the route to fully chaotic behavior in succession via direct and inverse cascades of period-doubling bifurcations, and has been found in many nonlinear systems from different fields such as physics, chemistry, biology, etc. We show experimentally in a loss-modulated CO_2 laser operating in this regime that a chaotic system can serve as an amplifier of NRS's via a mechanism of deterministic stochastic resonance. The gain factor for the NRS depends on the signal amplitude and, for small enough signals, it can be more than twice larger than that obtained near the first period-doubling bifurcation point. Simultaneously, we observed SR at the idler frequency. All our experimental results are in good agreement with a numerical simulation done in the framework of a simple two-level rate-equation model.

II. EXPERIMENT

The experimental investigations are based on a loss-modulated CO_2 laser which can be broadly classified as a nonautonomous oscillator with an asymmetric Toda potential [22]. A loss-modulated CO_2 laser has a very rich variety of nonlinear behavior, including a route to chaos via period-doubling bifurcations [23], and can be described qualitatively in the framework of a simple two-level rate-equation model [23]. The experimental setup is essentially the same as in our previous studies on the effects of NRS's on the dynamics of a CO_2 laser [7]. Two acousto-optic modulators inserted in the laser cavity were used to provide the time-dependent cavity losses. The driving signal $V_d(t) = V_d \cos(2\pi f_d t)$ had frequency $f_d = 100$ kHz and amplitude V_d and served as a bifurcation parameter. In what follows, we used the bifurcation parameter μ defined as $\mu = V_d / V_{1/2}$, where $V_{1/2} \cong 2.27$ V is the first period-doubling threshold. The near-resonant signal $V_s(t) = V_s \cos(2\pi f_s t)$ had the frequency $f_s = f_d/2 + \Delta$. Further, along with Δ we used a normalized detuning frequency δ defined as $\delta = 2\Delta/f_d$. The laser responses were detected with a Cd-Hg-Te detector and recorded by a digital oscilloscope or analog-to-digital converter coupled with a PC to store and process the data.

As mentioned above, the route to chaos via period doubling is one of the scenarios inherent in many nonlinear systems in physics, chemistry, biology, etc., including such an optical system as a CO_2 laser with modulated losses. As the bifurcation parameter increases, these nonlinear systems exhibit sequential formation of period-doubling bifurcations which terminates at an accumulation point. Beyond this point there is a successive pairwise merging of chaotic subbands, which ends with a fully chaotic behavior when the two last chaotic subbands (or attractors) merge into a single large attractor [8]. In our case, these two chaotic subbands serve as two phase states which are separated by an unstable T-periodic orbit. After the band-merging crisis, when two chaotic subbands collide with the unstable T-periodic orbit, random switching between these two states takes place. In the presence of NRS's a new relevant time scale $T_\Delta = 1/\Delta$ appears in the system and correspondingly two phase states with respect to a translation in time by $T_\Delta/2 = 1/2\Delta$, which exist between the shifted first bifurcation point and the critical point for the band-merging crisis. When the signal amplitude is large enough to reach and cross the unstable T-periodic orbit, the laser system can change the phase state with respect to a chosen reference signal. This means that the presence of NRS's can synchronize otherwise random switching between two attractors after the band-merging crisis. As is well known, the physical process underlying SR in a bistable system is a synchronization between the noise-induced random two-state switching and the deterministic modulation.

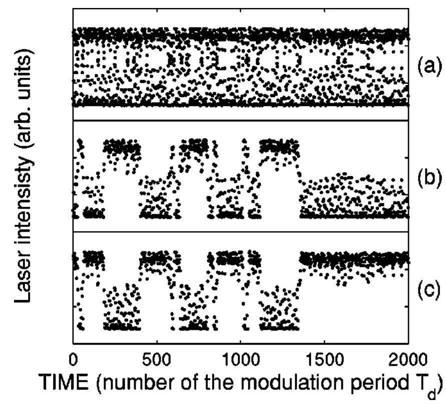

FIG. 1. Decomposition of the experimental stroboscopic laser response with NRS into a pair of repellers. (a) Initial laser response sampled with a period $T_d = 1/f_d$; (b) and (c) sampling with a period $2T_d$ starting from the first and the second points, respectively.

One of the tools that allow one to reveal synchronization aspects of SR is the residence-time distribution (RTD) between the switching events [24,25]. Experimentally, residence-time distributions have been investigated in different nonlinear systems and very different experimental conditions [13,26–28]. In our case residence times are the times between two subsequent crossings of the unstable T-periodic orbit. In order to reveal switching between two chaotic subbands we adopted a procedure similar to that used in [29] to study critical exponents for crisis-induced intermittency. We have decomposed all stroboscopic signals into a pair of repellers taking every second point from the laser response and starting from the first and the second point, respectively (Fig. 1). Thereafter, we subtracted from the responses the value of the amplitude of the unstable T-periodic orbit and applied a low-band filter to avoid false switching events. Then, we used a binary filter to obtain a pure two-state dynamics. Such a procedure allows us to count the number of switching events between two states during some period of time and find periods of time between switching events.

In the absence of NRS's an exponential distribution $P(t) \propto \exp(-\beta t)$ of the time period between switchings is observed, where the power β in the exponential distribution increases with increasing bifurcation parameter. We have estimated the power β in the exponent for two values of the bifurcation parameter μ, for example, for $\mu \cong 3.08$ ($V_d = 7.0$ V), $\beta \cong 2\times 10^3$ s^{-1} and for $\mu = 3.22$ ($V_d = 7.3$ V), $\beta \cong 3.48\times 10^3$ s^{-1}. In the presence of NRS's, for a value of bifurcation parameter μ above the onset of crisis, the RTD

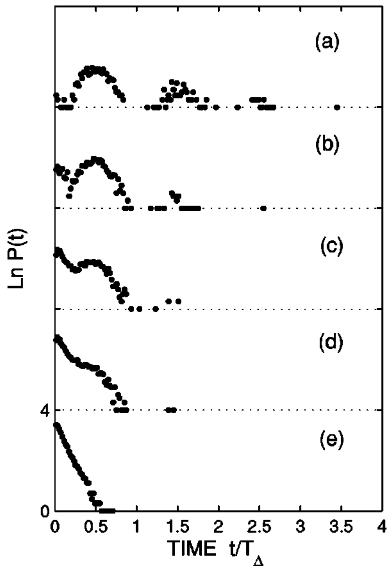

FIG. 2. The experimental time distributions between switching events shown for different values of the bifurcation parameter. $V_d = 6.9$ (a), 7.0 (b), 7.1 (c), 7.3 (d), and 7.5 (e) V. The critical value for the band-merging crisis $V_{dc} \cong 6.8$. The probability density $P(t)$ is in dimensionless form. Time is normalized to the period T_Δ. Every distribution was obtained from 200 stroboscopic laser responses (2000 points each) sampled with a driving period $T_d = 1/f_d$.

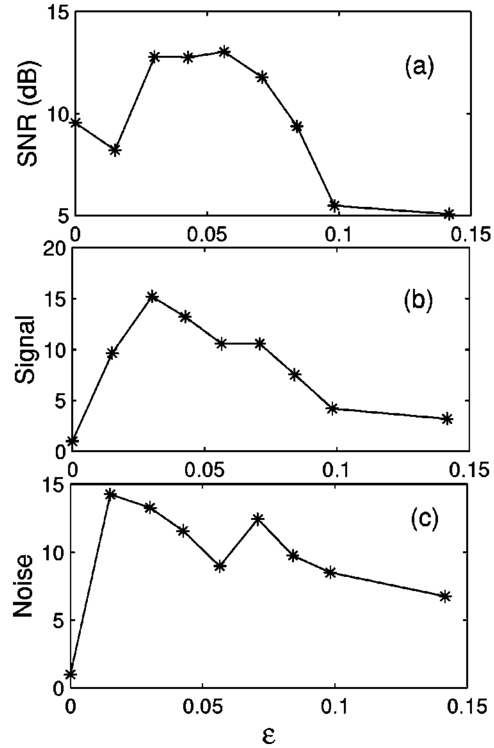

FIG. 3. (a) SNR, (b) the laser response at the signal frequency f_s normalized to the value just before crisis, and (c) the noise strength, versus ϵ, where $\epsilon = \mu/\mu_c - 1$ [μ_c is the critical value of the bifurcation for the band-merging crisis ($\delta = 10^{-2}$)]. Each point was obtained by averaging 200 data files.

displays a series of peaks well separated from the background, which are located at multiples of the half period of the detuning, i.e., $T_n = (n-1/2)T_\Delta$ [Fig. 2(a)]. Such a structure of the RTD testifies to a process of synchronization between random switching events and near-resonant signals. This synchronization takes place exactly at the detuning frequency. This means that the detuning frequency plays the same role as does a low frequency in conventional SR. It should be noted that the multipeaked structure of the RTD is a feature of both conventional [24,25] and deterministic SR [9]; here, it is due to the NRS synchronization at the detuning frequency of the random switching events between two repellers after the band-merging crisis. As the bifurcation parameter increases, which corresponds to a change of stochasticity, the RTD displays the appearance of an exponential decaying background with periodically occurring bumps on it. For larger values of the bifurcation parameter the majority of transitions occur during a much shorter time than a half period of the detuning and we observe only an exponential distribution between switching events [Fig. 2(e)]. This means that the synchronization is essentially lost.

For a quantitative characterization of SR we used such a standard measure as the signal-to-noise ratio which was obtained from the spectra of the Fourier transformed stroboscopic time series. We define here the signal-to-noise ratio as $10\log\{[I_N(f_s)+I_s(f_s)]/I_N(f_s)\}$, where $I(f)$ is the power spectrum density of the laser response, $I_N(f_s)$ is the interpolated level of the noise at the signal frequency, and $[I_N(f_s) + I_s(f_s)]$ is the superimposed power spectrum density of the signal and noise background at the same frequency. The SNR is shown in Fig. 3(a). The individual dependences of the amplitude of the signal and the noise at f_s versus the bifurcation parameter are shown in Figs. 3(b) and 3(c), respectively. The dependences of SNR (a) and the response at the signal frequency (RSF) (b) show well-pronounced maxima as the bifurcation parameter increases.

Along with the SNR, the gain factor that can be obtained in the system is one of the major characteristics of deterministic SR, in the context of practical use of chaos as an amplifier of periodic signals. As mentioned above, dynamical systems operating near the first period-doubling bifurcation are very sensitive to the effect of NRS's. This means that the gain factor near this bifurcation point can be taken as a reference for comparison purposes. In Fig. 4(a) the gain factor

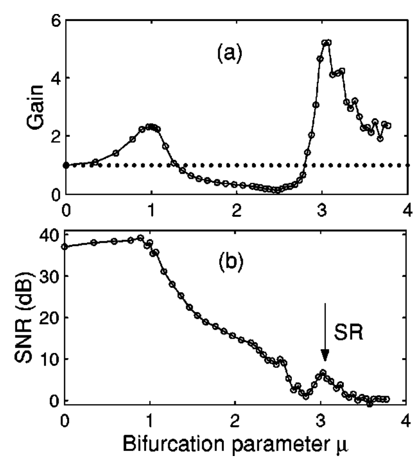

FIG. 4. (a) The gain factor G and (b) signal-to-noise ratio versus the bifurcation parameter μ ($\delta = 5\times10^{-2}$). Each point was obtained by averaging 30 data files.

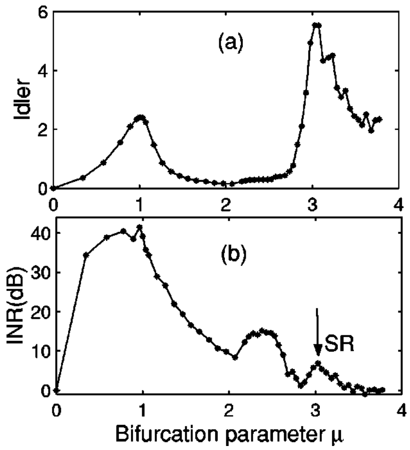

FIG. 5. (a) Response at the idle frequency $f_i = f_d/2 - \Delta$, normalized by I_0, and (b) idler-to-noise ratio (INR) versus the bifurcation parameter μ ($\Delta = 2500$ kHz, $\delta = 0.05$).

G is shown as a function of the bifurcation parameter in a wide range of μ. We define the gain factor as $G = I_s(f_s)/I_0$, where $I_s(f_s)$ and I_0 are the laser responses at the signal frequency in the presence and absence of the driving modulation, respectively. Simultaneously, the SNR [Fig. 4(b)] versus the bifurcation parameter is shown for the same range of μ. It is seen that maxima of the amplification and SNR take place in the range of the bifurcation parameter $\mu \approx 3$ shown in Fig. 4(b) by an arrow. One can see from Fig. 4(a) that the amplification of NRS's in chaos more than twice exceeds the amplification near the first period-doubling bifurcation. This

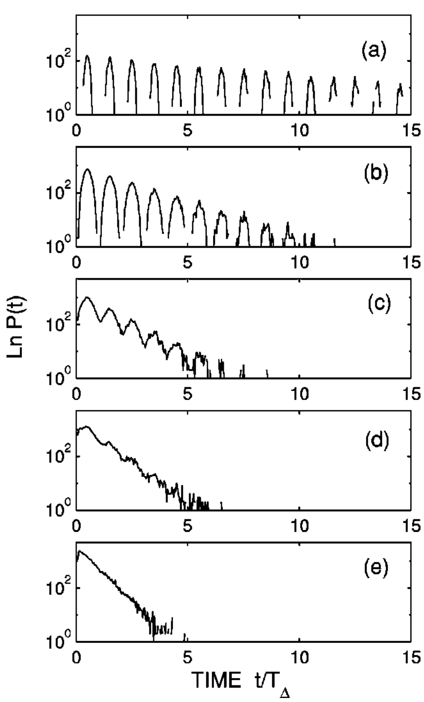

FIG. 6. The numerical time distributions between switching events shown for different values of the bifurcation parameter μ. $\mu = 3.672$ (a), 3.6809 (b), 3.6879 (c), 3.6933 (d), and 3.7021 (e). The probability density is in dimensionless form. Time is normalized to the period T_Δ. Each distribution was obtained from 100 stroboscopic laser responses (10^5 points in each) sampled with a driving period $T_d = 1/f_d$.

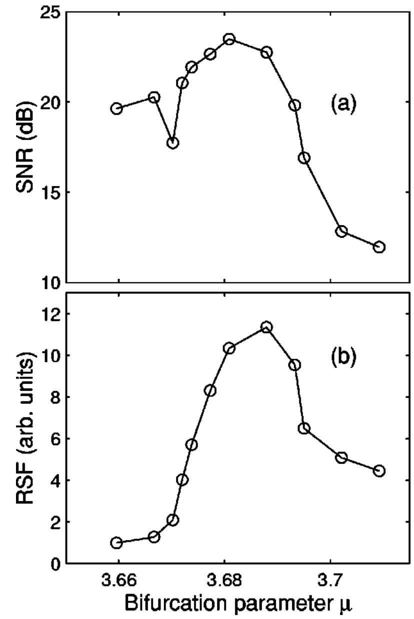

FIG. 7. Numerical (a) SNR and (b) RSF normalized to the value just before crisis versus μ ($\delta = 10^{-2}$).

is an important experimental result of this paper, which demonstrates that nonlinear systems operating in the region of crisis-induced intermittency can serve as amplifiers of NRS's with a gain factor even higher than near the first period-doubling bifurcation point. Simultaneously, SR at the idler frequency is also observed (Fig. 5). An additional maximum in the dependence of the idler-to-noise ratio on μ in Fig. 5(b) in the region of the bifurcation parameter between $\mu \approx 2$ and the critical value for the band-merging crisis reflects a transfer of power from the laser response at the signal frequency to the laser response at the idler frequency.

It should be noted that in these experiments the detuning frequency was large ($\delta = 5 \times 10^{-2}$). Because of this, the gain factor near the first period-doubling threshold is not so large as, for instance, in [7] where the detuning frequency δ was equal to 10^{-2} and the corresponding gain factor G was equal to about 12. Generally, a large amplification factor is achieved when the detuning frequency is small [1]. We shall show numerically that a similar effect of the detuning frequency δ is also obtained in the case of the amplification of NRS's in the region of crisis-induced intermittency.

III. NUMERICAL SIMULATION

In numerical simulations we used the following two-level rate-equation laser model [23]:

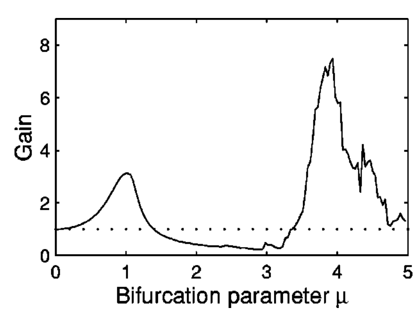

FIG. 8. The numerical gain factor G versus the bifurcation parameter μ ($\delta = 5 \times 10^{-2}$).

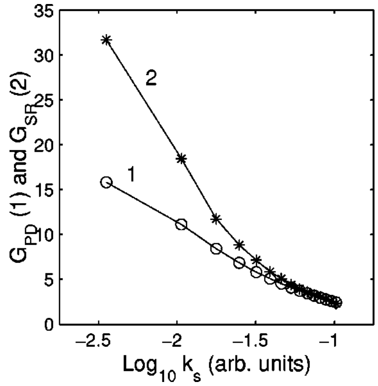

FIG. 9. The numerical gain factors G_{PD} (1) and G_{SR} (2) versus the signal amplitude k_s.

$$\frac{du}{dt}=\tau^{-1}(y-k)u,$$

$$\frac{dy}{dt}=(y_0-y)\gamma-uy, \tag{1}$$

where

$$k=k_0+k_d\cos(2\pi f_d t)+k_s\cos(2\pi f_s t). \tag{2}$$

Here u is proportional to the radiation density, y and y_0 are the gain and the unsaturated gain in the active medium, respectively, τ is half the round-trip time of light in the resonator, γ is the gain decay rate, k is the total cavity loss, k_0 is the constant part of the loss, k_d is the driving amplitude, k_s is the signal amplitude, and $f_s=f_d/2+\Delta$, where Δ is the detuning ($\delta=2\Delta/f_d$). The following fixed parameters were used throughout our calculations: $\tau=3.5\times10^{-9}$ s, $\gamma=1.978\times10^{5}$ s^{-1}, $y_0=0.175$, $k_0=0{,}173\,03$, $f_d=100$ kHz. The parameters k_d, k_s, and Δ were varied in our numerical simulations.

In Fig. 6 we present the residence-time distributions between switching events obtained from Eqs. (1) by numerical simulation for different values of the bifurcation parameter μ (as in the experiment we used the normalized bifurcation parameter μ defined as $\mu=k_d/k_{1/2}$, where $k_{1/2}=2.4675\times10^{-3}$ is the first period-doubling threshold). In order to obtain these distributions we used the same procedure described above for the experimental data. One can see in the RTD a strong input-output synchronization for values of μ slightly above the onset of the crisis. As in the experiment, a

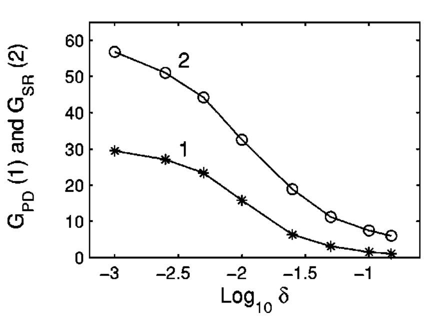

FIG. 10. The numerical gain factors G_{PD} (1) and G_{SR} (2) versus δ ($k_s=8.75\times10^{-6}$).

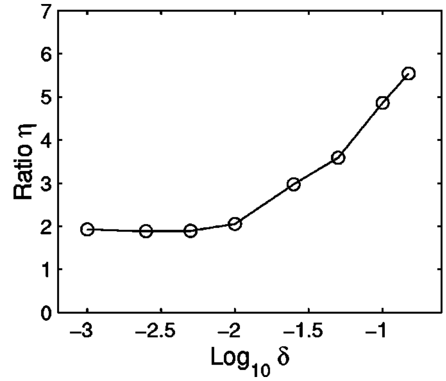

FIG. 11. The ratio η versus δ ($k_s=8.75\times10^{-6}$).

multiple-peaked structure of the RTD is seen exactly at multiples of the half period of the detuning, i.e., at $T_n=(n-1/2)T_\Delta$. The heights of the peaks decay exponentially, which is a common feature of systems displaying SR. Further increase of the bifurcation parameter μ leads to the appearance of an exponentially decaying background in the RTD with small peaks on it [Fig. 6(c)]. For large values of μ this synchronization is practically lost. Comparing Fig. 2 and Fig. 6 one can see a good agreement between experimental and numerical results. In Figs. 7(a) and 7(b) we show the numerical dependences of the SNR and RSF, respectively, on the bifurcation parameter μ. Both dependences pass through a maximum with increasing μ, which is a signature of deterministic SR. The SNR and RSF were defined and found in the same manner as for the experiment.

As we noted above, the gain factor that can be obtained in chaos is a very important characteristic of deterministic SR from the standpoint of possible practical applications. In the rest of the paper we shall analyze how the gain factor depends on the signal amplitude and the detuning frequency of the NRS from the viewpoint of obtaining its maximal value. In Fig. 8 the numerical dependence of the gain factor G_{SR} for NRS's on μ is shown in a wide range of the bifurcation parameter. It is clearly seen that the gain factor in chaos is approximately two times higher than near the first period doubling. This is in good agreement with the experimental results shown in Fig. 4(a). Generally speaking the gain factor G_{SR} in chaos substantially depends on the amplitude of the NRS (Fig. 9, curve 2). For the purpose of comparison we show also the gain factor G_{PD} near the first period doubling (Fig. 9, curve 1). In the range of the signal amplitude k_s (k_s is normalized to the value of $k_{1/2}$) going from 0.01 up to 0.1, the gain factor in both cases obeys the following scaling law but with different exponents: $G_{\mathrm{PD}},G_{\mathrm{SR}}\propto[I(f_s)]^{-\alpha}$. For the case of chaos $\alpha\cong0.9$ and near the first period doubling $\alpha\cong2/3$ [30].

In Fig. 10 we show the numerical dependences of G_{SR} and G_{PD} on δ. It is seen that a large amplification factor G_{ch} is achieved for small detuning. The ratio η, defined as $\eta=G_{\mathrm{SR}}/G_{\mathrm{PD}}$, versus δ is shown in Fig. 11. One can see that for small detuning δ (in the range from 10^{-3} up to 10^{-2}) the ratio $\eta\cong2$ and it is almost constant. For large values of δ, the ratio η rapidly increases with increasing δ. For example, for $\delta=0.1$ the gain factor G_{SR} exceeds G_{PD} by 5 times. This means that chaotic systems operating in the region of crisis-induced intermittency are much more sensitive to near-resonant signals than near the first period-doubling bifurcation point in a very wide range of the detuning frequencies.

IV. CONCLUSIONS

In summary, we have reported on the experimental observation of *deterministic* stochastic resonance with a *near-resonant* signal in a chaotic CO_2 laser operating in the region of crisis-induced intermittency resulting from band-merging crisis. All distinguishing features of SR with NRS's such as the RTD, SNR, and gain factor appear to be similar to those of conventional SR with a low-frequency signal. While the underlying mechanism of SR is a synchronization between NRS's at the detuning frequency and two-state random switching events, SR takes place at the frequency of the NRS as well as at the idler frequency. It is particularly remarkable that in this case chaos can serve as an amplifier with a gain factor more than two times (several times, as shown numerically) larger than near the first period-doubling threshold. The numerical simulation performed confirms all our experimental observations and predicts some additional features of deterministic SR with NRS's, such as a very high sensitivity of chaos to a NRS with a large detuning frequency.

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