Detecting unstable periodic orbits from transient chaotic time series

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We address the detection of unstable periodic orbits from experimentally measured *transient* chaotic time series. In particular, we examine recurrence times of trajectories in the vector space reconstructed from an ensemble of such time series. Numerical experiments demonstrate that this strategy can yield periodic orbits of low periods even when noise is present. We analyze the probability of finding periodic orbits from transient chaotic time series and derive a scaling law for this probability. The scaling law implies that unstable periodic orbits of high periods are practically undetectable from transient chaos.

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I. INTRODUCTION

In many biological and physical experiments, the measured signals often exhibit irregular behavior during an initial interval before finally settling into an asymptotic state that is nonchaotic. The traditional wisdom may simply be to disregard the transient portion of the data and to concentrate on the final state. By doing this, however, information about the system may be lost because the irregular part of the data may contain important hints about the system dynamics. This is particularly true when the underlying dynamics is deterministic and exhibits transient chaos [1–3]. While there has been a tremendous amount of work on analyzing time series from chaotic attractors [4], to our knowledge the problem of analyzing *transient* chaotic time series has not been addressed [5]. The purpose of this paper is to address one aspect of this important problem: detecting unstable periodic orbits embedded in the underlying chaotic invariant set that is responsible for transient chaos.

It has been known that *nonattracting chaotic saddles* are the dynamical invariant sets that give rise to transient chaos [1–3]. Because such a saddle is chaotic but nonattracting, a trajectory starting from a random initial condition in a phase-space region containing the saddle typically stays near the saddle for a time (exhibiting chaotic behavior), then exits the region and asymptotes to a final state. Physically, chaotic saddles lead to observable phenomena such as chaotic scattering [6], fractal basin boundaries [7], fractal concentrations of passive particle advected in open hydrodynamical flows [8], and fractal distribution of chemicals in environmental flows [9]. Mathematically, chaotic saddles are closed, bounded, and invariant sets with dense orbits. Like chaotic attractors, a chaotic saddle has embedded within an infinite number of unstable periodic orbits that constitute its ''skeleton'' [10]. Thus, successful detection of unstable periodic orbits from transient chaotic time series means that (1) the underlying dynamics is not stochastic but deterministic, and (2) dynamical invariants of the chaotic saddle may be estimated because unstable periodic orbits can be related to the natural measure of the chaotic saddle [11].

The primary difficulty of dealing with a transient chaotic system is that the chaotic phases, which contain the essential information about the chaotic saddle, are usually short. It is usually difficult to obtain long chaotic time series from measurements. Methods that rely on long time series and are proven to be effective for detecting periodic orbits in chaotic attractors [12,13] are applicable to transient chaos only if sufficiently long time series can be constructed from a large number of short, transient chaotic time series. In this paper we are not interested in constructing long time series. Instead, we work with short time series directly and we assume that an *ensemble* of such time series can be obtained from measurements. A question is, can unstable periodic orbits be detected from an ensemble of transient chaotic time series? In this paper we shall demonstrate that the answer to the above question is affirmative. Specifically, we find that the method originally developed by Lathrop and Kostelich (LK) [12] can be adapted to detect periodic orbits from transient chaotic time series, and we demonstrate that the method is quite robust for the detection of periodic orbits of low periods. For periodic orbits of high periods, we provide a theoretical argument suggesting that they are practically undetectable from transient chaos. In particular, we find that the probability of detecting unstable periodic orbits decreases *exponentially* with the period, and we obtain an explicit expression for the exponential decay exponent in terms of the dynamical invariants associated with the chaotic saddle. The theory is verified by numerical examples.

The rest of the paper is organized as follows. In Sec. II we describe how the LK algorithm can be used to extract recurrent orbits from transient chaotic time series. In Sec. III we discuss the effect of noise in the detection of periodic orbits by using the LK algorithm. In Sec. IV we analyze the probability of finding periodic orbits from transient chaotic time series and we derive a scaling law for this probability. In Sec. V we present conclusions.

II. LK ALGORITHM FOR DETECTING UNSTABLE PERIODIC ORBITS FROM TRANSIENT CHAOTIC TIME SERIES

The LK algorithm [12] to extract unstable periodic orbits from experimental chaotic time series is based on identifying sets of recurrent points in the reconstructed phase space. To do this, one first reconstructs a phase-space trajectory $\mathbf{x}(t)$ from a measured scalar time series $\{s(t)\}$ by using the delay-coordinate embedding method [14]: $\mathbf{x}(t)=(s(t),s(t+\tau),\ldots,s(t+(d-1)\tau)$, where d is the embedding dimension and τ is the delay time. To identify unstable periodic orbits, one follows the images of $\mathbf{x}(t)$ under the dynamics until a value $t_1>t$ is found such that $\|\mathbf{x}(t_1)-\mathbf{x}(t)\|<\epsilon$, where ϵ is a prespecified small number that defines the size of the recurrent neighborhood at $\mathbf{x}(t)$. In this case, $\mathbf{x}(t)$ is called an (m,ϵ) recurrent point, and $m=t_1-t$ is the *recurrence time*. A recurrent point is not necessarily a component of a periodic orbit of period m. However, if a particular recurrence time m appears frequently in the reconstructed phase space, it is likely that the corresponding recurrent points are close to periodic orbits of period m. The idea is then to construct a histogram of the recurrence times and identify peaks in the histogram. Points that occur frequently with are taken to be, approximately, components of the periodic orbits. The LK algorithm successfully detected unstable periodic orbits such as those from measurements of chaotic chemical reactions [12].

While the original algorithm was developed for chaotic attractors from which long time series can be obtained [12], it can be adapted to detect unstable periodic orbits from transient chaotic time series as well. The reason lies in the statistical nature of this method, as a histogram of recurrence times can be obtained even with short time series. Provided that there is a large number of such time series so that good statistics of the recurrence times can be obtained, unstable periodic orbits embedded in the underlying chaotic saddle can be identified. It is not necessary to concatenate many short time series to form a single long one (such concatenations are invariably problematic [5]). Intuitively, since the time series are short, we expect to be able to detect at least periodic orbits of short periods (the issue of long periods will be addressed later).

We have implemented and tested the LK algorithm for detecting unstable periodic orbits from various model chaotic systems. Here we report numerical results with the following Rössler system [15]: $dx/dt=-y-z, dy/dt=x+ay, dz/dt=b+(x-c)z$, where a,b, and c are parameters. There is transient chaos when the set of parameter values yields a periodic window in which a stable periodic attractor and a chaotic saddle coexist. For instance, for $a=b=0.2$ and $c=5.3$, the system falls in a periodic window of period 3. A typical measurement of a dynamical variable, say $x(t)$, exhibits chaotic behavior for a finite amount of time before settling in the period-3 attractor. We generate ten such time series by integrating the Rössler system from ten different initial conditions, and record the x coordinate for $0\leqslant t\leqslant 40$, the approximate lifetime of the transients. These time series are assumed to be the only available data about the system. For each time series, a seven-dimensional vector space is reconstructed by using the delay time $\tau=0.2$. To obtain recurrence times, it is necessary to determine ϵ, the size of the recurrent neighborhood. The value of ϵ must not be so large that many ''false positives'' are reported, but ϵ must not be so small that genuine recurrences are missed. Typically, we find in numerical experiments that the number of recurrences $N(\epsilon)$ usually increases with the length and the number of the invidual transient trajectories, and with ϵ. It tends to saturate when ϵ is too large. The value of ϵ at which $N(\epsilon)$ saturates is taken to be an appropriate size of the recurrent neighborhood. For the Rössler system, we use $\epsilon=2\%$ of the root-mean-square (rms) value of the chaotic signal. Figure 1(a) shows the histogram of the recurrence times for the ten transient chaotic time series from the period-3 window. Figures 1(b)–1(d) show, in the plane of $x(t)$ versus $x(t+\tau)$, three recurrent orbits. The orbit in Fig. 1(b) has the shortest recurrence time, so we call it a ''period-1'' orbit. Figures 1(c) and 1(d) show a period-3 and a period-8 orbit. The orbits were selected from the set of recurrent points comprising the corresponding peak in the histogram. In general, we find that the LK algorithm is capable of yielding many periodic orbits of low periods (say, a period less than 10).

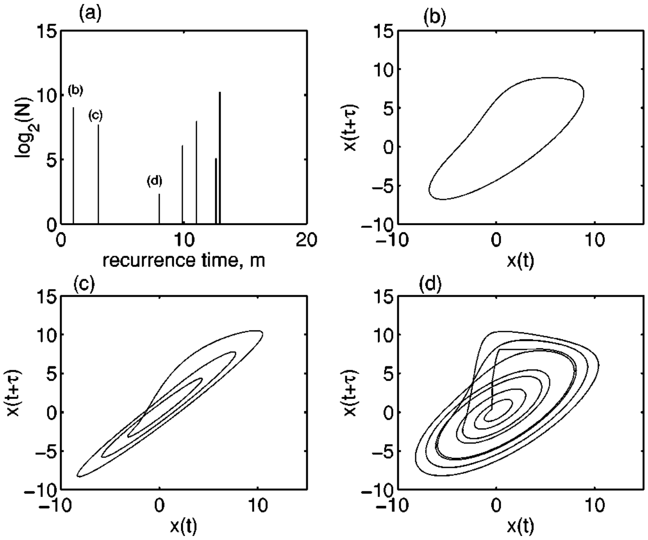

FIG. 1. For the Rössler system: (a) histogram of the recurrence time m (normalized with the recurrence time of the first peak), (b)–(d) a period-1, a period-3, and a period-8 recurrent orbit extracted from the histogram in (a), respectively.

III. EFFECT OF NOISE

In an experimental setting, time series are usually contaminated by dynamical and/or observational noise. A question is whether periodic orbits can still be extracted from noisy transient chaotic time series. Qualitatively, under the influence of noise, the effective volume of recurrent region in the phase space decreases and, hence, we expect to see a decrease in the number of recurrences. Figures 2(a)–2(d) show the number of recurrent points (a) and three periodic orbits extracted from ten transient chaotic time series with the additive noise of the form $G(0,0.01)$, where $G(0,0.01)$ is the normal (Gaussian) distribution centered at 0 with variance 0.01. This noise level represents a rms value that is approximately 0.5% of that of the chaotic signal. We see that at this low noise level, periodic orbits can still be reliably detected. We find, however, that for the Rössler system at $\epsilon=2\%$ of the rms value of the chaotic signal with rms value of the noise beyond 1% of rms value of the chaotic signal, no periodic orbits can be extracted from the histogram of recurrences. To be systematic, we compute, at several fixed values of ϵ, how the number of recurrent points decreases as the

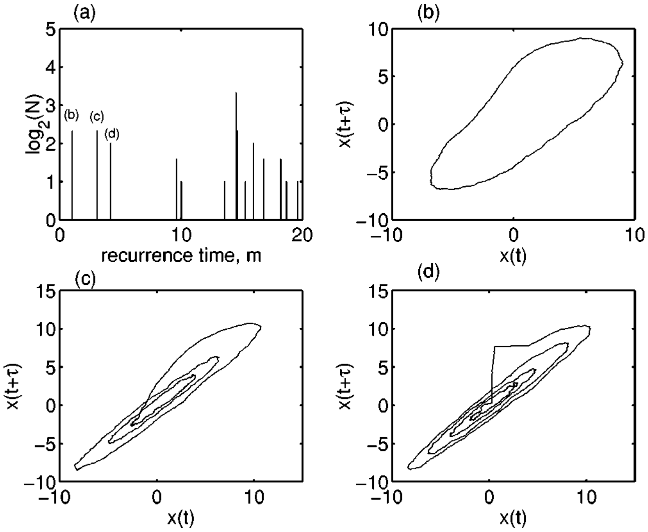

FIG. 2. For a noisy Rössler system: (a) histogram of the recurrence time m (normalized with the recurrence time of the first peak), (b)–(d) a period-1, a period-2, and a period-4 recurrent orbit extracted from the histogram in (a), respectively, where $\epsilon = 6\%$ and the rms value of the noise is at about 0.5% of that of the chaotic signal.

noise amplitude (η) is increased. Figures 3(a) and 3(b) show the result of such computations for $\epsilon = 2\%$ (a) and $\epsilon = 6\%$ (b) of the rms value of the signal. We see that the number of recurrent points goes to zero at $\eta \approx \epsilon/2$, which can be understood as follows. Under the noise of amplitude η, both the center and the boundary of the recurrent region are uncertain within η. Thus, the effective phase-space volume in d dimensions in which two points can still be considered within distance ϵ (recurrent) is proportional to $(\epsilon - \eta)^d - \eta^d$, which vanishes at $\eta = \epsilon/2$. Since ϵ should be small to guarantee

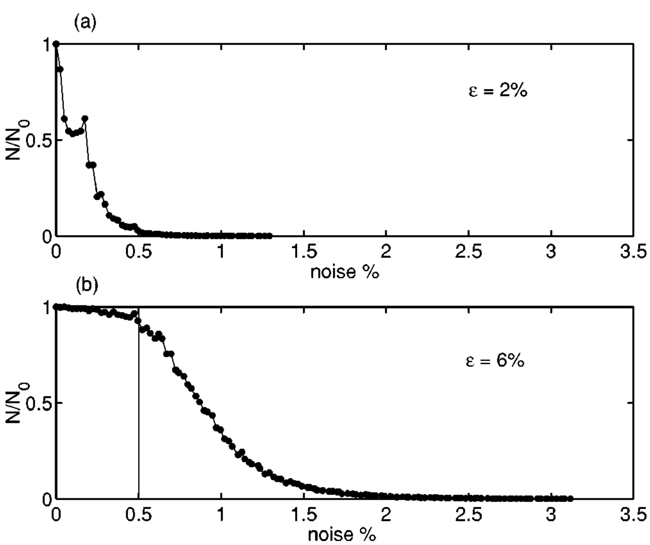

FIG. 3. For the noisy Rössler system, the relative number (N/N_0) of recurrent points versus the amplitude of noise for two values of the size of the recurrent neighborhood: (a) $\epsilon = 2\%$ and (b) $\epsilon = 6\%$ of the rms value of the signal, where N_0 is the number of recurrent points at zero amplitude of noise. The vertical line in (b) denotes the noise level at which periodic orbits in Fig. 2 are extracted.

recurrence, we see that the noise level that can be tolerated is also small.

IV. PROBABILITY OF DETECTING PERIODIC ORBITS FROM TRANSIENT CHAOTIC TIME SERIES

We now consider the probability of detecting periodic orbits from transient chaotic time series [16]. This is particularly relevant for transient chaos because trajectories on a chaotic saddle have an average lifetime time τ staying near the saddle and, hence, it is difficult for a typical trajectory to contain periodic orbits of period larger than, say, τ. Effort may then be devoted to connect short time series so that the resulting long time series would contain periodic orbits of larger period [5]. Such a task may be difficult. If one fails to detect periodic orbits of high periods, the question is whether one should attempt to increase the number of measurements so that more time series are available. Or, one may attempt to improve techniques to link these time series, a computationally demanding task because it is essentially a problem of optimizing many time series and the computation required in any optimization problem typically increases dramatically as the number of elements involved is increased. Our main point here is that in detecting unstable periodic orbits from transient chaos, the probability of detecting orbits of higher periods is typically exponentially small. This is an intrinsic dynamical property of the underlying chaotic saddle and, hence, increasing the number of measurements or improving techniques of detection will not help to enhance the chance to detect these orbits.

We derive a scaling relation for $\Phi(p)$, the probability to detect any period-p orbit. Note that $\Phi(p)$ is actually the probability for a trajectory to stay in a small neighborhood of any periodic orbit of period p. For a trajectory to stay in a ν-neighborhood of all p points of the ith orbit of period p, the trajectory must come within $\delta = \nu e^{-\lambda_i(p)p}$ of any of the p points when it first encounters with the periodic orbit, where $\lambda_i(p) > 0$ is the Lyapunov exponent of this orbit. The probability for this event is $\phi_i(p) \sim \delta^{D_i}$, where D_i is the pointwise dimension of any one of the p points of the this periodic orbit. The exponential factor $e^{-\lambda_i(p)p}$ is proportional to the natural measure associated with this periodic orbit [11]. The probability $\Phi(p)$ is the accumulative probability of all $\phi_i(p)$:

$$\Phi(p) = \sum_{i=1}^{K(p)} \phi_i(p) \sim \sum_{i=1}^{K(p)} \nu^{D_i} \exp[-\lambda_i(p) D_i p], \qquad (1)$$

where $K(p)$ is the total number of periodic points of period p. Since $\lambda_i(p)$ and D_i are the local positive Lyapunov exponent and pointwise dimension of periodic orbits of period p, for large p we expect them to obey distributions centered at λ_1 and D_1, respectively, where λ_1 and D_1 are the positive Lyapunov exponent and the information dimension of the chaotic saddle. Thus, the main dependence of $\Phi(p)$ on p is

$$\Phi(p) \sim e^{-\lambda_1 D_1 p} K(p) \sim e^{(-\lambda_1 D_1 + h_T)p} = e^{-\gamma p}, \qquad (2)$$

where γ is the exponential scaling exponent and h_T is the topological entropy. Using the Kaplan-Yorke formula for chaotic saddles [17] to express D_1 in terms of the Lyapunov

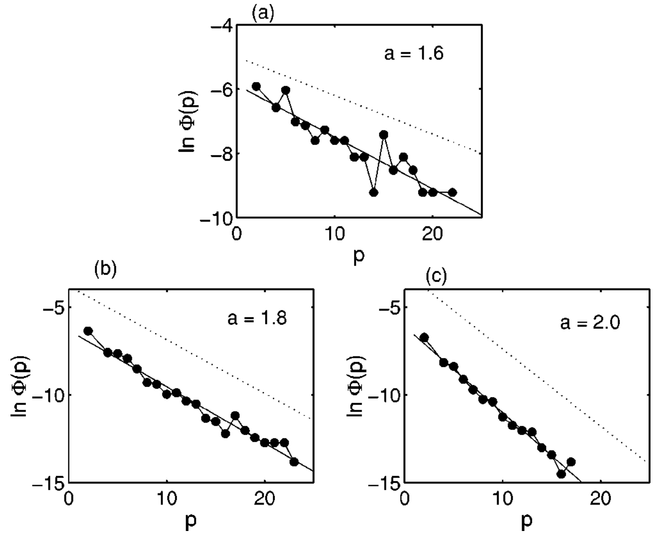

FIG. 4. (a)–(c) For the Hénon map at $a=1.6,1.8$, and 2.0, $\ln\Phi(p)$ versus p. The dotted lines indicate the theoretically predicted slopes of $\ln\Phi(p)$ versus p.

exponents $\lambda_2<0<\lambda_1$ and the lifetime τ: $D_1=(\lambda_1-1/\tau)(1/\lambda_1-1/\lambda_2)$, or since $\lambda_2<0, D_1=(\lambda_1-1/\tau)(1/\lambda_1+1/|\lambda_2|)$, we obtain the following scaling exponent:

$$\gamma=\lambda_1-h_T+\frac{\lambda_1^2}{|\lambda_2|}-\frac{1}{\tau}\left(1+\frac{\lambda_1}{|\lambda_2|}\right). \tag{3}$$

The scaling relation Eq. (2), together with the scaling exponent in Eq. (3), is applicable to chaotic saddles in two-dimensional invertible maps or in three-dimensional flows. Note that for chaotic attractors ($\tau\to\infty$), we have $\gamma\approx\lambda_1-h_T+\lambda_1^2/|\lambda_2|$.

To test Eqs. (2) and (3) numerically, we use chaotic saddles in the Hénon map [18]: $(x,y)\to(a-x^2+0.3y,x)$ for which unstable periodic orbits can be computed systematically [19]. We choose the following set of three parameter values for which there is transient chaos [20]: $a=1.6,1.8$, and 2.0. For each value of a, we choose 10^6 initial conditions in the region $[-2,2]\times[-2,2]$ containing the chaotic saddle, which yield 10^6 transient time series. For a given period p, we then compute the fractions of times that these 10^6 time series get close to every periodic orbit of period p. These fractions are then accumulated to yield the probability $\Phi(p)$. From our numerical experiments, we see that this probability usually increases with the number of transient time series and also with the length of the individual trajectories. Figures 4(a)–4(c) show $\ln\Phi(p)$ versus p for $a=1.6,1.8$, and 2.0, respectively. These plots indicate behavior of exponential decay, and the decay exponents are given by the slopes of the plots. To compute the theoretical scaling exponents in Eq. (3), it is necessary to compute the Lyapunov exponents, the topological entropy, and the lifetime of the chaotic saddles. The following techniques are used in our computation: (1) we use the PIM-triple algorithm [21,22] to compute the Lyapunov exponents; (2) we use the method in Ref. [23] to compute the topological entropy; and (3) we use the sprinkler method to compute τ [17]. The slopes of the dashed straight lines in Figs. 4(a)–4(d) are the theoretical slopes for the corresponding chaotic saddles. We see that the numerical slopes agree reasonably well with the theoretical ones, as shown further in Table I, where the numerical and theoretical slopes, together with the values of other quantities involved in Eq. (3), are listed.

TABLE I. Theoretical and numerical values of the scaling exponent γ at three different parameters for the Hénon map.

a	1.60	1.8	2.0
λ_1	0.58	0.81	0.87
λ_2	-1.78	-2.01	-2.07
h_T	0.53	0.54	0.53
τ	11.2	4.7	5.4
γ (theoretical)	0.12	0.31	0.44
γ (numerical)	0.13 ± 0.04	0.32 ± 0.03	0.47 ± 0.04

V. CONCLUSION

In this work we demonstrate that unstable periodic orbits of low periods can be detected reliably from an ensemble of transient chaotic time series by using the LK algorithm. Our numerical analysis indicates that the LK algorithm is powerful for extracting unstable periodic orbits at low noise level. We further give a theoretical justification for the difficulty of detecting periodic orbits of high periods from transient chaos. The theoretical scaling law is verified by numerical examples. Since the probability of detecting these orbits is exponentially small, as a matter of practicality it is perhaps worthless to obtain long time series or to improve technique to detect periodic orbits from transient chaos. We remark that although there has been a tremendous amount of work on analyzing time series from chaotic attractors, the analysis of transient chaotic time series remains a far less explored area.

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