PHYSICAL REVIEW E VOLUME 61, NUMBER 6 JUNE 2000

Complete reduction of oscillators in resonance $p:q$

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(Received 25 October 1999)

This paper extends to any type of resonance $(p:q)$ the Lissajous transformation that handles the resonance (1:1) in a Hamiltonian composed of two harmonic oscillators. The manifolds of constant energy for such a system are two-dimensional surfaces of revolution that are spheres for the resonance 1:1, spheres pinched once for the resonances $(1:q)$ when $1<q$, and spheres pinched twice for the resonances $(p:q)$ when $1<p<q$. The extended Lissajous transformation is valid for resonant pseudo-oscillators (a nondefinite quadratic form), which allows us to find that the reduced phase flow lies on an unbounded surface of revolution.

PACS number(s): 05.45.−a, 45.05.+x, 33.40.+f, 95.10.Ce

I. INTRODUCTION

Under consideration in this paper are Hamiltonians of the type

$$\mathcal{H}(X,Y,x,y)=\mathcal{H}_0(X,Y,x,y)+\varepsilon\,\mathcal{P}(X,Y,x,y,\varepsilon),$$

whose principal part

$$\mathcal{H}_0=\frac{1}{2}(X^2+\omega_1^2x^2)+\frac{1}{2}(Y^2+\omega_2^2y^2), \tag{1}$$

consists of two harmonic oscillators, while $\mathcal{P}$ is a perturbation proportional to a small parameter ε. Hamiltonians of this kind appear often in nonlinear dynamics. Indeed, they are among the most studied in molecular spectroscopy, galactic dynamics, and celestial mechanics (see, e.g., [1] and references therein).

One typical example of Hamiltonians of the kind given by Eq. (1) is the resonance originally described by Fermi [2] in the molecule of CO_2, this type of resonance has been observed to be important in experimental studies of photoisomerization, excited ions, tunneling effect, as well as other experimental and theoretical works (see, e.g., [3,4]). In galactic dynamics, this type of perturbed Hamiltonians have been used for describing the motion of a star under the gravity field of a galaxy [5–9]. In most of the cases, the oscillators are in resonance 1:1, like the famous Hénon-Heiles Hamiltonian [10], but dealing with less symmetric galaxies, like elliptical or barred galaxies, we meet resonances of the type $p:q$ [11,12].

Such systems have been profusely studied because the dynamical systems they represent show chaos above a certain threshold of energy. As it has been indicated [13,1], incipient chaos creeps in at low energies depending on the existence of unstable equilibria and the homoclinic solutions emanating from them. On the one hand, numerical integration is not fit to reveal the fine details of the phase space near those singularities; analytical studies are preferable. On the other hand, the usual techniques of normalization fail in the presence of resonances due to the resulting small divisors [14], and the resonant and near resonant cases must be treated in an specific way [5–7,15–19].

These systems are called *semisimple* [20] because their dominant term leads to a linear Hamiltonian vector field that is semisimple. The concept of normalization for semisimple systems in equilibrium at the origin must be credited to Whittaker [21–23] who applied Poincaré's *nouvelle méthode* [24]; Birkhoff [25] provides a different method for the normalization (for an automated development of the generating function, see, e.g., [15]). The introduction of the Lie transformation in the 1960s [26,27] made easier the automatization of the normalization, and a good example of it is given by Giorgilli [16], where the normalization is carried out for several resonances and several degrees of freedom. Albeit the problem may be considered as a classical one, it still attracts the attention of many authors, see, e.g., [28–38]

When the normalization is carried out by a Lie method, especially when high orders are required, one must be aware of the fact that the simpler is the Lie derivative associated with the unperturbed Hamiltonian, the easier is the application of the method. With this aim, and for the resonance (1:1), Deprit defined the Lissajous transformation [39]. Indeed, in the Lissajous variables, the Hamiltonian is simply one of the conjugate moments (L), which simplifies drastically the computation of the normalized Hamiltonian. Besides, for the resonance (1:1) Deprit [39] proved that the most salient feature of the reduced phase space is that each manifold of $L=$ constant is a two-dimensional sphere, and each point of this sphere represents an orbit. This fact is very important when one is interested in finding the global picture of the phase portrait; indeed, the classical Mercator map presents polelike singularities, and quite often [13,40,32,41,42] bifurcations occur precisely at these points.

Bearing in mind the advantages of the Lissajous transformation we proposed to extend this transformation. We faced the problem head on the way it has been met for the resonance (1:1) with the Lissajous transformation [39].

Recently, Elipe and Deprit [43] obtained an extension of the Lissajous transformation to any finite combination of harmonic oscillators and this transformation is valid for any resonance mode. Indeed, for the Hamiltonian of the type

$$\mathcal{H}_0=\frac{1}{2}\sum_{1\leqslant i\leqslant n}(X_i^2+\omega^2p_i^2x_i^2),\quad \text{with } p_i\in\mathbf{R}, \tag{2}$$

the canonical transformation

$$\lambda:(\boldsymbol{\psi},\boldsymbol{\Psi})\mapsto(\boldsymbol{x},\boldsymbol{X}):T^n\times(\mathbf{D}\subset\mathbf{R}^n)\mapsto\mathbf{R}^{2n}$$

given by

$$x_1=\frac{1}{\omega^{1/2}p_1}[(n-2)\Psi_1+\Sigma]^{1/2}\sin p_1\sigma,$$

$$X_1=\omega^{1/2}[(n-2)\Psi_1+\Sigma]^{1/2}\cos p_1\sigma,$$

$$\text{and for } 1<j\leqslant n, \tag{3}$$

$$x_j=\frac{1}{\omega^{1/2}p_j}(\Psi_1-\Psi_j)^{1/2}\sin p_j(-2\psi_j+\sigma),$$

$$X_j=\omega^{1/2}(\Psi_1-\Psi_j)^{1/2}\cos p_j(-2\psi_j+\sigma),$$

where we used the shorthand $\Sigma=\Sigma_{1\leqslant i\leqslant n}\Psi_i$ and $\sigma=\Sigma_{1\leqslant i\leqslant n}\psi_i$, reduces the Hamiltonian (2) to the function

$$\lambda^{\#}\mathcal{H}_0=\omega\Psi_1.$$

Details about how this transformation has been obtained will appear elsewhere [43,44]. It was proved, too, that the transformation is valid for linear combinations of oscillators and, with slight modifications, for diffusers.

In this paper we deal exclusively with the resonance $p:q$. Let us assume that the Hamiltonian (1) corresponds to two oscillators in resonance $p:q$, that is to say, there exist two coprime integers, p,q and a frequency ω such that $\omega_1=p\omega$ and $\omega_2=q\omega$. Without loss of generality, we can assume that $1\leqslant p\leqslant q$.

Let us consider the convex set $\Gamma=\{(\Psi_1,\Psi_2)\in\mathbf{R}^2:\Psi_1>0,|\Psi_2|<\Psi_1)\}$, then a torus $T^2=\{(\psi_1,\psi_2)\in\mathbf{R}^2:\psi_1\mathrm{mod}(2\pi),\psi_2\mathrm{mod}(2\pi)\}$. For this two degrees of freedom problem, the generalized Lissajous transformation (3)

$$f:(\psi_1,\psi_2,\Psi_1,\Psi_2)\mapsto(x,y,X,Y):T^2\times\Gamma\mapsto\mathbf{R}^4$$

is defined by

$$x=\sqrt{\frac{\Psi_1+\Psi_2}{\omega_1 p}}\sin p(\psi_1+\psi_2)=s/p\sin p(\psi_1+\psi_2),$$

$$\tag{4}$$

$$y=\sqrt{\frac{\Psi_1-\Psi_2}{\omega_2 q}}\sin q(\psi_1-\psi_2)=d/q\sin q(\psi_1-\psi_2),$$

$$X=\sqrt{\frac{\omega_1(\Psi_1+\Psi_2)}{p}}\cos p(\psi_1+\psi_2)=\omega s\cos p(\psi_1+\psi_2),$$

$$Y=\sqrt{\frac{\omega_2(\Psi_1-\Psi_2)}{q}}\cos q(\psi_1-\psi_2)=\omega d\cos q(\psi_1-\psi_2),$$

where we put $s^2=(\Psi_1+\Psi_2)/\omega$ and $d^2=(\Psi_1-\Psi_2)/\omega$ to remove the irrational expressions from the definition. For the resonance 1:1, this set of Lissajous variables $(\psi_1,\psi_2,\Psi_1,\Psi_2)$ coincides with the second set of Lissajous variables (ℓ',g',L',G') defined by Deprit [39].)

In this set of Lissajous variables, the pullback $f^{\#}\mathcal{H}_0$ takes the simple form

$$f^{\#}\mathcal{H}_0=\omega\Psi_1, \tag{5}$$

and henceforth, the Lissajous variables are a set of action-and-angle variables.

When the perturbation $\mathcal{P}$ is

$$\mathcal{P}=\sum_{n\geqslant n}\varepsilon^n\mathcal{H}_n(\boldsymbol{x},\boldsymbol{X}),$$

a power series in the algebra of real polynomials in $(\boldsymbol{x},\boldsymbol{X})$, the normalization takes place within that algebra. The Lissajous transformation (4) converts $\mathcal{P}$ into a Fourier series of the type

$$f^{\#}\mathcal{P}=F=\sum_{j\geqslant 0,|k|\geqslant 0,n\geqslant 1}\varepsilon^n C_{j,k,n}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(j\,\psi_2+k\,\psi_1), \tag{6}$$

with coefficients in the real algebra of polynomials in s and d. Even more, if G is a homogeneous polynomial of degree n in the Cartesian variables (x,y), the coefficients in the Fourier series $f^{\#}G$ are homogeneous polynomials of degree n in s and d.

The Lie derivative associated with $\mathcal{H}_0$ is the partial differential operator

$$L_0:F\to(F,\mathcal{H}_0)$$

mapping F onto its Poisson bracket to the right with $\mathcal{H}_0$. The *kernel* of L_0 is the set of functions F such that $L_0(F)=0$; the *image* of L_0, the set of functions F of the form $F=L_0(G)$. Normalization of a Hamiltonian of type

$$\mathcal{H}(\boldsymbol{p},\boldsymbol{P},\boldsymbol{\epsilon})=\sum_{n\geqslant 0}\boldsymbol{\epsilon}^n\mathcal{H}_n(\boldsymbol{p},\boldsymbol{P}),$$

we recall [27], is a one-parameter family of canonical transformations

$$\nu:(\boldsymbol{p}',\boldsymbol{P}',\epsilon)\to(\boldsymbol{p},\boldsymbol{P})$$

that changes $\mathcal{H}$ into a function

$$\nu^{\#}\mathcal{H}(\boldsymbol{p}',\boldsymbol{P}',\epsilon)=\mathcal{H}(\boldsymbol{p}(\boldsymbol{p}',\boldsymbol{P}',\epsilon),\boldsymbol{P}(\boldsymbol{p}',\boldsymbol{P}',\epsilon),\epsilon)$$

in the kernel of L_0.

In the Cartesian variables (x,y,X,Y), the Lie derivative associated with the elliptic oscillator (1) is represented by the partial differential operator

$$L_0=X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}-\omega^2\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}\right), \tag{7}$$

however, in the Lissajous variables $(\psi_1,\psi_2,\Psi_1,\Psi_2)$, the Lie derivative associated with the Hamiltonian (5) is very simple, namely, the operator

$$L_0=\omega\frac{\partial}{\partial\psi_1}.$$

Thus, any series of the type (6) may be decomposed into its components

$$F^{\natural}=\sum_{j\geqslant 0,n\geqslant 1}\varepsilon^n C_{j,0,n}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} j\,\psi_2$$

in the kernel of L_0 and

$$F^{\flat}=\sum_{j\geqslant 0,|k|>0,n\geqslant 1}\varepsilon^n C_{j,k,n}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(j\,\psi_2+k\,\psi_1)$$

in the image of L_0.

Thus, when the perturbation is periodic in the Lissajous variable ψ_1, normalizing a perturbed elliptic oscillator amounts to averaging the dynamical system over ψ_1.

Normalization has a second interpretation: it is a *reduction* [45]. In the process of normalization, the number of degrees of freedom has fallen by one unit. Upon analyzing this fact from a geometric standpoint, one will recognize that ignoring the coordinate ψ_1 and holding its conjugate moment Ψ_1 as a parameter amounts to partitioning the phase space into leaves consisting of all states for which the parameter (the moment Ψ_1) has a given value, collecting into classes all points within that leaf which are images of one another by the canonical transformations generated by the integral Ψ_1, and then ''reducing'' the phase space on each leaf $\Psi_1=\text{constant}$ by handling each class as an individual phase space.

We find (Sec. II) a set of functions that play an analogous role to the Hopf variables in the 1:1 resonance [39]; with these functions, we show that the phase portrait of the reduced Hamiltonian has the structure of a two-dimensional revolution surface, namely, a sphere for the 1:1 resonance; a *single pinched* sphere for resonances $1{:}q$ $(1<q)$, and a *double pinched* sphere for resonances $p{:}q$ $(1<p<q)$. Similar results are obtained by the subtraction of two oscillators (Sec. III) and for two diffusers (Sec. IV); in these cases, as it is expected, the phase portrait is an unbounded surface.

II. TWO-DIMENSIONAL PINCHED SPHERES

After normalization, the reduced problem is one degree of freedom in (ψ_1,Ψ_1). In those variables the phase space is a cylinder, but as it happened for elliptic oscillators (resonance 1:1) where the reduced phase space was made of two-dimensional spheres, we will show that for anharmonic oscillators a Mercator map does not provide a good representation, and that actually, the reduced phase space is made of two-dimensional pinched spheres.

To prove that, we proceed from scratch. Our first step consists of finding a set of integrals, analogous to the ones found in [39], and which eventually will lead us to the adequate orbit space where the reduced Hamiltonian should be studied. After several trials, we found that this task was easier to be accomplished in complex variables. Let us define the following canonical transformation $(z,w,Z,W)\mapsto(x,y,X,Y)$ from complex to Cartesian variables (a modification of Birkhoff's transformation)

$$x=\frac{1}{\sqrt{2}}\left(z+\frac{1}{\omega p}\,i\,Z\right),\quad X=\frac{1}{\sqrt{2}}(\omega p\,i\,z+Z),\tag{8}$$

$$y=\frac{1}{\sqrt{2}}\left(w+\frac{1}{\omega q}\,i\,W\right),\quad Y=\frac{1}{\sqrt{2}}(\omega q\,i\,w+W),$$

and its inverse

$$z=\frac{1}{\sqrt{2}}\left(x-\frac{1}{\omega p}\,i\,X\right),\quad Z=\frac{1}{\sqrt{2}}(X-\omega p\,i\,x),\tag{9}$$

$$w=\frac{1}{\sqrt{2}}\left(y-\frac{1}{\omega q}\,i\,Y\right),\quad W=-\frac{1}{\sqrt{2}}(Y-\omega q\,i\,y),$$

where $i=\sqrt{-1}$.

In those variables, the Hamiltonian (1) yields

$$\mathcal{H}_0=\omega\,i\,(p\,zZ+q\,wW),\tag{10}$$

and now, its associated Lie derivative is the operator

$$L_0=\omega\,i\left[p\left(z\frac{\partial}{\partial z}-Z\frac{\partial}{\partial Z}\right)+q\left(w\frac{\partial}{\partial w}-W\frac{\partial}{\partial W}\right)\right].$$

The image by the Lie derivative of an expression of the type $z^a w^b Z^c W^d$ is

$$L_0(z^a w^b Z^c W^d)=i\,\omega[p(a-c)+q(b-d)]z^a w^b Z^c W^d,$$

hence, the monomial $z^a w^b Z^c W^d$ belongs to the kernel of L_0 if and only if

$$p(a-c)+q(b-d)=0.\tag{11}$$

Taking into account this relation, there follows immediately that

$$L_0(zZ)=L_0(wW)=0,$$

which proves that the functions M_1, M_2 defined by

$$M_1(p,q)=\frac{i}{2}(pzZ+qwW),\tag{12}$$

$$M_2(p,q)=\frac{i}{2}(p\,zZ-q\,wW),$$

are integrals. Observe that $M_2\leqslant M_1$; observe further that $M_2=M_1$ if and only if $q=0$.

By virtue of (11), the functions z^qW^p and w^pZ^p are integrals; hence, their linear combinations will be integrals too. However, we are not interested in obtaining collections of integrals, but in finding integrals that will provide some insight to the Lissajous transformation (4). After several trials and with the help of a computer algebra system, we choose the combinations

$$C_2(p,q)=\tfrac{1}{2}\,\omega^{-(p+q)/2}[(i\,\omega\,q)^p Z^q w^p+(i\,\omega\,p)^q z^q W^p],\tag{13}$$

$$S_2(p,q)=i\,\tfrac{1}{2}\,\omega^{-(p+q)/2}[(i\,\omega\,q)^p Z^q w^p-(i\,\omega\,p)^q z^q W^p].$$

The four integrals (12) and (13) cannot be independent. Indeed, as we have the integrals expressed in complex variables, one can easily check that

$$C_2^2+S_2^2=i^{p+q}p^q q^p w^p W^p z^q Z^q$$

and that

$$(M_1+M_2)^q(M_1-M_2)^p=i^{p+q}p^q q^p w^p W^p z^q Z^q,$$

hence, they are bound by the constraint

$$C_2^2+S_2^2=(M_1+M_2)^q(M_1-M_2)^p. \tag{14}$$

In an analogous way, we can define the functions $C_1(p,q)$ and $S_1(p,q)$, which do not belong to the kernel of the Lie derivative L_0, as

$$C_1(p,q)=\tfrac{1}{2}\,\omega^{-(p+q)/2}[W^pZ^q+(i\,\omega\,q\,w)^p(i\,\omega\,p\,z)^q], \tag{15}$$

$$S_1(p,q)=i\,\tfrac{1}{2}\,\omega^{-(p+q)/2}[W^pZ^q-(i\,\omega\,q\,w)^p(i\,\omega\,p\,z)^q].$$

These functions satisfy the relation

$$C_1^2+S_1^2=(M_1+M_2)^q(M_1-M_2)^p. \tag{16}$$

By composition of the transformations (9) and (4), we obtain the explicit expressions of the four integrals (13) and the two functions (15) in terms of the Lissajous variables

$$M_1(p,q)=\tfrac{1}{2}\,\Psi_1, \tag{17}$$

$$M_2(p,q)=\tfrac{1}{2}\Psi_2,$$

$$C_2(p,q)=2^{-(p+q)/2}(\Psi_1-\Psi_2)^{p/2}(\Psi_1+\Psi_2)^{q/2}\cos 2pq\,\psi_2,$$

$$S_2(p,q)=2^{-(p+q)/2}(\Psi_1-\Psi_2)^{p/2}(\Psi_1+\Psi_2)^{q/2}\sin 2pq\,\psi_2,$$

$$C_1(p,q)=2^{-(p+q)/2}(\Psi_1-\Psi_2)^{p/2}(\Psi_1+\Psi_2)^{q/2}\cos 2pq\,\psi_1,$$

$$S_1(p,q)=2^{-(p+q)/2}(\Psi_1-\Psi_2)^{p/2}(\Psi_1+\Psi_2)^{q/2}\sin 2pq\,\psi_1,$$

and conversely, by inverting these formulas, we can determine unambiguously the Lissajous variables (Ψ_1,Ψ_2, ψ_1,ψ_2) from the integrals M_1, M_2, C_2, S_2 and the state functions C_1, S_1 as

$$\Psi_1=2M_1,\quad \Psi_2=2M_2, \tag{18}$$

$$\cos 2pq\,\psi_2=C_2\,(M_1-M_2)^{-p/2}(M_1+M_2)^{-q/2},$$

$$\sin 2pq\,\psi_2=S_2\,(M_1-M_2)^{-p/2}(M_1+M_2)^{-q/2},$$

$$\cos 2pq\,\psi_1=C_1\,(M_1-M_2)^{-p/2}(M_1+M_2)^{-q/2},$$

$$\sin 2pq\,\psi_1=S_1\,(M_1-M_2)^{-p/2}(M_1+M_2)^{-q/2}.$$

Taking into account the relations (17), it is a matter of computing Poisson brackets to discover that the subjacent Lie algebra is determined by the relations

$$\{M_2;C_2\}=p\,q\,S_2, \tag{19}$$

$$\{C_2;S_2\}=\tfrac{1}{2}\,p\,q\,(M_1+M_2)^{q-1}(M_1-M_2)^{p-1}[(q+p)\,M_2+(p-q)\,M_1],$$

$$\{S_2;M_2\}=p\,q\,C_2.$$

In the particular case of the resonance 1:1, we recover the classical *rigid body* Poisson structure [46] in $\mathbf{R}^3$, already found in [39].

The functions M_2, C_2, and S_2 just found, determine a transformation

$$(x,y,X,Y)\mapsto(M_2,C_2,S_2)$$

mapping the three-dimensional sphere

$$S^3(L):X^2+Y^2+\omega^2(p^2x^2+q^2y^2)=2\,\omega\Psi_1=4\,\omega M_1$$

in the phase space $(\omega\,p\,x,\omega\,q\,y,X,Y)$ onto the two-dimensional surface

$$\mathcal{G}(M_1):C_2^2+S_2^2=(M_1+M_2)^q(M_1-M_2)^p.$$

The phase space of of the normalized Hamiltonian system may be regarded as a foliation of invariant manifolds. Indeed, the reduced phase space above each point along the axis Ψ_1 (equivalently M_1) exhibits itself as a two-dimensional surface Eq. (14). This surface $\mathcal{G}(M_1)$ is a surface of revolution generated by rotating the function $g(M_2)=(M_1+M_2)^{q/2}(M_1-M_2)^{p/2}$ (for a fixed value of M_1) about the axis M_2. Some of these figures for several resonances are shown in Fig. 1.

For the resonance 1:1, the surface $\mathcal{G}(M_1)$ is the two-dimensional sphere

$$S^2(M_1):M_2^2+C_2^2+S_2^2=M_1^2,$$

such as Deprit [39] proved. However for other resonances, the surface is no longer a sphere. It is a *single pinched* sphere when either $p=1$ or $q=1$; and a *double pinched* sphere [47] when both $p,q\neq 1$.

Indeed, since $|M_2|\leqslant M_1$ (remember that by definition $|\Psi_2|\leqslant\Psi_1$, $\Psi_1>0$), the function $g(M_2)=(M_1+M_2)^{q/2}(M_1-M_2)^{p/2}$, the generator of the surface of revolution, is derivable everywhere except, perhaps, at the extrema of the interval of definition, that is to say, at the points $M_2=-M_1$ and $M_2=+M_1$. The derivative of the function $g(M_2)$ is

$$\frac{d\,g(M_2)}{d\,M_2}=-\frac{p}{2}(M_1-M_2)^{(p/2-1)}(M_1+M_2)^{q/2}+\frac{q}{2}(M_1-M_2)^{p/2}(M_1+M_2)^{(q/2-1)}, \tag{20}$$

and

$$\lim_{M_2\to+M_1}\frac{d\,g(M_2)}{d\,M_2}=\infty \text{ if and only if } p=1,$$

$$\lim_{M_2\to-M_1}\frac{d\,g(M_2)}{d\,M_2}=\infty \text{ if and only if } q=1.$$

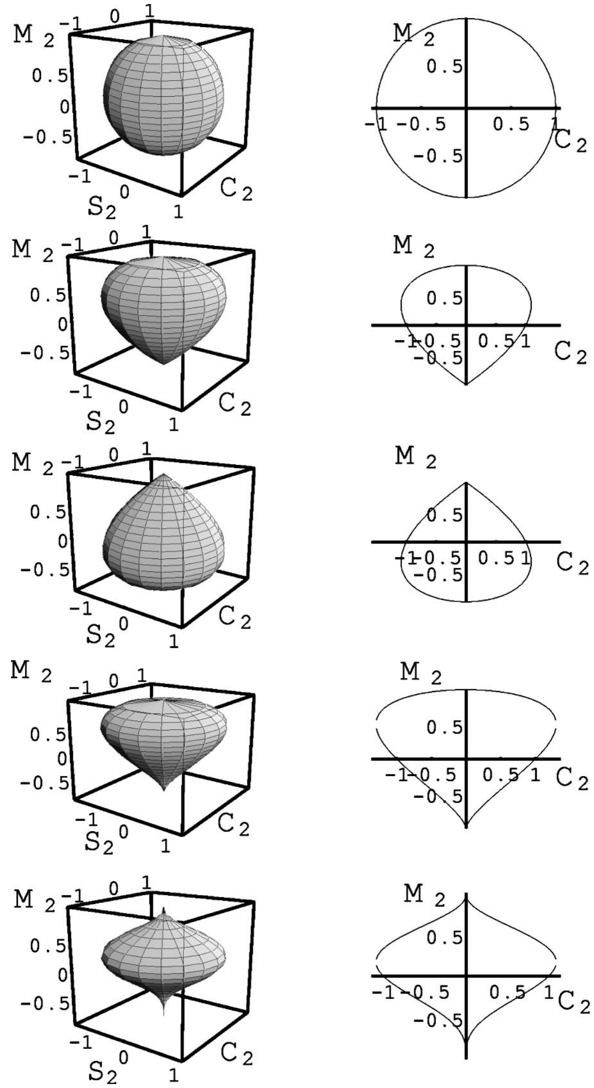

FIG. 1. Left column: Surface $[C_2^2+S_2^2=(M_1+M_2)^q(M_1-M_2)^p]$ with $M_1=1$ and for several resonances (from top to bottom 1:1, 1:2, 2:1, 1:3, 5:7). Right column: Sections $S_2=0$ of these surfaces.

For the rest of the cases, these limits are finite. Hence, the surface is completely smooth for the resonance 1:1 (sphere), and since we consider resonances $p{:}q$ with $p\leqslant q$, there results that for resonances $1{:}q$ $(q>1)$, there is only one singular point (single pinched sphere), namely, the point $(M_2=-M_1,\ C_2=0,\ S_2=0)$. For resonances p:$q$ $(1<p<q)$, there are two singular points (double pinched sphere) $(M_2=\pm M_1,\ C_2=0,\ S_2=0)$.

Incidentally, let us mention, too, that the tangent to the curve $g(M_2)$ at the pinched points $(M_2=\pm M_1)$ is zero, except for $p=1$ or $q=1$ (where the tangent is ∞, as we just have seen) and for $p=2$ or $q=2$. Indeed, in the latter, from (20), there results that

$$(p=2)\quad \lim_{M_2\to +M_1}\frac{d\,g(M_2)}{d\,M_2}=-(2M_1)^{q/2},$$

$$(q=2)\quad \lim_{M_2\to -M_1}\frac{d\,g(M_2)}{d\,M_2}=+(2M_1)^{p/2}.$$

III. PSEUDO-OSCILLATORS

The generalized Lissajous transformation (4) above presented is not only valid for Hamiltonians of the type (1), but also for the Hamiltonian made by the subtraction of two harmonic oscillators

$$\mathcal{H}_0=\tfrac{1}{2}(X^2+\omega_1^2x^2)-\tfrac{1}{2}(Y^2+\omega_2^2y^2). \tag{21}$$

This type of Hamiltonians appears in some problems of cosmology like the study of the dynamics of a Friedmann-Robertson-Walker universe [48,49], but it also appears in more general problems, for instance, in finding the orbital stability of perturbed Hamiltonians in which the principal part is made of a nondefinite sign quadratic in the form of (21), and the perturbation is a series made of homogeneous polynomials in the coordinates. A typical example is the problem of the stability of the Lagrangian points in the restricted three body problem [50,51], where a normalization must be carried out in order to apply the well-known theorem of Arnold [52]. By means of our Lissajous transformation, the Hamiltonian may be normalized up to any arbitrary high order, even at the resonant cases [53].

For this problem of pseudo-oscillators, the Lissajous transformation (4) is valid, but now, the pullback of the Hamiltonian (21) is now

$$f^{\#}\mathcal{H}_0=\omega\Psi_2=2\omega M_2.$$

For this Hamiltonian, the functions M_1, M_2 are integrals, but now C_2, S_2 are not. However, for the Hamiltonian (21) the functions C_1, S_1 are integrals. Thus, by switching (C_2, S_2) with (C_1, S_1), the preceding section is valid for this case. Hence, the functions M_1, C_1, and S_1 determine a transformation

$$(x,y,X,Y)\mapsto(M_1,C_1,S_1)$$

mapping the nondefinite form

$$X^2-Y^2+\omega^2(p^2x^2-q^2y^2)=2\omega\Psi_2=4\omega M_2$$

in the phase space $(\omega\,p\,x,\omega\,q\,y,X,Y)$ onto the two-dimensional surface

$$\mathcal{F}(M_1):C_1^2+S_1^2=(M_1+M_2)^q(M_1-M_2)^p. \tag{22}$$

The Lie algebra in this case is obtained by the Poisson brackets

$$\{M_1;C_1\}=p\,q\,S_1,$$

$$\{C_1;S_1\}=-\tfrac{1}{2}\,p\,q\,(M_1+M_2)^{q-1}(M_2-M_1)^{p-1}\times[(q+p)\,M_1+(p-q)\,M_2],$$

$$\{S_1;M_1\}=p\,q\,C_1.$$

Normalization of a perturbed Hamiltonian system in which the unperturbed part is made of pseudo-oscillators as given in (21), consists of averaging over the Lissajous variable ψ_2. Hence, in the normalized Hamiltonian, M_2 is an

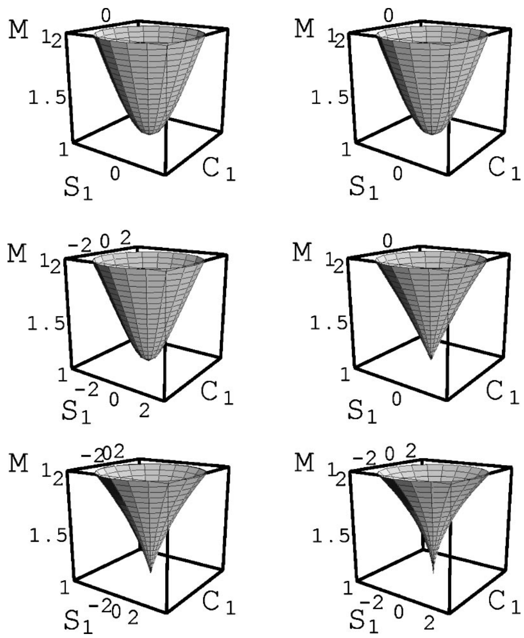

FIG. 2. The surface $[C_1^2+S_1^2=(M_1+M_2)^q(M_1-M_2)^p]$ corresponding to two pseudo-oscillators for several resonances (from top to bottom, resonances 1:1, 1:2, and 2:3). Left column: Positive values of the integral $M_2(=+1)$. Right column: Negative values of the integral $M_2(=-1)$.

integral. Again, the phase space of the normalized Hamiltonian system may be regarded as a foliation of invariant manifolds.

At first glance, it seems that the surface (22) is identical to the one appearing in Fig. 1. However, this is not the case, since when the Hamiltonian is normalized, the moment M_2 becomes an integral (instead of M_1), and two things must be taken into account. First, M_2 may be positive or negative, and second, M_1 is such that $|M_2|\leqslant M_1$. In Fig. 2 we present the surface (22) for several values of M_2 (positive and negative) and several resonances. As we could guess, we have unbounded surfaces.

IV. TWO DIFFUSERS

Let us end this paper by defining a transformation that has the same effect as the Lissajous transformation, but now for Hamiltonians that are combinations of diffusers like

$$\mathcal{H}_0=\tfrac{1}{2}(X^2-p^2\omega^2x^2)+\tfrac{1}{2}(Y^2-q^2\omega^2y^2). \tag{23}$$

By analogy with the anharmonic oscillator, the Lissajous transformation sought is

$$X=\sqrt{\omega(\Psi_1+\Psi_2)}\cosh p\,(\psi_1+\psi_2), \tag{24}$$

$$Y=\sqrt{\omega(\Psi_1-\Psi_2)}\cosh q\,(\psi_1-\psi_2),$$

$$x=\sqrt{\frac{\Psi_1+\Psi_2}{\omega p^2}}\sinh p\,(\psi_1+\psi_2),$$

$$y=\sqrt{\frac{\Psi_1-\Psi_2}{\omega q^2}}\sinh q\,(\psi_1-\psi_2).$$

Indeed, with this transformation, the Hamiltonian (23) reads

$$\mathcal{H}_0=\omega\Psi_1.$$

In order to find integrals of the Hamiltonian system (23), we define a symplectic transformation with multiplier $i=\sqrt{-1}$ in complex variables, given by the equations

$$x=\frac{1}{\sqrt{2}}\left(i\,z-\frac{1}{\omega p}Z\right),\quad X=\frac{1}{\sqrt{2}}(\omega p\,i\,z+Z), \tag{25}$$

$$y=\frac{1}{\sqrt{2}}\left(i\,w-\frac{1}{\omega q}\,W\right),\quad Y=\frac{1}{\sqrt{2}}(\omega q\,i\,w+W),$$

and its inverse

$$z=-\frac{i}{\sqrt{2}}\left(x+\frac{1}{\omega p}X\right),\quad Z=\frac{1}{\sqrt{2}}(-\omega p\,x+X), \tag{26}$$

$$w=-\frac{i}{\sqrt{2}}\left(y+\frac{1}{\omega q}Y\right),\quad W=\frac{1}{\sqrt{2}}(-\omega q\,y+Y).$$

The Hamiltonian (23) after this transformation becomes

$$\mathcal{K}_0=i\,\mathcal{H}_0=-\omega(p\,z\,Z+q\,w\,W);$$

hence, a monomial $z^aw^bZ^cW^d$ belongs to the kernel of L_0 if and only if

$$p(a-c)+q(b-d)=0.$$

Taking into account this relation, we readily check that the functions

$$M_1(p,q)=\frac{i}{2}(pzZ+qwW), \tag{27}$$

$$M_2(p,q)=\frac{i}{2}(p\,zZ-q\,wW),$$

$$C_2(p,q)=\tfrac{1}{2}\,\omega^{-(p+q)/2}[(i\,\omega\,q)^pZ^qw^p+(i\,\omega\,p)^qz^qW^p],$$

$$S_2(p,q)=\frac{i}{2}\,\omega^{-(p+q)/2}[(i\,\omega\,q)^pZ^qw^p-(i\,\omega\,p)^qz^qW^p]$$

are integrals. Besides, we can define the functions

$$C_1(p,q)=\tfrac{1}{2}\,\omega^{-(p+q)/2}[W^pZ^q+(i\,\omega\,q\,w)^p(i\,\omega\,p\,z)^q],$$

$$S_1(p,q)=\tfrac{1}{2}\,\omega^{-(p+q)/2}[W^pZ^q-(i\,\omega\,q\,w)^p(i\,\omega\,p\,z)^q]$$

that for this Hamiltonian are not integrals.

These functions, when expressed in Lissajous variables (26) become

$$M_1(p,q)=\tfrac{1}{2}\,\Psi_1,$$

$$M_2(p,q)=\tfrac{1}{2}\,\Psi_2,$$

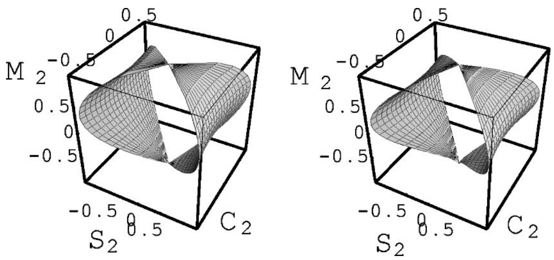

FIG. 3. Surface $[C_2^2-S_2^2=(M_1+M_2)^q(M_1-M_2)^p]$ for the resonances 1:2 and 2:3 of two diffusers.

$$C_2(p,q)=2^{-(p+q)/2}(\Psi_1-\Psi_2)^{p/2}\times(\Psi_1+\Psi_2)^{q/2}\cosh 2pq\,\psi_2,$$

$$S_2(p,q)=-2^{-(p+q)/2}(\Psi_1-\Psi_2)^{p/2}\times(\Psi_1+\Psi_2)^{q/2}\sinh 2pq\,\psi_2,$$

$$C_1(p,q)=2^{-(p+q)/2}(\Psi_1-\Psi_2)^{p/2}\times(\Psi_1+\Psi_2)^{q/2}\cosh 2pq\,\psi_1,$$

$$S_1(p,q)=-2^{-(p+q)/2}(\Psi_1-\Psi_2)^{p/2}\times(\Psi_1+\Psi_2)^{q/2}\sinh 2pq\,\psi_1.$$

Again, it is easy to see that

$$C_2^2-S_2^2=(M_1+M_2)^q(M_1-M_2)^p$$

and that

$$C_1^2-S_1^2=(M_1+M_2)^q(M_1-M_2)^p.$$

Hence, the phase flow of the normalized system (that is a perturbed Hamiltonian where the angle ψ_1 has been eliminated by a Lie transformation) takes place in the unbounded surface (see Fig. 3)

$$C_2^2-S_2^2=(M_1+M_2)^q(M_1-M_2)^p. \tag{28}$$

After some algebra, we find that the Poisson brackets of the integrals M_2, C_2, S_2 satisfy the properties

$$\{M_2;C_2\}=p\,q\,S_2,$$

$$\{C_2;S_2\}=-\tfrac{1}{2}\,p\,q\,(M_1+M_2)^{q-1}(M_1-M_2)^{p-1}\times[(q+1)\,M_2-(q-1)\,M_1],$$

$$\{S_2;M_2\}=-p\,q\,C_2.$$

Analogously to the case of the pseudo-oscillators (Sec. III), the transformation (24) is also valid for Hamiltonians of the kind

$$\mathcal{H}_0=\tfrac{1}{2}(X^2-p^2\omega^2x^2)-\tfrac{1}{2}(Y^2-q^2\omega^2y^2). \tag{29}$$

Indeed, the transformation (24) converts this Hamiltonian into

$$\mathcal{H}_0=\omega\Psi_2.$$

Now, the functions M_1, M_2, C_1, and S_1 are integrals, whereas C_2 and S_2 are not.

The phase flow of the normalized system (that is a perturbed Hamiltonian where the angle ψ_2 has been eliminated by a Lie transformation) takes place on the manifold

$$C_1^2-S_1^2=(M_1+M_2)^q(M_1-M_2)^p, \tag{30}$$

where C_1, S_1, and M_1 are variables and M_2 is constant.

ACKNOWLEDGMENTS

The author is extremely indebted to Dr. A. Deprit at the U.S. National Institute of Standards and Technology, whose comments and suggestions inspired this work. This work was supported by the Spanish Ministry of Education and Science (Project Nos. PB98-1576 and ESP99-1074-CO2-01) and by the Centre National d'Études Spatiales at Toulouse.

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