Quantum algorithmic integrability: The metaphor of classical polygonal billiards

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We study the algorithmic complexity of motions in classical polygonal billiards, which, as the number of sides increases, tend to curved billiards, both regular and chaotic. This study unveils the equivalence of this problem to the procedure of quantization: the average complexity of symbolic trajectories in polygonal billiards features the same scaling relations (with respect to the number of sides) that govern quantum systems when a semiclassical parameter is varied. Two cases, the polygonal approximations of the circle and of the stadium, are examined in detail and are presented as paradigms of quantization of integrable and chaotic systems.

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I. INTRODUCTION

Chaos is certainly the most significant concept that has issued from the theory of dynamical systems and yet its true meaning, most concisely and universally encompassed in the equation *chaos equals deterministic randomness*, has not been fully adopted in the literature and in the scientific community. This is somehow paradoxical, for even in popular magazines the idea has spread that the discovery of chaos might be considered the third scientific revolution of the century—after relativity theory and quantum mechanics. In my opinion, there are two main reasons behind this failure: first, the information-theoretical concepts implied in the notion of deterministic randomness are unfamiliar to most scientists; second, for the vast majority of physicists the ''true'' mechanics is not classical—where chaos is commonly found—but quantum, where chaos is, quite significantly as we know, absent.

A clash is implied in this last statement: if chaos is absent in quantum mechanics, should it not be also absent in classical mechanics, which is just the limiting case of the former? This clash has led people to draw all sorts of conclusions. Many have claimed that classical chaos must imply quantum chaos, via the *correspondence principle* [1], while others on the same basis have pretended that classical chaos theory should be derived from quantum dynamics. Others yet have seen in the collapse postulate (or any other addition to Schrödinger equation required to make physical predictions) the origin of randomness—but randomness alone is not chaos. And on the opposite side of the radical fringe others have argued that, since chaos is absent in the quantum mechanical theory of nature, it should be absent in nature altogether, in particular at the macroscopic level, and so farewell classical chaos. Taken at face value, this last statement implies a logical inconsistency, for quantum mechanics is just a theory, a description of nature. Yet many physicists consider it a very good description in all respects, including chaos or its lack. The fact remains that chaos in nature is an undisputed reality, quite well described by classical dynamics. In previous work we have put forward the idea that the gap between the two mechanics, classical and quantum, is wider than what can be naively expected from the correspondence principle, and that the former is more than the limiting case of the latter [2–6].

In this paper I shall show that a similar situation is met in the dynamics of purely classical billiards: here, rational polygonal billiards play the role of quantum systems, whose ''classical limit'' are curved billiards, to which they tend geometrically as the number of polygonal sides increases indefinitely. Indeed, while it has long been recognized that rational polygonal billiards are nonchaotic systems [7–11], one can use them to approximate chaotic curved billiard tables to arbitrary precision, and ask what happens then to the character of their motion. In addition, the observation has recently been made of positive ''effective'' Lyapunov numbers in polygonal billiards [12], and a paradox of the same flavor as ''quantum chaos'' seems to arise, where chaos appears and disappears at his wish, like the Cheshire cat.

To resolve this paradox, in this paper I shall introduce an elementary, physically motivated version of algorithmic complexity theory. In applying this theory, it will become clear that the procedure of approximating curved billiards by polygons is quite analogous to that of quantizing classical systems: understanding the complexity of the motion in polygons can then be used with profit to clarify the issues involved in the other, more important problem.

Our arguments are organized as follows: in the next section we review the fundamentals on integrable and chaotic billiards, and the notion of *algorithmic integrability*. To adapt algorithmic complexity theory to physical purposes, in Sec. III a simple *coding* of trajectories in billiards is introduced, which translates these into symbolic sequences: their *algorithmic complexity* is the object of this paper. We attribute special importance to the *scaling* of this quantity with respect to time, within certain time intervals: this leads to the concept of *randomness (or order) within a range*, which is presented in Sec. IV. Rather than studying orbital complexity

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directly, we define in Sec. V the *average coding length*, which has a clear physical meaning, and can be used to estimate the former. The theory is immediately applied to the case at hand: the circle (an integrable system), the stadium (a fully chaotic one), and their rational polygonal approximations, which by analogy to quantum mechanics we also call *polygonal quantizations*. The case of circle quantizations is studied in Sec. VI, where the paradox presented above is resolved. The stadium billiard is put to the same test in Sec. VII, and the problem of *correspondence* is addressed. This finally prompts the conclusions, where the position of Chirikov in this debate is briefly reviewed [59].

II. ORDER, CHAOS, AND COMPLEXITY IN BILLIARDS

Billiards are dynamical systems that require little introduction, and just a few formal definitions are necessary. A *billiard table B* is a bounded, connected domain of the plane, with piecewise smooth boundary. An *ideal billiard* in B is the dynamical system originating from the uniform motion of a point particle—a ball—inside B, with elastic reflections at the boundary, following the familiar law that the angle of incidence equals angles of reflection.

In many games that can be played with a billiard, attention is paid to such reflections: one discretizes time (the integer n meaning the time of the nth rebound) and considers the subset $\mathcal{S}$ of the tangent space of B, which consists of unit vectors, attached at boundary points and pointing inside B. $\mathcal{S}$ can easily be parametrized by the pair (l,ϕ), where l is the arc length along the boundary, and ϕ is the angle between the unit vector and the inner normal to the boundary, $-\pi/2<\phi<\pi/2$. In so doing, the dynamics is a function T from $\mathcal{S}$ to itself, that maps the bounce occurring at l_{n-1} with ''exit angle'' ϕ_{n-1} into the new collision point l_n and exit angle ϕ_n:

$$(l_n,\phi_n)=T(l_{n-1},\phi_{n-1}). \tag{1}$$

Since this mapping preserves the canonical measure $d\mu=\cos\phi\, dl\, d\phi$, billiards are among the simplest and most successful examples of Hamiltonian dynamics. But perhaps they mostly owe their success to the fact that a member of their family can be found at virtually all levels in the famous ergodic hierarchy: there are integrable ones—the circle—as well as *K*—the stadium, which is therefore also ergodic, and mixing. Gallavotti and Ornstein have shown [13] that billiards can also be Bernoulli, and Ornstein and Weiss [14] have conjectured that chaos in nature is mostly of this type: not without reason, we can say that billiards have served to shape our view of reality. In a sense, the present paper aims at the same ambitious goal.

An interesting subclass will be studied here, which covers part of the ergodic hierarchy, but falls short of producing chaotic representatives: the polygonal billiards [10]. Sinai has indeed proven [9] that these billiards have null metric entropy, and Ford has termed polygonal billiards with rational angles algorithmically integrable (A) [15], where the letter A was also intended to honor the memory of V. M. Alekseev and his work [16] in which orbits of null entropy systems are shown to have null algorithmic complexity also. The terminology makes it evident that a shift in perspective

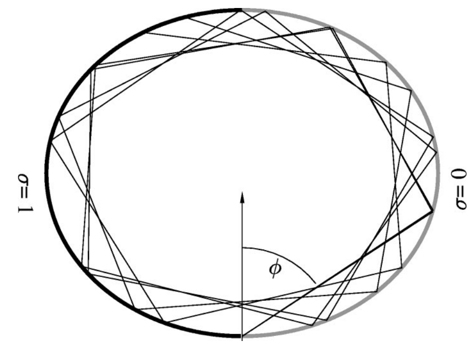

FIG. 1. The initial portions of two trajectories in a circular billiard table, characterized by slightly different initial angles ϕ. Their common initial position is at the bottom of the table. Also shown is a coding of rebounds with two symbols, $\sigma=0$ (light portion of the boundary) and $\sigma=1$ (dark portion).

has taken place: while the ergodic hierarchy is concerned with statistical properties of ensembles of orbits, algorithmic theory deals with a new object, the complexity of the description of the motion, which will be the basis of our investigation [17].

Seminal work [18] on A-integrable billiards is Zemlyakov and Katok's [8] study of polygons whose vertex angles are all rational multiples of π: it shows that these billiards satisfy the conditions for integrability except for the effect of vertices. Rationality of the angles provides a second constant of the motion (the angle of reflection ϕ times a suitable integer multiple of π), but *splitting* of trajectories heading on a vertex provides the *error* [19] which prevents the system from being integrable. Notwithstanding these errors, trajectories are still computable, in the sense that effective algorithms can be devised, in such a way that the number of informational bits in the output (the trajectory) is much greater than the number of informational bits in the input (the algorithm plus the initial condition of the motion). Echardt *et al.* in [7] call such trajectories algorithmically meaningful. Therefore, A-integrable systems are computationally akin to Liouville-Arnol'd (L-A) integrable ones [20,21]. We shall come back to these concepts later.

A few years after Ref. [7] Vega, Uzer, and Ford returned to the theme of rational billiards, presenting a seemingly different set of conclusions [12]: they examined the rate of divergence of nearby trajectories and found that this rate is exponential, even for rational billiards. The key factor in their derivation is the fact that nearby trajectories differ at time zero by a finite, fixed amount, and they are reinitialized to this fixed amount at each iteration of the Benettin-Strelcyn algorithm for computing Lyapunov numbers. In the mind of these authors, this upper bound to precision stands for the human limitation to the dogma of infinite precision. If we take this limitation into account, they claim, trajectories effectively live on a multisheeted surface to which splitting at rational vertices gives an average negative curvature: the resulting motion is, practically speaking, chaotic.

The first aim of this paper is to put order in these conflicting observations, by utilizing a scaling approach to algorithmic complexity theory. But first, let us define the rules of our game.

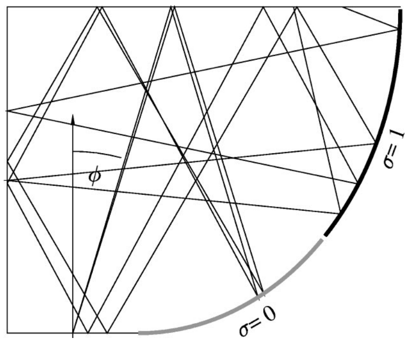

FIG. 2. Same as Fig. 1, now for the quarter stadium billiard table studied in this paper. Notice that rebounds are coded only on the circular part of the boundary, by $\sigma=0$ (light thick portion of the boundary) and by $\sigma=1$ (dark thick portion).

III. SYMBOLIC SEQUENCES IN BILLIARDS: A GAME

The billiard in a circle (Fig. 1) is a noticeable example of Liouville-Arnol'd integrability: the second, smooth integral of the motion being the angular momentum with respect to the center. Cutting the circle into two equal pieces and inserting a rectangular strip between the two halves gives rise to a fully chaotic billiard: the stadium [22]. We shall get rid of all symmetries in this geometrical figure, and study the *quarter stadium* (Fig. 2). Let us now replace the circular sides in both billiards by a polygonal approximation with equal sides: it is apparent that this can be done so as to form a rational billiard. In both these cases, Vega, Uzer, and Ford [12] have found exponentially divergent trajectories, within their approximation scheme of course. In the following, I shall show that the two cases are nonetheless profoundly different. To do this, I first need to introduce a symbolic coding of this problem.

How to code a dynamical object into a symbolic sequence is something that follows from physical insight, or practical convenience, or mathematical efficacy [23]. In our case, we elect to code trajectories according to the bounce coordinate l alone: for this, we assume that the circular parts of the boundary (or their polygonal approximations) are divided into a finite number S of equal regions, each of which corresponds to a symbol σ which can be taken to be a natural number from 0 to $S-1$. We also decide that bounces on the straight segments of the stadium are not registered. This coding is indicated in Figs. 1 and 2, for $S=2$. For simplicity, we shall always present the results for $S=2$ in this paper, although in the Appendix the general case is considered.

In a billiard table like this, we may think of putting detectors all around the boundaries, which are set off whenever the ball hits them. In the polygons, nearby sides can be connected to the same detector, so that the total number of output channels is S also when M—the number of sides—is much larger than S and is allowed to increase while keeping S fixed as we shall do in the following. The history of a trajectory is then coded in the record $\sigma_1,\sigma_2,\ldots$ of boundary reflections.

Now, let us play a game: the ball is initially set still at a fixed point, and a test player—chosen appropriately among our friends—can aim it by hitting it properly with the familiar cue. As the departing angle ϕ is varied different trajectories are initiated, and different symbolic sequences $\{\sigma_j\}$ are recorded. Let us now ask the player to do it twice, that is, to aim the ball a second time so that the sequence of bounces obtained in the first shot is repeated [24]. It comes as no surprise that the player will not be able to set the initial angle ϕ to exactly the same value in both tries: the difference in this quantity causes the two resulting symbolic trajectories to agree only over a finite time-span. Trajectories of this kind are pictured in Figs. 1 and 2. But how is this sport related to algorithmic complexity and to our problem?

IV. DETERMINISTIC RANDOMNESS OF FINITE TRAJECTORIES

The symbolic coding just described has reduced any finite trajectory into a sequence of symbols $\sigma_1,\ldots,\sigma_N$, and the billiard game can be thought of as a computer program designed to output this sequence. Of course, this computer program can also be translated into a sequence of bits, $p_1,\ldots,p_L$. The relation between a sequence and its ''generating'' programs is the object of algorithmic complexity theory. In fact, the algorithmic complexity $K(N)$ of a sequence $\{\sigma_j\}_{j=1,\ldots,N}$ is defined very roughly as the length of the shortest computer program capable of outputting the sequence, and stopping afterward [25,26]. Since this program uniquely defines its output, we can say that the complexity of a sequence is the length of its shortest definition:

$$p_1,\ldots,p_L \overset{\text{outputs}}{\Rightarrow} \sigma_1,\ldots,\sigma_N \text{ implies } K(N)\leqslant L. \tag{2}$$

Algorithmic complexity theory teaches us that most sequences—in a probabilistic sense—of length N have complexity close to maximal, i.e., N. Moreover, in the limit of increasing N, almost all of them are random, in the sense that their complexity $K(N)$ grows as N [27]. At the same time, computable sequences exist, for which $K(N)$ is much less than N, and grows less than linearly. When σ is generated by a dynamical system, as in our case, we shall follow Alekseev and Yakobson [16], Chirikov *et al.* [28], and Ford [29], and identify *order* with *computability*, and *chaos* with *randomness*, whence the definition put forward in the Introduction.

Let us return now to our billiard problem, and study the complexity $K(N,\phi)$ of its dynamical sequences [30]. We have explicitly indicated that complexity may depend on the initial condition of the motion, the angle ϕ. The central issue is then to find an optimal computer code to output the specified sequence. A possible candidate is obtained—in any abstract language—by an encoding of (a) the geometrical rules of the game, which requires a fixed number of bits C_{machine} (which depends only on the machine on which the rules are coded); (b) the instructions set for fixing the billiard boundaries, of coding length C_{boundary}; (c) the number of rebounds N; and (d) a certain number of digits of ϕ. Accordingly, the length $L(N,\phi)$ of this program can be estimated as

$$L(N,\phi)\simeq C_{\text{machine}}+C_{\text{boundary}}+\log_2 N+\Lambda(N,\phi), \tag{3}$$

where the function $\Lambda(N,\phi)$ is defined as the *number of bits of ϕ necessary and sufficient to determine the first N symbols in the sequence σ*.

The function $\Lambda(N,\phi)$ can serve to estimate the complexity $K(N,\phi)$ in its most relevant aspect: the N dependence.

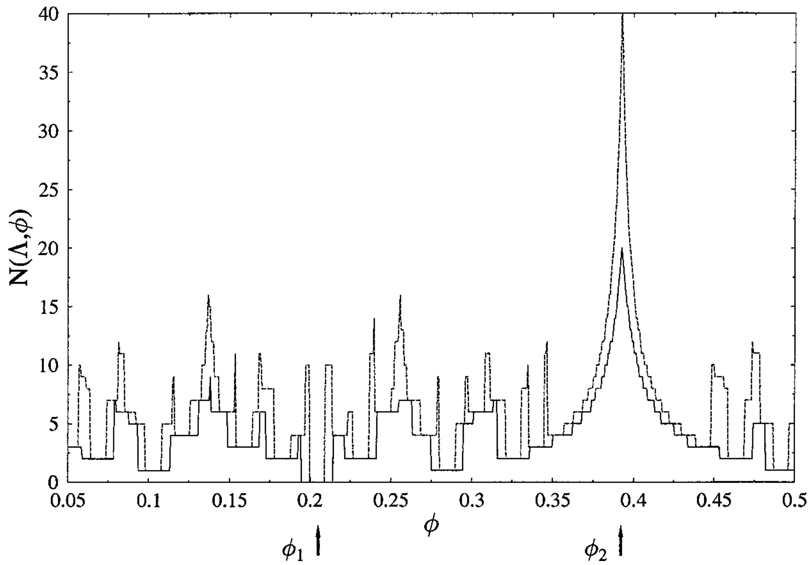

FIG. 3. Coding length $N(\Lambda,\phi)$ vs ϕ for an octagonal billiard, with $S=8$. The starting point is at the center of one of the sides. Two values of ε are used: $\varepsilon=0.01$ (bottom continuous line) and $\varepsilon=0.005$ (top broken line). The arrows mark the values ϕ_1 and ϕ_2 described in the text.

On the one hand, it is clear that any computer program for σ must include this contribution; on the other hand, this function features the most relevant N dependence in Eq. (3), and serves therefore to discriminate between order and chaos. In fact, when in a range of sequence length $N_l \leqslant N \leqslant N_u$ the function $\Lambda(N,\phi)$ grows linearly, $\Lambda(N,\phi)\sim\lambda N$, the complexity $K(N,\phi)$ has the same leading behavior, and the sequence $\{\sigma_j\}$ should duly be termed random, or chaotic. When, in a similar interval, $\Lambda(N,\phi)$ grows less than linearly, $\{\sigma_j\}$ should be called computable, or ordered. We therefore introduce the notion of order (and randomness) within a range. The physical significance of such order (or randomness) will then be proportional to the importance of such range.

We have emphasized this scaling approach in [3] to rebut a common objection to the application of complexity theory to finite dynamical sequences [31]. The rationale behind our idea is evident and is quite similar to that adopted in the independent theory of computational time complexity: not so much the value of complexity is relevant, but the way it increases as the problem size grows, for this may render it quickly unfeasible. In fact, there is no point in practicing for our billiard player if he is playing a chaotic billiard: for each additional bounce he wants to set correctly, his aiming precision in the initial angle ϕ must increase geometrically. Leaving the metaphor, when the role of the billiard is played by a system whose symbolic dynamics we want to predict, to obtain a linear increase in forecast precision we must exponentially increase the accuracy in the initial conditions [32]. In most instances this demands an exponential increase of resources.

V. CODING UNDER FINITE PRECISION

The algorithmic theory explained so far rests on the question, ''How many bits of information do I need to describe a symbolic sequence?'' The answer is mostly given by the function Λ. Clearly, this theory is in line with common usage in dynamical systems, and the letter Λ can be thought of as the L in Lyapunov. Yet we now pretend that this theory is still inadequate for physical purposes. In fact, if we are to describe a world under finite precision, it is then more significant to study the *inverse* function of $\Lambda(N,\phi)$, that is, $N(\Lambda,\phi)$, which certainly exists, for Λ is a nondecreasing function of N.

This function quantifies the number N of symbols σ_j that can be predicted knowing the first Λ digits of the binary expansion of ϕ. It is clear that the computability or randomness—hence the order or chaos—of a dynamical sequence can also be inferred from the study of $N(\Lambda,\phi)$ and, at this level, nothing has been lost by adopting this new definition. Moreover, the new function has a transparent physical meaning: if we let $\varepsilon := 2^{-\Lambda}$, in this new independent variable the function $\tilde{N}(\varepsilon,\phi) := N(-\log_2 \varepsilon,\phi)$ can be called the length of the codable trajectory under *uncertainty* ε.

Let us go back to our example. The player is capable of setting the angle ϕ only with a certain error ε. Under these circumstances, only finitely many dynamical symbols will typically coincide in the initial portions of the symbolic sequences σ and σ' of two independent shots—their number is $N(\Lambda,\phi)$: in other words, $N(\Lambda,\phi)$ is the smallest integer k such that $\sigma_1=\sigma'_1, \ldots, \sigma_k=\sigma'_k, \sigma_{k+1}\neq\sigma'_{k+1}$.

Following [34] we also say that $\tilde{N}(\varepsilon,\phi)$ (plus 1) is the *first error time*, when the information on ϕ is no longer sufficient to compute (i.e., forecast) the symbolic sequence. This function is rather straightforward to compute numerically, and its theoretical analysis can be carried out in full detail (see the Appendix and below). It is plotted in Fig. 3 versus ϕ at fixed uncertainty ε, for the $M=8$ quantization of the circle (a regular octagonal billiard). One observes that low values, like that seen at ϕ_1, are associated with trajectories heading very early in history toward a vertex, where a symbolic error may occur [7,34]. Large values of $N(\Lambda,\phi)$ are found when this happens much later: ϕ_2 is associated with a periodic trajectory that stays far away from the vertices [35].

It can be shown that the behavior of N at the periodicity values is at the root of the dynamical properties of the system. In fact, let the ideal, frictionless ball run for an infinite time. A coding function F of the initial angle ϕ can be defined as $F(\phi)=\Sigma_{j=1}^{\infty}\sigma_j S^{-j}$. This function represents the translation dictionary between the trivial code (the infinite sequence of digits of ϕ), and the dynamical code (the infinite sequences of bounces). In chaotic systems, like the motion over surfaces of constant negative curvature and the anisotropic Kepler problem, the relation between the properties of N and the function F can be fully exposed. This relation justifies the multifractal properties of the coding function F, which were originally investigated by Gutzwiller and Mandelbrot [36,37]. We defer the study of $F(\phi)$ to further publications.

Let us consider Fig. 3 again. In players' terms, the angle ϕ_2 is an easy shot and ϕ_1 a tough one. Recall now that we are not in a position to specify ϕ exactly: we find it necessary, and convenient, to average over initial angles, so defining the average coding length $A(\varepsilon)$:

$$A(\varepsilon) := \int N(-\log_2 \varepsilon, \phi) \cos \phi d\phi, \tag{4}$$

which will become our main indicator of the complexity of the motion. The fundamental question now becomes (roughly): ''How many symbols can I compute on average with $N = -\log_2 \varepsilon$ bits of program?''

A canonical average over the full phase space can also be defined and computed, with quantitatively similar results to those we are going to describe. Let us pause no more, and compare the behavior of this quantity in the circular billiard, the stadium, and their polygonal approximations.

VI. ORBITAL COMPLEXITY IN INTEGRABLE AND POLYGONAL BILLIARDS

To draw a parallel with quantum mechanics [33] suppose now that nature, by means of our observations, clearly shows that curved billiard boundaries are a mathematical idealization and that physically we can just have polygonal billiards with an arbitrary but finite number of sides, M. These sides will be inscribed in the circle and in the stadium, and I shall call the resulting polygonal billiards a quantization of the curved ones. In this context, M is a crucial quantity, which plays the role of a semiclassical parameter. Clearly, when we let M go to infinity we regain geometrically the original table. The question is, will we obtain the same kind of dynamics?

We start now to answer this question in the case of the circle and its polygonal quantizations: the stadium will be treated in the next section. Before resorting to exact analysis, let us have a look at the numerical data. Figure 4 reports the average coding length $A_M(\varepsilon)$ of a regular M-gon versus ε, for various values of M. The scale is doubly logarithmic: according to Eq. (3) the power-law behavior of these curves for small ε is a manifestation of the ordered character of the motion, which is well assessed in the literature [9,7,16].

Yet the physical picture evident in Fig. 4 is much richer. First, as M grows at fixed (large) ε, $A_M(\varepsilon)$ tends to the coding function for the exact circular billiard, the earlier the larger the value of ε. This is easy to understand: at large ε the coding length is short, and differences between polygonal boundaries with large M and the circle are not significant.

If we turn our attention now to the left part of Fig. 4, where relatively smaller values of ε are plotted, a less expected phenomenon appears: with increasing M at fixed ε the difference between $A_M(\varepsilon)$ and $A_{\text{circle}}(\varepsilon)$ seems to grow

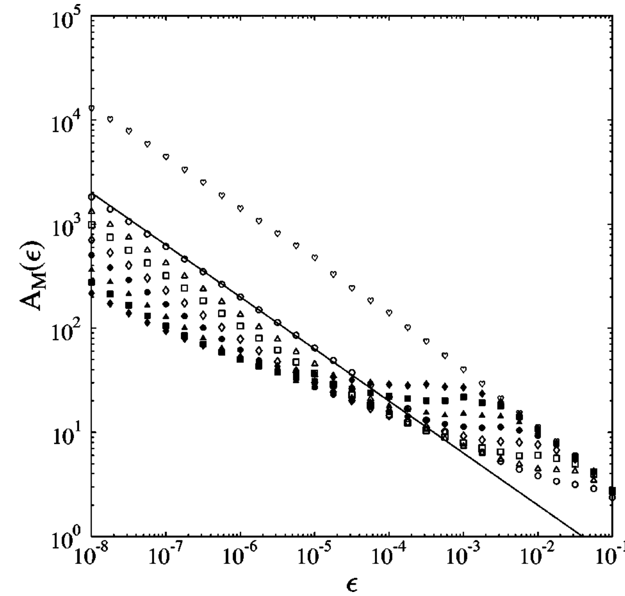

FIG. 4. Average coding length $A_M(\varepsilon)$ vs ε with $S=2$ for the circle (hearts) and its polygonal quantizations. The number of sides of M is chosen to be twice a prime, in a roughly geometrically increasing sequence: $M=62$ (open circles), $M=134$ (open triangles), $M=254$ (open squares), $M=514$ (open diamonds), $M=1042$ (filled circles), $M=2062$ (filled triangles), $M=4106$ (filled squares), $M=8198$ (filled diamonds). The continuous line is the second formula in Eq. (5) with $M=62$.

rather than vanish. Moreover, this discrepancy is brought about by a decreasing $A_M(\varepsilon)$. Recall that $A_M(\varepsilon)$ is the average of the inverse function of $\Lambda(N,\phi)$ (if you rotate the graph clockwise by 90°, N appears plotted on the horizontal axis and $\varepsilon = 2^{-\Lambda(N)}$ on the vertical): in this region the complexity of a trajectory of fixed length $N = A_M$ grows when M is increased. This is the phenomenon observed by Vega, Uzer, and Ford: polygonal billiards have their own brand of instability, generated by splitting of trajectories at vertices. The more vertices, the larger the Lyapunov numbers computed in [12]. The paradox presented in Sec. II is thus demonstrated.

But we are now equipped to resolve it. Observe first that the increase in complexity obtained by raising M at fixed ε is only temporary; if we keep going, we find that $A_M(\varepsilon)$ reaches a minimum, and then inverts its course to reach quickly (we shall come back to the rate of this convergence later on) the circle value A_{circle} [38].

Furthermore, a deeper argument must be made: remaining at fixed ε is not a proper thing to do, not even in the presence of human limitations to finite precision. In fact, we cannot increase precision *indefinitely*, but we certainly can over a finite, physically reasonable range. For instance, this may be dictated by the computational power of the machine on which Fig. 4 has been computed. Exploring this range, from larger to smaller uncertainties, we discover that $A_M(\varepsilon)$ starts off like the L-A integrable circle, $A_{\text{circle}}(\varepsilon) \sim \varepsilon^{-1/2}$, then ''feels'' the effect of vertices, and successively redirects its course on a different line, with the same exponent: a transition from L-A to A integrability has taken place.

These numerical observations can be derived in full detail from the analysis developed in the Appendix. We are able to prove that the average coding length is given by

$$A_M(\varepsilon) \simeq \begin{cases} \dfrac{\pi}{2} \dfrac{1}{\sqrt{2\varepsilon}} & \text{for} \quad M\varepsilon \gg H \\ \dfrac{\pi}{2} \dfrac{1}{\sqrt{M\varepsilon}} & \text{for} \quad M\varepsilon \ll H, \end{cases} \tag{5}$$

where the crucial functional dependence on M and ε is apparent, and where H is a constant, which plays a similar role to Planck's constant in the ''usual'' quantum mechanics.

In the above equation, the first behavior coincides with the result we find for the circular billiard:

$$A_{\text{circle}}(\varepsilon) \simeq \frac{\pi}{2} \frac{1}{\sqrt{2\varepsilon}}, \tag{6}$$

so that at fixed ε we have $\lim_{M\to\infty} A_M(\varepsilon) = A_{\text{circle}}(\varepsilon)$. This is clearly a form of correspondence, to the relevance of which we shall return in the next section.

The average coding length computed in the second Eq. (5) and in Eq. (6) leads to an estimate for the average complexity which increases only logarithmically with N and M: first, the function Λ satisfies

$$\Lambda(N) \leq C + 2 \log_2 N + \log_2 M, \tag{7}$$

where C is a positive constant independent of M and N, and where $M=2$ must be set for the circular billiard, which therefore turns out to be the simplest member in the family, as it should. Following suit, from Eqs. (3) and (7) one obtains [39] that the average program length is

$$L(N) \leq C' + 3 \log_2 N + 2 \log_2 M, \tag{8}$$

where C' is another positive constant. This result should be compared with the existing literature on the *topological* complexity of symbolic dynamics [40].

In conclusion, we see that vertices add logarithmically to the complexity, but only as long as $M\varepsilon \ll H$, i.e., $\log M < \log H + \Lambda$. We have thus reconciled the observation of Ref. [12] and the established knowledge on polygonal billiards. We can now turn to a more delicate problem: what happens to orbital complexity in the polygonal approximations of a chaotic billiard.

VII. ORBITAL COMPLEXITY IN CHAOTIC BILLIARDS AND CORRESPONDENCE

In the previous section we have established that rational approximations of the circle cannot be called chaotic, not even under the assumption of finite precision. Is this achievement possible to their more sophisticated relatives, those inscribed in the stadium? After all, as M tends to infinity, these billiards tend to a fully chaotic system, and *one thing is certain, the correspondence principle must be obeyed* [43].

Figure 5 is the analog of Fig. 4, now for the quarter stadium billiard table. Differences and similarities between the two are evident: while $A_M(\varepsilon)$ is ultimately a power law in both cases, $A_{\text{stadium}}(\varepsilon)$ is never such. The logarithmic character of this curve clearly reveals that the complexity of trajectories grows linearly with length: chaos is here manifest in its essence. For large values of $M\varepsilon$, $A_M(\varepsilon)$ is approximately equal to $A_{\text{stadium}}(\varepsilon)$. For instance, over the interval $(10^{-5}, 10^{-2})$, the coding length A_{32768} is a logarithmic function of ε. We can therefore expect that a corresponding range in N exists so that the average complexity $K(N)$ grows linearly in N: here, trajectories are random, in the sense explained in Sec. IV. Notice that the existence of this range is what permits ''sensible'' numerical experiments of chaotic motion on finite digital computers.

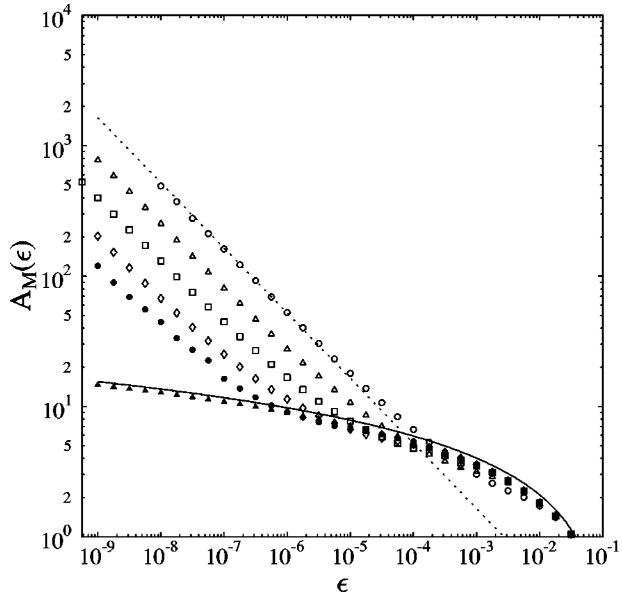

FIG. 5. Average coding length $A_M(\varepsilon)$ vs ε for the quarter stadium (filled triangles) and its polygonal quantizations, with $S=2$. The values of M are powers of 2: $M=128$ (open circles), $M=512$ (open triangles), $M=2048$ (open squares), $M=8192$ (open diamonds), $M=32\,768$ (filled circles). The solid and dotted curves are given in Eq. (9) with $M=\infty$ and $M=128$, respectively, and $C=-1.731$, $D=0.831$, $\rho=1.78$.

The effect of vertices noticed in the circle quantization (Sec. VI) appears again when following the curves $A_M(\varepsilon)$ to the left, as they leave $A_{\text{stadium}}(\varepsilon)$ staying slightly lower than this latter, i.e., showing a relative increase in the complexity of the motion. As before, this increase is quantified by the logarithmic contribution $\log M$. Alas, this excess of zeal rapidly turns into failure: the algorithmic simplicity of the motion within polygonal boundaries is soon detected, and $A_M(\varepsilon)$ starts to grow like $\varepsilon^{-1/2}$. What has happened is that the complexity estimate (7), previously dominated by that of the curved stadium, takes over for good.

An exact analysis can be performed here too (see the Appendix), showing that the average coding length satisfies

$$A_M(\varepsilon) \simeq \begin{cases} C - D \log \varepsilon & \text{for} \quad M\varepsilon \gg H \\ \dfrac{\pi}{4} \dfrac{1}{\sqrt{\rho M \varepsilon}} & \text{for} \quad M\varepsilon \ll H, \end{cases} \tag{9}$$

where C, D, ρ, and H are suitable constants, the last playing the same role as in the previous section. Consequently, in the region $M\varepsilon \ll H$ complexity estimates of the form (7) and (8) still apply, and the motion is clearly ordered in this regime.

The end of this paper is in sight, and we must start drawing conclusions. There is a significant difference between the case of the circle [Fig. 4 and Eq. (5)] and that of the stadium quantization [Fig. 5 and Eq. (9)]: while the former, as we noted, shows a transition from L-A to A integrability, the second matches *A* integrability with chaos, and this mating is troublesome, to say the least. Perhaps the most evident consequence is the following. Let me ask how long a symbolic sequence must be to perceive the finiteness of the number of boundary sides via its algorithmic manifestations. The answer is instructive. In the polygonal quantization of the circle, and of the stadium as well, the critical value of ε at which the functions A_M and their limits A_{circle} and A_{stadium} start to differ significantly (call it ε_M) scales as $\varepsilon_M \sim H/M$. Yet, at the same time, the different structure of the two problems implies that the length of polygonal-curved accordance is proportional to $\sqrt{M}$ in the integrable case and only to the logarithm of *M* for the chaotic stadium [44].

Of course, as in the previous section, we observe that at fixed ε the sequence $A_M(\varepsilon)$ tends to the curved billiard value $A_{\text{stadium}}(\varepsilon)$, when the number of sides *M* tends to infinity. This is certainly a form of correspondence. Nonetheless, according to the reasoning of Sec. IV, what is important is the scaling of complexity with respect to the length of dynamical sequences, and the relevance of the range in which this scaling is validated. But this is precisely what we have just computed. Therefore, we are inevitably led to the conclusion that the range of correspondence in the *M* quantization of the stadium is negligibly small, in physical terms.

It would look rather awkward now to try to avoid this conclusion by clinging to the weak form of correspondence at fixed ε and appealing to the perused observation that the ''classical limit (here, $M \to \infty$) and the infinite-time limit (here, $\varepsilon \to 0$) do not commute.'' This is irrelevant (recall note [31]): chaos can and must be detected over finite time scales of physical relevance, and over these time scales we can perceive the inadequacy of polygonal billiards to describe the richness of motions in the stadium [46].

These results have been obtained by the explicit calculation reported in the Appendix. Yet I believe that their truest meaning lies within their algorithmic nature: in a sense, they could not have been different, given the premises. In fact, we must realize that both circle and stadium polygonal quantizations are dynamical systems endowed with an amount of complexity which scales as the logarithm of *M*. Yet, while the *M*-circle dynamics unfolds this complexity slowly, at a logarithmic pace, its *M*-stadium relative does it eagerly, linearly in time, so that the complexity reservoir is quickly exhausted. I shall now briefly present my views on the physical implications of these facts.

VIII. CONCLUSIONS: CHAOS IN NATURE, IS THERE ANY?

Algorithmic complexity theory, used here in a physicist's fashion via the concept of average coding length, clarifies the simplicity of the dynamics of polygonal billiards, and offers a solution of the paradox that arises from the juxtaposition of Refs. [7] and [12]. At the same time, this theory provides a complete characterization, in all ranges of parameters, of the dynamical properties of the families of billiards studied in this paper. We have thus discovered that even under the assumption of finite precision measurements the motion in polygonal billiards is to be considered mostly ordered: the effect of vertices can be quantified by a logarithmic contribution to the average algorithmic complexity of symbolic trajectories.

Since our technique is based on a very general theory, algorithmic complexity, and on a basic assumption about the finite precision of our measurements, which can also be used as a testing hypothesis in ''exact'' situations, we believe that it is well suited to treat the general case of the approximation of a complex theory by a sequence of simpler descriptions, with results that we expect to be fairly similar to those presented in this paper [45]. In fact, the algorithmic estimates derived in this paper are manifestations of the same phenomenon observed in the Schrödinger quantization of the Arnol'd cat [47,2,3,5] [see Eqs. (8) and (9) of Ref. [3]] and of bounded systems [4], in the sequence of energy levels of integrable and non-integrable Hamiltonians [48], and also in the classical dynamics of discrete systems (see Sec. IV of Ref. [34]). Truly then, the nature of chaos and order is information content.

Yet, if we admit this, we must be prepared to bear the consequences when going back to the problem of quantum mechanics: the metaphor of rational billiards used here permits us to understand the claim that the correspondence principle is validated for integrable systems and violated for chaotic ones. It was shown in the last section, in fact, that in the first case a discrete nature—in which only polygons are allowed—would let us play with our curved theoretical models for a long time. For the same reason, in the quantum mechanics of an integrable system, the action of increasing a semiclassical parameter draws complexity from a reservoir that keeps up with the classical description for a time span that grows appreciably, as a power law, in this parameter. In the second case, on the contrary, the time of chaotic freedom is logarithmically short, and the essence of correspondence to regain classical/curved complexity by quantum/polygonal computations is exponentially remote in the semiclassical parameter [46].

In the end, what we have rediscovered might even seem trivial: the correspondence principle demands that classical objects be ''computed'' quantum mechanically—this is certainly possible, as we have seen in the metaphor of classical billiards. The point is that for most classical objects this computation is unfeasible, in the algorithmic sense explained in this paper. In a sentence: quantum dynamics is a computable theory [49] and can ''correspond'' [50] to the uncomputable classical mechanics only as far as its algorithmically simple nature allows it [51].

Notwithstanding this evidence, people have been reluctant for a long time to admit that there is not such a thing as ''quantum chaos,'' or that this chaos is something different from, and less than, deterministic randomness. This hindrance is now history. On the contrary, Chirikov has appreciated this fact since the very beginning, and his notions of transient chaos [28] and pseudochaos [52,53] show this very clearly. Moreover, he has put a strong emphasis on this latter concept: largely simplifying, but I believe appropriately, one could say that Chirikov views the pseudochaos (that we call *A* integrability) of discrete systems (and as a consequence of digital computers as well) as a faithful representation of what takes place in quantum mechanics [54] and in nature, at all levels [55]. At this very last point we part company: in fact, an unsettled debate [56,57] still holds on the meaning and validity of the correspondence principle and on the related and more crucial question of whether quantum *A* integrability is enough to cope with a world in which chaos seems to be essential. In my work with Ford a negative answer to this question has been presented, suggesting that in this sense quantum dynamics is incomplete.

Certainly, classical mechanics is inadequate to describe microscopic reality, but it contains the gene of noncomputability, which we believe should be present in a complete theory of nature; at the same time, the more fundamental quantum mechanics inherited this gene in too tame a form to be effective. This is, *in nuce*, the point of contention. Will time settle the matter—or will it be that the general indeterminacy principle foreseen by Ford [58] will change everything around?

ACKNOWLEDGMENTS

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APPENDIX: AVERAGE FIRST ERROR TIME CALCULATION

We shall now justify the formulas presented in Secs. VI and VII via a direct calculation of the average coding length/first error time $A(\varepsilon)$. First, let us consider the case of the (integrable) circular billiard. The angle ϕ is a constant of the motion: letting $\omega := \pi - 2\phi$ the motion is $l_n = l_{n-1} + \omega$, the usual ergodic (for irrational ω/π) rotation of the circle. Trajectories with different initial angle ϕ separate at a linear speed 2ε, where ε has the same meaning as in the main body of the paper. Consequently, a bunch of trajectories opens up linearly in time like a slightly unfocused light beam (see Fig. 1). An error occurs when a boundary point of the circle partition that determines the coding enters this light cone. Let us now estimate the average time required for this to happen.

Let S be the number of cells of the partition of the boundary, and $\delta_S := 2\pi/S$ their common length. Let also $\theta_n := l_n \bmod \delta_S$, $n = 1,2,\ldots$. According to what we have just said, an error occurs when the angle θ_n of the reference trajectory (the center of the beam) comes within $n\tilde{\varepsilon}/2$ of zero or 1, where $\tilde{\varepsilon} := (4/\delta_S)\varepsilon$. Let $p_n(\varepsilon)$ be the probability that the first error occurs at time n. Ergodicity implies that $p_1(\varepsilon) = \tilde{\varepsilon}$. Certain approximations are required to evaluate p_n for $n > 1$: we assume that $\{\theta_n\}$, $n = 1,2,\ldots$, are uncorrelated random variables, so that $p_n(\varepsilon) = n\tilde{\varepsilon}\prod_{j=1}^{n-1}(1 - j\tilde{\varepsilon})$. Estimating then the asymptotic (small $\tilde{\varepsilon}$) behavior of $A(\varepsilon) = \sum_n n p_n(\varepsilon)$ by standard techniques gives $A(\varepsilon) \simeq \sqrt{\pi/2}\,\tilde{\varepsilon}^{-1/2}$, and therefore

$$A_{\text{circle}}(\varepsilon) \simeq \frac{\pi}{2} S^{-1/2} \varepsilon^{-1/2}, \qquad \text{(A1)}$$

which accurately reproduces the behavior found numerically in Sec. VI, both in the exponent and in the prefactor.

For the stadium, one can simply invoke the positivity of the Lyapunov exponent λ, which implies that the transverse dimension δ of a beam of trajectories grows on average geometrically in the number of bounces: $\delta(n) \sim \text{const} \times \lambda^{\eta n}\varepsilon$, where the constant η takes into account the average time elapsed between two coded bounces. Under these circumstances, we do not need a detailed analysis as in the circular case, but we can safely assume that for the large majority of initial angles the first symbolic error occurs when $\delta(n)$ is of the order of the length of boundary partitions, δ_S. We therefore obtain

$$A_{\text{stadium}}(\varepsilon) \simeq C - D \log(S\varepsilon), \qquad \text{(A2)}$$

where C and D are constants.

Surprisingly, the same ε dependence of the average coding length found for the circle can be shown to hold for the polygonal billiards considered in this paper. The presence of the vertices imposes a more sophisticated analysis than in the two cases considered so far. When the beam bounces fully on the same polygonal side, its angular amplitude is conserved, and its transverse dimension grows linearly in time on average, exactly as it does in the circular billiard. To the contrary, when the beam impinges on a vertex of amplitude α, its angular amplitude is increased by the amount $2(\pi - \alpha)$. In the polygonal approximations of the circle, we have $\alpha = (1 - 2/M)\pi$, while $\alpha = (1 - 1/2M)\pi$ holds for the quarter stadium, so that $\phi_M^{(c)} := 4\pi/M$ is the resulting perturbation in the circular case, and $\phi_M^{(s)} := \pi/M$ in the quarter stadium.

Two regimes must now be considered. When ε is much larger than ϕ_M, the effect of vertices is negligible, trajectories behave as in the corresponding curved billiards, and $A_M(\varepsilon)$ coincides to a good approximation with Eqs. (A1) and (A2).

In the opposite regime, when ϕ_M is much larger than ε, impinging on a vertex causes a major enlargement of the beam. We can assume that this leads almost immediately to an error: a similar analysis to that performed in the circular case can then be carried out. We let $\delta_M = 2\pi/M$ be the approximate length of the polygonal sides: this quantity plays here the same role that δ_S did above. The result is then

$$A_M(\varepsilon) \simeq \frac{\pi}{2} M^{-1/2} \varepsilon^{-1/2}, \qquad \text{(A3)}$$

for the M-circle, and

$$A_M(\varepsilon) \simeq \frac{\pi}{4} \rho^{-1/2} M^{-1/2} \varepsilon^{-1/2} \qquad \text{(A4)}$$

for the M-stadium, where ρ is a parameter that measures the effective length of trajectories between two coded bounces.

Finally, the intermediate region between the two regimes is determined by the condition $\phi_M \sim \varepsilon$ described above, which becomes the ''indeterminacy principle'' $M\varepsilon \sim H$, where H is a constant.

[1] The correspondence principle states that classical mechanics gives the right (i.e., quantum) results when the action per cycle is large compared to Planck's constant [P.A.M. Dirac, *The Principles of Quantum Mechanics* (Clarendon, Oxford, 1947).

[2] J. Ford, G. Mantica, and G. Ristow, Physica D **50**, 493 (1991).

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[4] J. Ford and M. Ilg, Phys. Rev. A **45**, 6165 (1992).

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[11] P.J. Richens and M.V. Berry, Physica D **2**, 495 (1981).

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[16] V.M. Alekseev and M.V. Yakobson, Phys. Rep. **75**, 287 (1981).

[17] For a general introduction to complexity and dynamics, see R. Badii and A. Politi, *Complexity, Hierarchical Structures and Scaling in Physics* (Cambridge University Press, Cambridge, 1996).

[18] I agree here with Echardt *et al.*'s appreciation in Ref. [7].

[19] Errors, i.e., different paths of nearby trajectories rebounding at vertices in polygonal billiards, have been studied, e.g., in A. Hobson, J. Math. Phys. **16**, 2210 (1975); G. Casati and J. Ford, J. Comput. Phys. **20**, 97 (1976); F. Henyey and N. Pomphrey, Physica D **6**, 78 (1982).

[20] P.J. Richens and M.V. Berry in [11] call these systems *pseudo-integrable*.

[21] A. Shudo [Phys. Rev. A **46**, 809 (1992)]; A. Shudo and Y. Shimizu, [Phys. Rev. E **47**, 54 (1993)] study the properties of the sequence of quantum eigenvalues in almost integrable billiards.

[22] L.A. Bunimovich, Commun. Math. Phys. **65**, 295 (1979).

[23] In particular, much attention is devoted to assuring that the original system and the resulting shift in the space of symbols are isomorphic. For this, a careful coding procedure is essential. In this paper, I shall assume that the coding imposes itself for its physical significance.

[24] This game is not as uncommon as it might seem: who has not been challenged, after a good shot, to *do it again if you can?*

[25] A.N. Kolmogorov, Probl. Peredachi Inf. **1**, 3 (1965).

[26] G. Chaitin, *Information, Randomnessand Incompleteness* (World Scientific, Singapore, 1987).

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[31] It is frequently claimed that chaos, especially when intended as deterministic randomness, is definable only in the infinite time limit. For finite sequences the indeterminacy due to the necessary recourse to a universal computer would prevent us from making a clearcut distinction between ordered and random sequences and (it is then pretended) no statements about the order or degree of chaos of a finite time evolution could be substantiated. Counting the number of sequences of length N with complexity less than j, $j \leq N$ (see, e.g., [5]) provides a first answer to this objection. The scaling solution of the problem adopted in this paper was first presented in [3] and then applied to the definition of chaos in systems with a finite number of states, where similar problems arise, in [34].

[32] This hints at the relation between complexity and Lyapunov exponents; see A. Brudno, Russ. Math. Surv. **33**, 197 (1978); [30].

[33] For a discussion of the proper quantum mechanical correspondence, see L.A. Ballentine, *Quantum Mechanics* (Prentice Hall, Englewood Cliffs, NJ, 1990). Ballentine has compared quantum dynamics with the classical dynamics of Liouville distributions: Phys. Rev. A **47**, 2592 (1993); **50**, 2854 (1994); **54**, 3813 (1996).

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[35] The periodic orbit theory of this function is interesting but will not be developed in this paper.

[36] M.C. Gutzwiller and B.B. Mandelbrot, Phys. Rev. Lett. **60**, 673 (1988); M.C. Gutzwiller, Physica D **38**, 160 (1989).

[37] D. Bessis and G. Mantica, Phys. Rev. Lett. **66**, 2939 (1991).

[38] The same result was within reach of Vega *et al.* [12], had they continued their circle calculations, Fig. 10, beyond their "saturation" value.

[39] When computing K from Λ, observe that the coding length C_{boundary} of the set of boundaries can be easily shown to be only logarithmically increasing in M.

[40] Establishing rigorous bounds on the algorithmic complexity of orbits in billiards is an important and difficult enterprise. Yet rigorous analysis cannot, at the moment, catch most of the phenomena outlined in this paper. Consider, for instance, the topological complexity of a symbolic dynamics, which is defined as the number of words of length less than or equal to $N$and is denoted by $|\Sigma_N|$ [10]. Controlling this number can give a bound on the algorithmic complexity of any orbit, via a lexicographic argument: $K(N)$ does not exceed a constant plus $\log_2(|\Sigma_N|)$. For rational polygonal billiards, perhaps the best available result [41] is $|\Sigma_N| \leq \text{const} \times N^3$; see also [42]. It must be observed that in this paper we group different sides in the same letter and we do *not* code certain rebounds (in the stadium, see text). This situation differs from that in which the estimate above has been derived. Yet, in adapting these estimates to the M-circle, one would get $K(N) \leq \text{const} + 3\log_2 N$, to be compared with the result following from Eq. (8), which

involves an average, in the sense explained in this paper. It appears that while our result agrees with the lexicographical estimate, this latter bears no evidence of the number of sides of the polygon and the related analysis has limited significance.
[41] P. Hubert, Thèse, Institut de Mathématiques de Luminy, Université de Marseille, France, 1995.
[42] S. Troubetzkoy, Chaos **8**, 242 (1998).
[43] Capitalized statement of a colleague that Joe Ford had provocatively posted on a wall in his office.
[44] This algorithmic fact, which is also typical of Schrödinger quantization, gives reason to the so-called Zaslavsky time scale in chaotic systems, $\tau \sim (1/\lambda)\log(S/\hbar)$, where S is the action and λ the maximum Lyapunov exponent: G.P. Berman and G.M. Zaslavsky, Physica A **91**, 450 (1978); G.M. Zaslavsky, Phys. Rep. **80**, 157 (1981). The expectation values of physical quantities in classical and quantum maps also agree on a finite time scale, whose scaling has been studied by M.V. Berry, N.L. Balasz, M. Tabor and A. Voros, Ann. Phys. (Leipzig) **122**, 26 (1979).
[45] The first example that comes to mind here is the dynamics of computation in digital machines.
[46] R.P. Feynman, when discussing computer simulations of physical systems in Int. J. Theor. Phys. **21**, 467 (1982), *rejects* as against the rules the outcome that computer size might grow exponentially with the size of the physical problem. *Mutatis mutandis*, this is precisely what we do when we reject polynomial-curved correspondence in the stadium billiard and quantum-classical correspondence in classically chaotic systems.
[47] J.H. Hannay and M.V. Berry, Physica D **1**, 267 (1980).
[48] G. Casati, I. Guarneri and F. Valz-Gris, Phys. Rev. A **30**, 1586 (1984); A. Shudo, Phys. Rev. Lett. **63**, 1897 (1989).
[49] Precisely because its algorithmic content is low, the quantum dynamics of chaotic systems *can* be computed efficiently (for times larger than the Zaslavsky time scale) by semiclassical algorithms: S. Tomsovic and E.J. Heller, Phys. Rev. E **47**, 282 (1993). It is interesting to read this paper on the quantum stadium in the informational-theoretical setting of algorithmic complexity proposed here and to compare it with K.M. Christoffel and P. Brumer, Phys. Rev. A **33**, 1309 (1986), also on the stadium billiard. Of course, this comparison must also include Ref. [4], which studies in full generality the complexity of the motion in bounded quantum systems and shows its computability.
[50] We have focused attention here on trajectories. When going back to the proper quantum-classical correspondence recall that it should be discussed in terms of many-to-many relations (for example, many classical trajectories to a set of quantum eigenstates), as in [2,3].
[51] Notice that we do not consider quantum systems open to external perturbations of stochastic nature, as, e.g., in R. Schack and C.M. Caves, Phys. Rev. E **53**, 3257 (1996) and in the theory of quantum decoherence, W. H. Zurek and J. P Paz, Physica D **83**, 300 (1995). One could say that in these theories the external environment is a reservoir of the ''missing'' complexity required for correspondence.
[52] B.V. Chirikov, in *Chaos and Quantum Physics*, edited by M.-J. Giannoni, A. Voros and J. Zinn-Justin (Elsevier, Amsterdam, 1991).
[53] Following Ford, I do not favor the term pseudochaos, for historical [8,15] and semantic reasons make *A* integrability preferable: pseudochaos and quantum chaology [57] as well, are negative definitions, in the sense that they vaguely hint at something *less*, or *other* than chaos; *A* integrability, on the contrary, positively says what replaces the simple Liouville-Arnol'd notion of integrability. On the side, it is interesting to remark that the term pseudointegrability [11] has also been proposed to denote the motion in rational billiards. The amusing note here is that the same dynamics is pseudointegrable for some authors and pseudochaotic for some others.
[54] The common usage is now to call quantum ''chaos'' the quantum dynamics of chaotic classical systems. Of the original problem, just the name has survived. For an introduction to the interesting physical aspects of this theory, see M.C. Gutzwiller, *Chaos in Classical and Quantum Systems* (Springer-Verlag, Berlin, 1980) and B.V. Chirikov and G. Casati, *Quantum Chaos* (Cambridge University Press, Cambridge, England, 1995).
[55] B.V. Chirikov and F. Vivaldi have further clarified this view in Physica D **129**, 223 (1999).
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