Simple model for mixing at accelerated fluid interfaces with shear and compression

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A simple model was recently described for predicting linear and nonlinear mixing at an unstable planar interface between two fluids of different density subjected to an arbitrary time-dependent variable acceleration history [J. D. Ramshaw, Phys. Rev. E **58**, 5834 (1998)]. Here we generalize this model to include the Kelvin-Helmholtz (KH) instability resulting from a tangential velocity discontinuity Δu, as well as the effects of a uniform anisotropic compression or expansion of the mixing layer as a whole. The model consists of a second-order nonlinear ordinary differential equation of motion for the half-width h of the mixing layer. This equation is derived by combining the wavelength renormalization hypothesis used in the earlier model with a suitable expression for the rate of change of the kinetic energy of the mixing layer. The resulting generalized model contains no additional free parameters, and reduces to the previous model in the absence of tangential velocities and compression. It also reduces in the linear regime to the correct linearized stability equation for an accelerated shear layer with compression [J. D. Ramshaw, Phys. Rev. E **61**, 1486 (2000)]. For a pure incompressible KH instability in the nonlinear regime, the model predicts that $h=\eta|\Delta u|t$, where $\eta=[\alpha(2-\theta)/\sqrt{\theta(1-\theta)}]\sqrt{\rho_1\rho_2}/(\rho_1+\rho_2)$, and α and θ are parameters appearing in the nonlinear Rayleigh-Taylor and Richtmyer-Meshkov growth laws. For equal densities and the same parameter values previously used to match variable-acceleration experimental data, we find $\eta=0.10$, in close agreement with experimental data for free shear layers.

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I. INTRODUCTION

There is a continuing lively interest in unstable fluid interfaces, both because of their intrinsic fascination and because they provide a mechanism for rapidly mixing together two fluids that would otherwise remain separated. Such mixing can be either desirable or undesirable depending on the circumstances. The degree to which the two fluids are mixed together by an instability can be characterized by the half-width $h(t)$ of the mixing layer as a function of the time t. The instability is typically seeded, at least in theoretical treatments, by introducing a small sinusoidal perturbation at $t=0$. As long as the perturbation remains small enough to permit linearization, it remains sinusoidal, and $h(t)$ may be identified with its amplitude. As the perturbation grows larger, the problem enters the nonlinear regime and the mixing layer becomes irregular and asymmetrical, with the penetration depth of the heavier fluid generally exceeding that of the lighter one. To preserve backward compatibility with earlier treatments, we shall adhere to the conventional definition of h as the visual penetration depth of the lighter fluid into the heavier one. It should be noted that negative values of h must be allowed in order to describe situations in which the interface undergoes stable oscillations and the initial displacement suffers periodic reversals in direction. When this occurs, the positive half-width of the mixing layer may be identified with $|h|$.

There are three classical interfacial instabilities, which are associated with the names of Rayleigh-Taylor (RT) [1–3], Richtmyer-Meshkov (RM) [4,5], and Kelvin-Helmholtz (KH) [1,2]. In practical situations these instabilities are rarely encountered in pure form; they usually occur in various hybrid combinations, of which an arbitrary variable acceleration history [6,7] is of particular interest. More generally, when a plane shear layer with a tangential velocity discontinuity Δu and density discontinuity $\Delta\rho$ is simultaneously subjected to a variable normal acceleration $a(t)$, the three basic instabilities become intermingled. As long as the disturbances remain small, the resulting motion can be completely described by means of a conventional linear stability analysis [1,2,8,9]. For larger values of h, however, the problem becomes nonlinear and can no longer be solved analytically. Approximations then become necessary, of which perhaps the simplest are models that take the form of heuristic nonlinear generalizations of the linear results. Models of this type have recently been described for accelerated interfaces between two incompressible fluids with $\Delta u=0$ (no KH instability) in both planar [7] and spherical [10] geometry. Our purpose here is to generalize the model of Ref. [7] to include the KH instability, as well as the effects of a slow uniform anisotropic compression or expansion of the mixing layer as a whole. The linear stability analysis for this situation was recently presented [9], and serves as a cornerstone for the development of the present nonlinear model. The resulting generalized nonlinear model then encompasses all three of the basic interfacial instabilites, either alone or in arbitrary combinations, including a fully consistent treatment of compression effects. It also reduces to the correct linear stability equation [9] in the linear regime.

The previous models [7,10] were developed by the following procedure. First, the kinetic energy T of the system is evaluated from the linearized potential flow solution for a single-mode perturbation of wavelength λ. The resulting expression for T is then extended into the nonlinear regime by means of a wavelength renormalization hypothesis (WRH), whereby λ is replaced by an effective wavelength which is postulated to be proportional to h. Finally, a nonlinear evolution equation for h is derived from the nonlinear expression for T by means of Lagrange's equations. The use of Lagrange's equations preserves the essential property of energy conservation (in the absence of dissipation), while the WRH captures the essential self-similar scaling behavior that such mixing layers are expected to exhibit. It was shown in Ref. [7] that the resulting simple evolution equation for h properly represents the known behavior of the RT and RM special cases in both the linear and nonlinear regimes, and produces solutions in good agreement with available experimental data for several different time-dependent variable acceleration histories [6].

The present development follows essentially the same procedure, but with one important difference: Lagrange's equations cannot be used in the present context, because h is no longer a proper generalized coordinate when $\Delta u \neq 0$. (That is to say, the positions of all the Lagrangian fluid particles in the system can no longer in principle be expressed as functions of h.) We therefore abandon Lagrange's equations in favor of a suitable expression for the rate of change of the kinetic energy of the mixing layer. This expression is directly derived from the local momentum equation.

The development proceeds along the following outline. In Sec. II we derive a suitable expression for the rate of change of the kinetic energy K of a nonuniform inviscid fluid subjected to an externally imposed uniform anisotropic compression or expansion. This expession is then specialized in Sec. III to the planar mixing layer of present interest, and the various quantities appearing therein are evaluated from the known linear potential flow solution [9]. These quantities are then heuristically extended into the nonlinear regime by means of the WRH, as discussed above. The nonlinear model equation of motion for h is derived in Sec. IV by requiring the resulting nonlinear expression for K to obey the kinetic energy equation of Sec. III, with a decay term introduced to represent the dissipation of kinetic energy into thermal energy in the nonlinear regime. It is remarkable that the resulting generalized model contains no new constants or parameters associated with the KH instability; it merely involves the same two parameters as the previous model [7], namely the WRH parameter b and the dissipation parameter c. These parameters are completely determined by pure incompressible RT and RM experiments in the absence of shear [7].

In Sec. V we examine the form and behavior of the model in various special cases. In particular, we verify that the present model properly reduces to the known linear evolution equation for h [9] in the linear regime, and to the previous model of [7] in the absence of shear and compression. The present model thereby inherits all of the special cases that the previous model [7] successfully represented. We also examine the asymptotic behavior of h in the nonlinear regime for a pure incompressible KH instability with no normal acceleration. In this case the model predicts that $h = \eta|\Delta u|t$, where the coefficient η reduces to 0.10 when $\rho_1 = \rho_2$, and b and c are taken to have the same values used to match variable-acceleration experimental data [6,7]. This is very close to the value of η inferred from experimental data for a free shear layer [11]. However, the significance of this agreement is somewhat uncertain, since there is no precise equivalence between spatially and temporally evolving shear layers. More detailed comparisons with experimental data are clearly required in any case. Nevertheless, this preliminary agreement, together with the other special cases already discussed, lends cause for optimism that the model will produce reasonable results in more compliciated situations involving simultaneous normal acceleration, transverse shear, and compression.

II. KINETIC ENERGY EQUATION

Our first task is to derive a suitable time evolution equation for the kinetic energy of an inhomogeneous inviscid fluid subjected to an externally imposed slow uniform but anisotropic compression or expansion. Such an equation can be derived from the local momentum equation for the fluid, which has the familiar form

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \rho\mathbf{G}, \tag{1}$$

where ρ and p are the local fluid density and pressure, respectively, and $\mathbf{G}$ is a uniform external body force per unit mass. The fluid velocity $\mathbf{u}$ is taken to be of the form $\mathbf{u} = \mathsf{D}\cdot\mathbf{r} + \mathbf{U}$, where $\mathbf{r}$ is the position vector, D is a symmetric dyadic which is constant in space, and $\nabla\cdot\mathbf{U} = 0$. Thus $\mathbf{U}$ is the incompressible part of the velocity field, from which the externally imposed uniform compression/expansion D has been removed. It follows that $\nabla\cdot\mathbf{u} = \mathsf{U}{:}\mathsf{D} \equiv D$, where U is the unit dyadic. Thus D is uniform in space, and this implies a restriction to slow compression or expansion; i.e., values of D which are much smaller than the rate at which pressure is equilibrated by acoustic waves. Under these conditions, the pressure field will be essentially uniform when $\mathbf{U} = \mathbf{G} = \mathbf{0}$, and it then follows from Eq. (1) that D must obey the condition

$$\dot{\mathsf{D}} + \mathsf{D}\cdot\mathsf{D} = 0. \tag{2}$$

A subset of this condition was obtained by a somewhat different argument in the linear theory [9].

Taking the dot product of $\mathbf{U}$ with Eq. (1), we obtain the local kinetic energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\mathbf{U}|^2\right) + \nabla\cdot\left(\frac{1}{2}\rho|\mathbf{U}|^2\mathbf{u}\right) + \rho\mathbf{U}\mathbf{U}{:}\mathsf{D} = -\nabla\cdot(p\mathbf{U}) + \rho\mathbf{U}\cdot\mathbf{G}, \tag{3}$$

where use has been made of Eq. (2) and the continuity equation $\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = 0$. The global kinetic energy equation is obtained by integrating Eq. (3) over a time-dependent volume V which is Lagrangian with respect to the compression velocities, so that the surface S of V moves with the local velocity $\mathsf{D}\cdot\mathbf{r}$. The resulting expression combines with the Reynolds transport theorem [12] to yield

$$\dot{K} = -2\mathsf{D}{:}\mathsf{K} + \mathbf{P}\cdot\mathbf{G} + \dot{K}_S, \tag{4}$$

where

$$\mathsf{K} = \int_V d\mathbf{r}\,\frac{1}{2}\rho\mathbf{U}\mathbf{U}, \tag{5}$$

$$K=\mathsf{K}:\mathsf{U}=\int_V d\mathbf{r}\,\frac{1}{2}\rho|\mathbf{U}|^2, \tag{6}$$

$$\mathbf{P}=\int_V d\mathbf{r}\rho\mathbf{U}, \tag{7}$$

$$\dot{K}_S=-\int_S dA\left(p+\frac{1}{2}\rho|\mathbf{U}|^2\right)\mathbf{U}\cdot\mathbf{n}_S, \tag{8}$$

and $\mathbf{n}_S$ is the outward unit normal to S. Equation (4) is the desired time evolution equation for the kinetic energy K of the incompressible part of the flow field. The term $\mathbf{P}\cdot\mathbf{G}$ represents the work done by the external body force, while the term $-2\mathsf{D}:\mathsf{K}$ represents the amplification of kinetic energy by compression. The latter term is analogous to a pdV work term, and to the similar terms that appear in the turbulent kinetic energy equation in compressible turbulence models.

III. ACCELERATED SHEAR LAYER WITH COMPRESSION

We now proceed to specialize the preceding general formulation to the physical situation of present interest, namely, an initially planar interface which separates two immiscible homogeneous fluids of different density and negligible surface tension. The density of fluid i is uniform with the value ρ_i, and the unperturbed velocity field of fluid i is $\mathbf{u}_i^D=\mathbf{u}_i^0+\mathsf{D}\cdot\mathbf{r}$, where $\mathbf{u}_i^0$ is uniform, $\mathbf{u}_i^0\cdot\mathbf{n}=0$, and $\mathbf{n}$ is the unit normal to the original unperturbed interface pointing from fluid 1 into fluid 2. The velocity gradient tensor D satisfies the conditions of the preceding section, including Eq. (2), and has the form [9]

$$\mathsf{D}=D_n\mathbf{nn}+\mathsf{D}_t, \tag{9}$$

where $D_n=\mathbf{n}\cdot\mathsf{D}\cdot\mathbf{n}$ and $\mathsf{D}_t\cdot\mathbf{n}=0$. Since D is uniform in space, the fluid densities ρ_i remain uniform within each fluid but depend upon time according to

$$\dot{\rho}_i=-D\rho_i. \tag{10}$$

Moreover, pressure equality at the unperturbed interface requires that the tangential velocities $\mathbf{u}_i^0$ obey the conditions [9]

$$\dot{\mathbf{u}}_i^0+\mathsf{D}\cdot\mathbf{u}_i^0=0. \tag{11}$$

The system is in zero gravity but is subjected to a normal acceleration $a(t)\mathbf{n}$. Just as in the linear theory [9], it is convenient to describe the system in a comoving accelerating Cartesian coordinate frame in which the unperturbed interface is stationary for all t. In this frame the system experiences an artificial external body force per unit mass of $\mathbf{G}=-a(t)\mathbf{n}$, the unperturbed interface is defined by $\mathbf{n}\cdot\mathbf{r}=0$ (with the understanding that the origin is located somewhere on the interface), and the unperturbed fluids 1 and 2 occupy the regions $\mathbf{n}\cdot\mathbf{r}<0$ and $\mathbf{n}\cdot\mathbf{r}>0$, respectively. It is also convenient to let the coordinate frame move parallel to the interface with the velocity of the linear KH surface waves [9]. These waves then become stationary in this system, and this implies [9]

$$\rho_1\mathbf{u}_1^0=-\rho_{12}\Delta\mathbf{u},\quad \rho_2\mathbf{u}_2^0=\rho_{12}\Delta\mathbf{u}, \tag{12}$$

where $\Delta\mathbf{u}=\mathbf{u}_2^0-\mathbf{u}_1^0$ is the tangential velocity discontinuity at the interface, and $\rho_{12}=\rho_1\rho_2/(\rho_1+\rho_2)$.

We now consider the effect of introducing a small sinusoidal perturbation of amplitude h into the interface location, so that the interface is now defined by $\mathbf{n}\cdot\mathbf{r}=hC$, where $C=\cos[\mathbf{k}(t)\cdot\mathbf{r}]$ and $\mathbf{k}\cdot\mathbf{n}=0$. Fluids 1 and 2 now occupy the regions $\mathbf{n}\cdot\mathbf{r}<hC$ and $\mathbf{n}\cdot\mathbf{r}>hC$, respectively. The time dependence of $\mathbf{k}$ is necessary to allow for the change in wavelength due to the transverse compression, which is determined by [9]

$$\dot{\mathbf{k}}=-\mathsf{D}\cdot\mathbf{k}=-\mathsf{D}_t\cdot\mathbf{k}. \tag{13}$$

For small h, the resulting potential flow fields $\mathbf{u}_i=\nabla\phi_i$, pressure distributions p_i, and interface dynamics can be analytically determined to first order in h by a conventional linear stability analysis [9]. We shall make use of these linear results to evaluate the various quantities in Eq. (4) as functions of h, $\mathbf{k}$, D, and the other parameters of the problem. To this end, we identify the volume V of Sec. II with a slab of cross-sectional area $\mathcal{A}$ bounded by the plane surfaces $\mathbf{n}\cdot\mathbf{r}=Z_1<0$ and $\mathbf{n}\cdot\mathbf{r}=Z_2>0$, where $|Z_i|\gg\max(2\pi/k,h)$, $\dot{\mathcal{A}}=(D-D_n)\mathcal{A}$ due to the transverse compression [9], and $\dot{Z}_i=D_nZ_i$ due to the normal compression. The masses $M_i=\rho_i\mathcal{A}|Z_i|$ are of course conserved and hence are constant in time, so that $\dot{M}_i=0$, which is easily verified by differentiation.

The quantities K, $2\mathsf{D}:\mathsf{K}$, $\mathbf{P}\cdot\mathbf{n}$, and $\dot{K}_S$ will be evaluated based on the known linearized potential flow solution [9]. This is done by setting $\rho=\rho_i$ and $\mathbf{U}=\mathbf{u}_i-\mathsf{D}\cdot\mathbf{r}=\mathbf{u}_i^0+\nabla\phi_i'$ in Eqs. (5)–(7), where ϕ_i' is given by Eq. (15) of Ref. [9], $i=1$ for $\mathbf{n}\cdot\mathbf{r}<hC$, and $i=2$ for $\mathbf{n}\cdot\mathbf{r}>hC$. Since K is quadratic in $\mathbf{U}$, these integrals must be evaluated to second order in h and/or $\dot{h}$ even to describe the linear regime. For this purpose it is essential to include the second-order effects of the perturbation on the integration limits. Just as in Ref. [7], however, it is unnecessary to evaluate the ϕ_i to second order, since the linearized interface dynamics is completely determined by the linear approximation to the ϕ_i [9]. The required integrations are tedious but straightforward, and the resulting second-order expressions for K, $2\mathsf{D}:\mathsf{K}$, and $\mathbf{P}\cdot\mathbf{n}$ are given by

$$K=\mathsf{K}_0:\mathsf{U}+\frac{\bar{\rho}}{2k}\mathcal{A}(\dot{h}-D_nh)^2-\frac{\rho_{12}}{4k}\mathcal{A}(\mathbf{kk}:\mathsf{T})h^2, \tag{14}$$

$$2\mathsf{D}:\mathsf{K}=2\mathsf{D}:\mathsf{K}_0+\frac{\bar{\rho}}{2k}\mathcal{A}(D_n+D_k)(\dot{h}-D_nh)^2+\frac{\rho_{12}}{4k}\mathcal{A}\mathbf{kk}:[(D_n+D_k)\mathsf{T}-4\mathsf{D}\cdot\mathsf{T}]h^2, \tag{15}$$

$$\mathbf{P}\cdot\mathbf{n}=-\frac{1}{2}\Delta\rho\mathcal{A}h(\dot{h}-D_nh), \tag{16}$$

where $2\mathsf{K}_0=M_1\mathbf{u}_1^0\mathbf{u}_1^0+M_2\mathbf{u}_2^0\mathbf{u}_2^0$, $2\bar{\rho}=\rho_1+\rho_2$, $\Delta\rho=\rho_2-\rho_1$, $\mathsf{T}=\Delta\mathbf{u}\Delta\mathbf{u}$, and $k^2D_k=\mathbf{k}\cdot\mathsf{D}\cdot\mathbf{k}$.

We now consider the surface term $\dot{K}_S$. The surfaces $\mathbf{n}\cdot\mathbf{r}=Z_i$ do not contribute to this term because $\mathbf{U}\cdot\mathbf{n}=0$ far from the interface. But the lateral portions of S do not contribute either due to the fact that both $\mathbf{U}$ and p are periodic in the $\mathbf{k}$ direction. [The periodicity of $\mathbf{U}=\mathbf{u}_i^0+\nabla\phi_i'$ is evident from Eq. (15) of Ref. [9], while the periodicity of p follows, after a little algebra, from Eqs. (9)–(11) and (16) of Ref. [9].] Thus there are no nonzero contributions to $\dot{K}_S$, so that

$$\dot{K}_S=0. \tag{17}$$

IV. DERIVATION OF THE NONLINEAR MODEL

The above expressions are based on the linearized flow field [9], so they clearly no longer strictly apply in the nonlinear regime. We shall nevertheless extend them into the nonlinear regime by means of the wavelength renormalization hypothesis discussed in Ref. [7], according to which $\lambda=2\pi/k$ retains the value $\lambda_0=2\pi/k_0$ for small $|h|$ [where k_0 now depends on time according to Eq. (13)] but becomes asymptotically proportional to $|h|$ for large $|h|$. The rationale for the WRH was discussed in detail in Ref. [7], and hence will not be repeated here. As emphasized in Ref. [7], the WRH does not lead to unique results in and of itself, and the manner in which it is introduced is crucial. By introducing this relation into Eqs. (15)–(17) and using Eq. (4) to determine the time evolution of $h(t)$, we automatically preserve the essential property of energy conservation, just as was done in Ref. [7] by the use of Lagrange's equations.

We shall take λ to have the same form as in [7], namely,

$$\lambda=\frac{2\pi}{k}=\max(\lambda_0,b|h|+(1-mb)\lambda_0), \tag{18}$$

where $\lambda_0(t)=2\pi/k_0(t)$ is the time-dependent wavelength of the initial perturbation in the linear regime, and $m\sim 1$ is the value of $|h|/\lambda_0$ at which the transition from linear to nonlinear behaviors occurs. As emphasized in Ref. [7], however, this is a primitive and highly oversimplified transition rule which should not be expected to be highly accurate, and more realistic alternatives should also be explored. When λ_0 is very small, however, the transition to the nonlinear regime occurs so quickly that the detailed manner in which it does so becomes relatively unimportant.

In the present context $\mathbf{k}$ is a vector, and it is necessary to specify its direction as well as its magnitude. This direction becomes ambiguous in the nonlinear regime, where the initial perturbation is presumably forgotten. In the nonlinear regime, however, λ becomes an effective wavelength which no longer literally represents the wavelength of a single-mode sinusoidal disturbance. The physical interpretation of λ in this regime is somewhat unclear; it presumably represents an appropriately weighted average over the unknown statistical distribution of length scales occurring in the mixing layer. This interpretation then implies that the vector $\mathbf{k}$ should not be regarded as having a unique but unknown direction in the nonlinear regime, but rather as having some statistical distribution of tangential directions. It then follows that the tensor $\mathbf{kk}$ appearing in Eqs. (15) and (16) should be interpreted as an appropriate average over the latter distribution, which we denote by $\langle\mathbf{kk}\rangle$. This distribution need not be isotropic, since $\Delta\mathbf{u}$ defines a preferred tangential direction. Indeed, during the early nonlinear stages of mixing, shear layers are known to be dominated by large-scale coherent structures aligned normal to $\Delta\mathbf{u}$ [11], and hence exhibit a high degree of tangential anisotropy. However, the degree to which this anisotropy persists in the asymptotic late-time regime of present interest is not known and has been controversial. In the absence of such information, we shall provisionally assume that this anisotropy can be neglected for purposes of evaluating $\langle\mathbf{kk}\rangle$, which can then be regarded as an isotropic average over all possible tangential directions. It is easy to verify that the result of this averaging is $\langle\mathbf{kk}\rangle=\frac{1}{2}k^2(\mathsf{U}-\mathbf{nn})$. Thus we let $\mathbf{kk}=\mathbf{k}_0\mathbf{k}_0$ when $|h|<m\lambda_0$, where $\mathbf{k}_0$ is the wave vector of the initial perturbation in the linear regime, which again depends on time according to Eq. (13), and

$$\mathbf{kk}=\langle\mathbf{kk}\rangle=\frac{1}{2}k^2(\mathsf{U}-\mathbf{nn}), \tag{19}$$

when $|h|>m\lambda_0$ in the nonlinear regime. In the latter case, $\dot{\mathbf{k}}$ is of course no longer determined by Eq. (13), but $\dot{\mathbf{k}}$ will not appear by itself; what is needed is $d\langle\mathbf{kk}\rangle/dt$, which is determined by Eqs. (18) and (19).

As discussed in Ref. [7], it is also necessary to allow for energy dissipation in the nonlinear regime. This can be done by introducing a suitable sink term into Eq. (4) to obtain

$$\dot{K}=-2\mathsf{D}:\mathsf{K}-a(t)\mathbf{P}\cdot\mathbf{n}-\Phi, \tag{20}$$

where use has been made of Eq. (17). Equation (20) is our final kinetic energy equation for the mixing layer. We shall assume that the dissipation rate Φ of kinetic energy in the nonlinear regime is controlled by the large scale motions, and is consequently independent of molecular viscosity, just as it is in turbulence [13]. In the absence of shear and compression, this implies that Φ must be of the form $\Phi=c\mathcal{A}\overline{\rho}|\dot{h}|^3$ [7]. In the present context, however, $\dot{h}$ is no longer the only velocity in the problem, so the form of Φ can no longer be determined by dimensional considerations alone. In this situation, it seems reasonable to base Φ on the velocity associated with the rate at which material masses are mixed together by the instability, so that no dissipation occurs in the absence of true mixing. The mass of fluid i which has moved across some Lagrangian area $\mathcal{A}$ of the original interface by time t is given by $m_i=\gamma\rho_i\mathcal{A}h$, where γ is of order unity and takes the value $1/\pi$ in the linear regime [9]. It follows that $\dot{m}_i=(m_i/h)(\dot{h}-D_nh)$, so that the velocity associated with the mixing of material masses is $(\dot{h}-D_nh)$. [As previously discussed, the second term subtracts out the purely geometric effects of the compression [9].] We therefore replace $\dot{h}$ in the previous expression for Φ by $(\dot{h}-D_nh)$ to obtain

$$\Phi=c\mathcal{A}\overline{\rho}|\dot{h}-D_nh|^3. \tag{21}$$

Since this form was obtained from inherently nonlinear considerations, Φ should be switched off in the linear regime by setting $c=0$ for $|h|<m\lambda_0$.

As shown in Ref. [7], the coefficients b and c are given by

$$b = \frac{\pi\theta}{\alpha(2-\theta)}, \tag{22}$$

$$c = \frac{2-3\theta}{4\alpha(2-\theta)}, \tag{23}$$

where α is the coefficient in the incompressible nonlinear RT growth law [14,15] $h = \alpha A a t^2$, and θ is the time exponent in the incompressible nonlinear RM growth law [16] $h \sim t^{\theta}$. Both α and θ may be measured experimentally, so the values of b and c may be regarded as known for present purposes.

We are now finally in a position to derive the model evolution equation for h by combining Eqs. (14)–(16) and (20). We thereby obtain

$$(\dot{h} - D_n h)\left[\frac{d}{dt}(\dot{h} - D_n h) + \frac{1}{2k}(D_k k - \dot{k})(\dot{h} - D_n h) - A k a(t) h - B(\mathbf{kk}{:}\mathsf{T})h + ck|\dot{h} - D_n h|(\dot{h} - D_n h)\right] = \Lambda, \tag{24}$$

where $A = (\rho_2 - \rho_1)/(\rho_2 + \rho_1) = \Delta\rho/(2\overline{\rho})$ is the Atwood number, $B = \rho_1\rho_2/(\rho_1 + \rho_2)^2 = \rho_{12}/(2\overline{\rho})$,

$$\Lambda = \frac{1}{2} B \mathsf{T}{:}\left[\frac{d}{dt}(\mathbf{kk}) + 2\mathsf{D}\cdot\mathbf{kk} - \frac{1}{k}(\dot{k} + D_k k)\mathbf{kk}\right] h^2, \tag{25}$$

and use has been made of Eqs. (2), (10), and (11). Equation (24) is the fundamental dynamical evolution equation of the model. It is a second-order nonlinear ordinary differential equation which determines $h(t)$ for an arbitrary given $a(t)$, $\Delta\mathbf{u}$, and D. Although Eqs. (2) and (11) imply a particular time dependence for the latter two quantities, these restrictions are not essential for reasons discussed in Ref. [9], so that Eq. (24) should still apply for any $\Delta\mathbf{u}$ and D that vary slowly in time. Similar considerations clearly apply to Eq. (9).

V. SPECIAL CASES

In the linear regime, Eq. (13) implies that $\dot{k} = -D_k k$, and it then follows immediately that $\Lambda = 0$ and $(D_k k - \dot{k}) = 2D_k k$. Equation (24) then reduces, when $c = 0$, to the known linear evolution equation for h, which is given by Eq. (22) of Ref. [9]. The model therefore behaves correctly in the linear regime.

In the case where shear and compression are both absent, $\Delta\mathbf{u} = \mathsf{D} = 0$ and Eq. (24) reduces, after a little algebra, to Eq. (13) of Ref. [7]. Equation (24) thereby inherits the full behavior of the previous model [7], which was shown to capture the known behavior of the pure RT and RM instabilities in both the linear and nonlinear regimes, and to agree reasonably well with available experimental data for several different variable acceleration histories [6].

We now proceed to examine the behavior of the model in the case of a pure incompressible KH instability in the nonlinear regime, where $a(t) = \mathsf{D} = 0$, $\lambda = 2\pi/k = b|h|$, and $\dot{k}/k = -\dot{h}/h$. Equations (24) and (25) then combine to give

$$h\ddot{h} + \frac{1}{2}\dot{h}^2 - B\left(\frac{\pi\Delta\mathbf{u}}{b}\right)^2 + \left(\frac{2\pi c}{b}\right)\frac{h}{|h|}|\dot{h}|\dot{h} = 0. \tag{26}$$

For $h, \dot{h} > 0$, Eq. (26) admits the asymptotic solution

$$h = \eta|\Delta\mathbf{u}|t, \tag{27}$$

where

$$\eta = \frac{\alpha(2-\theta)}{\sqrt{\theta(1-\theta)}}\frac{\sqrt{\rho_1\rho_2}}{\rho_1 + \rho_2}. \tag{28}$$

Equation (27) predicts that the mixing layer grows linearly in time with a rate proportional to $|\Delta\mathbf{u}|$, in agreement with elementary dimensional considerations. Equation (28) further predicts the value of the dimensionless coefficient η in terms of the density ratio and the parameters α and θ appearing in the corresponding growth laws for pure RT and RM instabilities, respectively [7]. The predicted dependence on density ratio is the same as that previously proposed on heuristic grounds by Youngs [17]. The predicted dependence on α and θ appears to be new and is somewhat remarkable, as it implies that the nonlinear growth behavior of a pure KH instability may be completely determined by performing pure RT and RM experiments, which might at first seem independent and unrelated. This prediction is subject to experimental verification. To this end, it is of interest to compare the value of η predicted by Eq. (28) with that inferred from experimental data on the growth rates of free shear layers [11].

Unfortunately, temporally evolving shear layers corresponding to a pure KH instability are difficult to produce in the laboratory. For this reason, most experiments are performed on spatially evolving shear layers occurring downstream of a splitter plate that separates two coflowing streams [11]. There is no precise equivalence between temporally and spatially evolving shear layers, so comparisons between them are subject to some uncertainty. However, it is widely believed that such comparisons are nevertheless meaningful, as it is clear that the two situations exhibit close similarities when an appropriate transformation between space and time is performed. The obvious transformation is $x = \overline{u}t$, where x is the downstream spatial coordinate and $\overline{u}$ is some appropriate mean disturbance velocity, which presumably must be a weighted average of the velocities u_1, u_2 of the two streams, i.e., $\overline{u} = \omega u_1 + (1-\omega)u_2$. However, the dependence of ω upon the density ratio ρ_1/ρ_2 is unclear. If $\overline{u}$ were the same as the linear KH wave speed, which seems intuitively reasonable, then we would have $\omega = \rho_1/(\rho_1 + \rho_2)$, but this identification is not supported by direct numerical simulations [17]. When the densities are equal, however, the value $\omega = 1/2$ is required by symmetry, thereby removing this source of uncertainty. We shall therefore restrict attention to the case $\rho_1/\rho_2 = 1$, in which the full visual width δ_{viz} of the mixing layer for large x was experimentally found to be $\delta_{\mathrm{viz}} = 0.19|\Delta u|x/\overline{u}$ [11]. This width corresponds to $2h$, and replacing x by $\overline{u}t$ then gives $h = 0.095|\Delta u|t$. Our best experimental estimate of η for $\rho_1/\rho_2 = 1$ is therefore $\eta = 0.095$.

To obtain a corresponding theoretical estimate from Eq. (28), we must choose values for the RT/RM parameters α and θ. We shall use the values $\alpha = 0.061$ and $\theta = 0.37$ determined in the variable-acceleration experiments of Dimonte and Schneider [6]. These values combine with Eq. (28) for $\rho_1 = \rho_2$ to yield $\eta = 0.103$, in very close agreement with the value inferred from δ_{viz} for the free shear layer. This is highly encouraging, especially since the model contains no new or additional adjustable parameters associated with the KH instability. However, this agreement may to some extent be fortuitous in view of the various uncertainties already discussed, particularly (a) the lack of equivalence between the temporally and spatially evolving shear layers, and (b) our use of an isotropic average over all tangential directions even though the shear layer exhibits highly anisotropic large-scale structures. In spite of these uncertainties, however, it is clear that the present generalized model provides a reasonably if not remarkably accurate representation of known interfacial instability behavior in a wide variety of special cases, and this lends some encouragement to the hope that it will continue to predict reasonable behavior in more complicated situations in which the features of these cases occur simultaneously in various combinations.

VI. CONCLUSION

We have presented a simple model, embodied in Eqs. (24) and (25), for predicting the half-width $h(t)$ of the mixing layer at an accelerated fluid interface with shear and compression. It was shown that the model correctly reproduces the known linear stability behavior for this case [9], and that it reduces to the previous model of Ref. [7] in the absence of shear and compression. The model thereby inherits the full behavior of the previous model, which was shown to correctly represent the known growth laws and scaling behavior for pure incompressible RT and RM instabilities in both the linear and nonlinear regimes and to agree reasonably well with experimental data for several different variable acceleration histories [6,7]. Using the same values for the RT and RM scaling parameters α and θ, the present model was also shown to provide an accurate description of the nonlinear growth of a free shear layer between two fluids of the same density. Thus the model correctly represents both the linear and nonlinear mixing behavior of unstable fluid interfaces in a wide variety of special cases. It is hoped that this model will provide a useful tool for making predictive estimates of mixing at unstable fluid interfaces where the canonical RT, RM, and KH instabilities occur in various combinations in the presence of anisotropic compression and/or expansion. Of course, a more definitive assessment of the accuracy and utility of the model will require more detailed comparisons with data from experiments and/or three-dimensional direct numerical simulations. Such comparisons will hopefully be forthcoming, and will most likely identify the need for further modifications and improvements. In particular, the model in its present form, like its predecessor [7], is especially simplistic in its treatment of demixing effects and the transition between the linear and nonlinear regimes. Other desirable enhancements would include the capability to represent ablation, multimode initial perturbations, and different compression rates in the two fluids. However, there are presumably limits to how much physics can be reasonably accommodated within simple models of this type. These limits are not yet clear, but may be expected to reveal themselves in due course.

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