Exact relationship for third-order structure functions in helical flows

T. Gomez,[1,2] H. Politano,[1] and A. Pouquet[1]
[1]*Observatoire de la Côte d'Azur, CNRS UMR 6529, Boîte Postale No. 4229, 06304 Nice Cedex 4, France*
[2]*Laboratoire de Modélisation en Mécanique, CNRS UMR 7606, Université Paris VI, Place Jussieu, Paris Cedex 5, France*
(Received 20 December 1999)

An exact law for turbulent flows is written for third-order structure functions taking into account the invariance of helicity, a law akin to the so-called ''4/5 law'' of Kolmogorov. Here, the flow is assumed to be homogeneous, incompressible and isotropic but not invariant under reflectional symmetry. Our result is consistent with the derivation by O. Chkhetiani [JETP Lett. **10**, 808, (1996)] of the von Kármán–Howarth equation in the helical case, leading to a linear scaling relation for the third-order velocity correlation function. The alternative relation of the Kolmogorov type we derive here is written in terms of mixed structure functions involving combinations of differences of all components for both the velocity and vorticity fields. This relationship could prove to be a stringent test for the measuring of vorticity in the laboratory, and provide a supplementary tool for the study of the properties of helical flows.

PACS number(s): 47.27.Gs, 47.27.Jv

I. INTRODUCTION

Turbulence, as studied in the laboratory and as observed in geophysical and astrophysical flows, resists analysis, contrary to critical phenomena where anomalous exponents of scaling laws can be computed using the renormalization group approach. In the context of intermittency, however, recent progress has been made for passive scalars [1] and passive vectors for the kinematic dynamo in magnetohydrodynamics (MHD) [2]; in both cases, the scaling exponents of structure functions (defined below) can be computed perturbatively for specified forcing and velocity statistics.

Thus, for the fully nonlinear problem of a turbulent velocity field, experimental and numerical investigations, as well as phenomenological models, still play a prevalent role. However, there are also exact laws, namely the so-called ''4/5'' law of Kolmogorov [3] and the equivalent relation [4] involving a functional of the total energy, advected by the longitudinal velocity increment δu_L (see below). Both laws show that the third-order structure functions of the velocity obey linear scaling relationships in the inertial range where self-similarity holds. The latter law writes

$$\langle \delta u_L(\mathbf{r})[\delta u_i(\mathbf{r})]^2\rangle = -\frac{4}{3}\overline{\epsilon} r \tag{1}$$

(where summation over repeated indices will be understood throughout the paper); $\overline{\epsilon} = -\dot{E}^V$ is the kinetic energy injection (and transfer) rate with $E^V = \frac{1}{2}\langle \mathbf{u}^2\rangle$. We hereafter call Eq. (1) the ''4/3'' law; it obtains rigorously from the Navier-Stokes equation written in terms of averages for velocity increments

$$\delta u_i(\mathbf{r}) \equiv u_i(\mathbf{x}+\mathbf{r}) - u_i(\mathbf{x}), \tag{2}$$

assuming homogeneity, isotropy, stationarity, and incompressibility. One further concentrates on the inertial range of turbulent flows where the dissipative terms can be neglected. The 4/3 law can also be derived starting from the 4/5 law of Kolmogorov which reads

$$\langle \delta u_L^3(\mathbf{r})\rangle = -\frac{4}{5}\overline{\epsilon} r; \tag{3}$$

$\delta u_L(\mathbf{r})$ is the longitudinal component of velocity differences, with $\delta u_L(\mathbf{r}) \equiv [\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})]\cdot\hat{\mathbf{r}}$ and $\hat{\mathbf{r}} = \mathbf{r}/r$. Hereafter, the subscript L denotes components of the field along the separation $\mathbf{r}$; similarly, n stands for any of its two transverse components, whereas the indices (2,3) are used when one needs to differentiate between these two transverse components. Indeed, starting from Eq. (3) and using the differential relationship between the fully longitudinal component of the velocity third-order structure function, and its mixed longitudinal-transverse component, stemming from the incompressibility condition, viz, $6\langle \delta u_L(\mathbf{r})\delta u_n^2(\mathbf{r})\rangle = \partial[r\langle \delta u_L^3(\mathbf{r})\rangle]/\partial r$, one arrives easily at Eq. (1), in the case of the full isotropy assumption.

The above exact laws derive from the conservation of energy in the absence of dissipation. Another invariant of the inviscid fluid equations is the kinetic helicity [5]:

$$H^V = \frac{1}{2}\langle \mathbf{u}\cdot\boldsymbol{\omega}\rangle, \tag{4}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity. This invariance is of a topological nature, dealing with the knottedness of vortex lines [5]. The role of helicity has been reviewed in [6]. For magnetized fluids, helicity is known to be important in the large-scale dynamo problem [7], and is also invoked in the study of solar flares and coronal mass ejection; in neutral fluids, it slows the dynamics as exemplified in [8], and it also plays a role in the evolution of hurricanes. Indeed, when helicity is maximal, as measured by the ratio $\tilde{\rho} = \langle \mathbf{u}\cdot\boldsymbol{\omega}\rangle/\sqrt{\langle \mathbf{u}^2\rangle\langle \boldsymbol{\omega}^2\rangle}$, with $|\tilde{\rho}| \leq 1$, and with a maximal state for $|\tilde{\rho}| = 1$, the nonlinear terms of the Navier-Stokes equations cancel exactly except for pressure. Note that such Beltrami flows with $\mathbf{u} = \pm\tilde{l}_0\boldsymbol{\omega}$ [9] are unstable (see, for example, [10]). More recently, the role of helicity was also studied in [11] in the context of drag reduction [12]. Thus, the purpose of this paper is to derive an exact law concerning third-order *struc-*

ture functions for helical flows, taking into account the presence of this second invariant, namely helicity. Chketiani [13] derived the von Kármán–Howarth equation for the helical case and deduced a linear scaling law for third-order correlation function of the velocity. Our work was done independently of the derivation in [13], and we use a different route, namely that of Antonia *et al.* [4], dealing directly with the temporal evolution of structure functions. Such functions, built on powers of the increments of the field components, are particularly useful in the study of turbulent flows because they allow for an elimination of the effect of large-scale motion such as sweeping: dealing with velocity differences, they stress the (presumably universal) small-scale statistical properties of such flows.

II. KINEMATICS OF HELICAL FLOWS

The first steps of our derivation are of a kinematical nature. With helicity, the structure of tensors differs from that in the full isotropic case. For example, a third-order tensor can no longer be assumed to be symmetric in its first two indices; indeed, the equation for helicity involves cross correlations between the velocity and the vorticity.

We begin with the second-order energy tensor $R_{ij}(\mathbf{r})$ which, assuming homogeneity and skew isotropy, can be written as [14]

$$\langle u_i(\mathbf{x})u_j(\mathbf{x}')\rangle = F(r)r_ir_j + G(r)\delta_{ij} + \tilde{H}(r)\epsilon_{ijk}r_k, \quad (5)$$

where $\mathbf{x}' \equiv \mathbf{x}+\mathbf{r}$ and ϵ_{ijk} is the unit alternating tensor. Helicity is included through the term proportional to $\tilde{H}(r)$, a pseudoscalar [15]. Using incompressibility [i.e., $\partial R_{ij}(\mathbf{r})/\partial r_i = 0$], it is well known that in the helical case, two defining functions remain, linked to energy and helicity. Hence, one can also write [14]

$$R_{ij}(\mathbf{r})/\overline{u}^2 = f(r)\delta_{ij} + \frac{rf'(r)}{2}P_{ij}(r) + \epsilon_{ijk}(r_k/r)\tilde{g}(r), \quad (6)$$

where a prime symbol in a function denotes a derivative with respect to r; as usual $P_{ij}(r) = \delta_{ij} - (r_ir_j)/r^2$ is the incompressibility projector, $\overline{u}^2 f(r)$ the longitudinal correlation function of the velocity with $\overline{u}^2 = \frac{1}{3}\langle \mathbf{u}^2\rangle$; one also introduces a new function $\tilde{g}(r)$ with

$$R_{23}(r) = r\tilde{H}(r)\epsilon_{23L} \equiv \overline{u}^2\epsilon_{23L}\tilde{g}(r).$$

A similar definition holds for the helicity tensor, with

$$\tilde{R}_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x})\omega_j(\mathbf{x}')\rangle = \epsilon_{jab}\frac{\partial R_{ib}(\mathbf{r})}{\partial x'_a}. \quad (7)$$

An equation for the temporal evolution of the energy and helicity will involve third-order tensors of the type

$$\phi^{(u)}_{ijl}(\mathbf{r}) = \langle u_i(\mathbf{x})u_j(\mathbf{x})u_l(\mathbf{x}')\rangle,$$

$$\phi^{(uu\omega)}_{ijl}(\mathbf{r}) = \langle u_i(\mathbf{x})u_j(\mathbf{x})\omega_l(\mathbf{x}')\rangle,$$

and

$$\phi^{(\omega uu)}_{ijl}(\mathbf{r}) = \langle \omega_i(\mathbf{x})u_j(\mathbf{x})u_l(\mathbf{x}')\rangle.$$

It can be shown that, as for the second-order tensor, only two basic functions remain to define the first two third-order tensors which are symmetric in their first two indices, using again incompressibility [i.e., $\partial\phi^{(u)}_{ijl}(\mathbf{r})/\partial r_l = 0$]. We thus define the two functions $k(r)$ and $\tilde{\alpha}(r)$ as

$$\phi^{(u)}_{LLL}(\mathbf{r}) = \langle u_L(\mathbf{x})u_L(\mathbf{x})u_L(\mathbf{x}')\rangle \equiv \overline{u}^3 k(r), \quad (8)$$

$$\phi^{(u)}_{L23}(\mathbf{r}) = \langle u_L(\mathbf{x})u_2(\mathbf{x})u_3(\mathbf{x}')\rangle \equiv -\overline{u}^3\epsilon_{L23}\tilde{\alpha}(r), \quad (9)$$

where $\tilde{\alpha}(r)$ is a pseudoscalar function. Thus

$$\phi^{(u)}_{ijl}(\mathbf{r})/\overline{u}^3 = \frac{k-rk'}{2r^3}r_ir_jr_l + \frac{2k+rk'}{4r}(r_i\delta_{jl}+r_j\delta_{il}) - \frac{k}{2r}r_l\delta_{ij} + \frac{\tilde{\alpha}}{r^2}(r_j\epsilon_{lim} - r_i\epsilon_{jlm})r_m. \quad (10)$$

All terms in Eq. (10) but the last one are standard [14], and the presence of skew-isotropy leads to the introduction in the expression for $\phi^{(u)}_{ijl}(\mathbf{r})$ and $\phi^{(uu\omega)}_{ijl}(\mathbf{r})$ of antisymmetric tensors, thus proportional to the unit alternating tensor.

Defining the tensor $\phi^{(\omega uu)}_{ijl}(\mathbf{r})$, on the other hand, is slightly more complex since it is not symmetric in its first two indices; one first writes [16], following for example the approach of Robertson [17]:

$$\phi^{(\omega uu)}_{ijl}(\mathbf{r}) = E^{(\omega)}(r)r_j\epsilon_{lim}r_m + F^{(\omega)}(r)r_i\epsilon_{jlm}r_m + G^{(\omega)}(r)r_l\epsilon_{ijm}r_m + \tilde{A}^{(\omega)}(r)r_ir_jr_l + \tilde{B}^{(\omega)}(r)r_i\delta_{jl} + \tilde{C}^{(\omega)}(r)r_j\delta_{il} + \tilde{D}^{(\omega)}(r)r_l\delta_{ij}, \quad (11)$$

where again the functions noted with a tilde symbol are pseudoscalars. Using incompressibility, one can show that one first needs to introduce two new functions $\tilde{k}^{(\omega uu)}(r)$ and $\alpha^{(\omega uu)}(r)$ defined similarly as in (8) and (9), but for $\phi^{(\omega uu)}_{ijl}(\mathbf{r})$. On the other hand, for that same tensor, two more basic functions, respectively taken to be proportional to $\phi^{(\omega uu)}_{23L}(\mathbf{r})$ and $\phi^{(\omega uu)}_{2L2}(\mathbf{r})$, are now necessary to fully define the coefficients of the tensor, namely,

$$\phi^{(\omega uu)}_{23L}(\mathbf{r}) = \langle\omega_2(\mathbf{x})u_3(\mathbf{x})u_L(\mathbf{x}')\rangle \equiv c_3\epsilon_{23L}\beta(r), \quad (12)$$

$$\phi^{(\omega uu)}_{2L2}(\mathbf{r}) = \langle\omega_2(\mathbf{x})u_L(\mathbf{x})u_2(\mathbf{x}')\rangle \equiv c_3\tilde{q}(r), \quad (13)$$

where $c_3 = \overline{u}^3/\ell_0$ where ℓ_0 is a characteristic length scale; in this manner, all functions $\tilde{k}^{(\omega uu)}$, $\alpha^{(\omega uu)}$, β, and $\tilde{q}$ are dimensionless.

With these new functions required in the definitions of helical tensors, the possibility arises that new scaling relationships emerge that may *a priori* be different from that of the fully isotropic case, and the question arises as to the relevance of such new scaling laws.

III. DYNAMICS OF HELICAL FLOWS

Note first that in the evolution equation of the velocity correlation tensor $R_{ij}(\mathbf{r})$, the fact of taking into account the presence of helicity in the flow does not result in altering the 4/5 and 4/3 laws, because helicity involves nondiagonal terms of the energy tensor, and the energy equation of course deals with the trace of $R_{ij}(\mathbf{r})$. We now move on to the dynamics of helicity in turbulent flows. One simply writes an equation for the temporal evolution of the scalar $\langle \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle$ starting with the incompressible Navier-Stokes equation, where P is the pressure and ν is the viscosity

$$\partial_t u_i(\mathbf{x}) = -\partial_{x_j}[u_j(\mathbf{x}) u_i(\mathbf{x})] - \partial_{x_i} P(\mathbf{x}) + \nu \nabla^2 u_i(\mathbf{x}), \tag{14}$$

with $\partial_{x_i} u_i(\mathbf{x}) = 0$, and where any partial derivative, $\partial/\partial y$ say, is denoted ∂_y. Its version for vorticity obtains immediately taking its curl.

An equation for the structure functions of the velocity and vorticity is then deduced using homogeneity and the fact that the positions $\mathbf{x}$ and $\mathbf{x}'$ are independent variables, viz.,

$$\begin{aligned}\partial_t\langle \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle &+ \partial_{r_k}\langle \delta u_k(\mathbf{r}) \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle \\ &= \frac{1}{2}\partial_{r_k}\langle \delta\omega_k(\mathbf{r})[\delta u_i(\mathbf{r})]^2\rangle + 2\nu \partial_{r_k}\partial_{r_k}\langle \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle \\ &\quad - 4\nu\langle \partial_{x_k}\omega_i(\mathbf{x}) \partial_{x_k} u_i(\mathbf{x})\rangle. \end{aligned}\tag{15}$$

The above equation can be simplified, using the hypothesis of skew isotropy, leading to first-order tensors [18] through a projection onto the longitudinal direction, which write

$$\langle \delta u_k(\mathbf{r}) \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle = \frac{r_k}{r}\langle \delta u_L(\mathbf{r}) \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle, \tag{16}$$

$$\langle \delta\omega_k(\mathbf{r})[\delta u_i(\mathbf{r})]^2\rangle = \frac{r_k}{r}\langle \delta\omega_L(\mathbf{r})[\delta u_i(\mathbf{r})]^2\rangle. \tag{17}$$

Note that it is useful to write the last term in Eq. (15) as $-\frac{4}{3}\partial_{r_k}(\tilde{\epsilon} r_k)$ using $\tilde{\epsilon} = \nu\langle \partial_{x_k}\omega_i \partial_{x_k} u_i\rangle$ where

$$\tilde{\epsilon} = -\dot{H}^V \tag{18}$$

is the rate of transfer of helicity.

After projection, this leads for Eq. (15) to

$$\begin{aligned}\partial_t\langle \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle &+ \left(\frac{2}{r} + \partial_r\right)\langle \delta u_L(\mathbf{r}) \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle \\ &= \frac{1}{2}\left(\frac{2}{r} + \partial_r\right)\langle \delta\omega_L(\mathbf{r})[\delta u_i(\mathbf{r})]^2\rangle + \left(\frac{2}{r} + \partial_r\right) \\ &\quad \times[2\nu\partial_r\langle \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle] - \left(\frac{2}{r} + \partial_r\right)\left[\frac{4}{3}\nu\tilde{\epsilon} r\right]. \end{aligned}\tag{19}$$

Neglecting the first term in the left-hand side (lhs) of Eq. (19) (see, e.g., [4]) and excluding the possibility that nonregular solutions at $r=0$ occur, a first integral of Eq. (19) gives the desired relationship which now readily obtains in the inertial range, namely,

$$\langle \delta u_L(\mathbf{r}) \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle - \frac{1}{2}\langle \delta\omega_L(\mathbf{r})[\delta u_i(\mathbf{r})]^2\rangle = -\frac{4}{3}\tilde{\epsilon} r. \tag{20}$$

This equation is the main result of the paper. It involves combinations of all the components of the velocity and vorticity fields. Two different terms are present on the lhs, contrary to the 4/5 and 4/3 laws; this stems from the fact that the nonlinearity in the vorticity equation can be decomposed into two parts, advection and stretching; the first term in Eq. (20) is similar to the energy law (1), appearing as an advection by δu_L of the helicity in terms of increments.

Note that for a Beltrami flow, the approach used to obtain this exact law, which relies on an evaluation of nonlinear transfer, does not apply since any pressure gradient term cancels out upon averaging. In fact, with the condition of invariance of the Loitsianskii integral $\mathcal{L} \sim \int_0^\infty r^4 f(r)$, the energy spectrum is constrained to behave in the large scales ($k \to 0$) as k^4 whereas the similar constraint on helicity is that it behaves as k^6, indicating that at least in that case and at large scale, helicity is not maximal and flows are far from being Beltrami-like [19]. Moreover, as stated before, such flows are unstable.

IV. COMPATIBILITY WITH THE VON KÁRMÁN-HOWARTH APPROACH

As mentioned in the Introduction, an earlier approach to arrive at an exact law in helical turbulence—that taken in [13]—consists of following the steps of the derivation of the 4/5 law of Kolmogorov, for which an essential intermediate step is the derivation due to von Kármán and Howarth (VKH) [20] of an equation, relating second and third order moments of the velocity. The two approaches are compatible, as already known for the 4/5 law of Kolmogorov, and as shown below for the helical case.

The VKH approach contrasts with the present one, the latter taking *ab initio* a formulation with Galilean-invariant expressions involving structure functions throughout. The VKH equation for helical flows is derived in [13] in terms of tensors involving only the velocity correlation functions, without obtaining the equivalent of a 4/5 or 4/3 law for structure functions in the helical case, because of the technical difficulty of expressing third-order structure functions in terms of correlation functions, when keeping a formulation that involves only the velocity, i.e. , in terms of $\phi_{ijl}^{(u)}(\mathbf{r})$ only, instead of employing, as done here, $\phi_{ijl}^{(uu\omega)}(\mathbf{r})$ and $\phi_{ijl}^{(\omega uu)}(\mathbf{r})$ as well.

The VKH equation for helical flows can also be obtained from Eq. (19); one first simply develops the mixed (velocity, vorticity) third-order structure functions in terms of correlation functions,

$$\frac{1}{2}\langle \delta\omega_L(\mathbf{r})[\delta u_i(\mathbf{r})]^2\rangle = 2\phi_{LLL}^{(\omega uu)} + 4\phi_{Lnn}^{(\omega uu)}, \tag{21}$$

and

$$\begin{aligned}\langle \delta u_L(\mathbf{r}) \delta u_i(\mathbf{r}) \delta\omega_i(\mathbf{r})\rangle = 4(&\phi_{Lnn}^{(uu\omega)} + \phi_{nLn}^{(\omega uu)} + \phi_{nnL}^{(\omega uu)}) \\ &+ 4\phi_{LLL}^{(\omega uu)} + 2\ \ \phi_{LLL}^{(uu\omega)}, \end{aligned}\tag{22}$$

with no summation on n. The equation equivalent to VKH but for helicity is now derived from Eq. (19), proceeding in two distinct steps: (i) express the temporal derivative in terms of correlation functions,

$$\partial_t\langle \delta u_i \delta\omega_i\rangle = +2\partial_t\langle u_i\omega_i\rangle - \frac{2}{r^2}\frac{\partial}{\partial r}(r^3\partial_t\tilde{R}_{LL}), \quad (23)$$

and note that $\partial_t\langle u_i\omega_i\rangle = -2\tilde{\epsilon}$; and (ii) in order to express all nonlinear terms, use Eqs. (21) and (22) together with a relationship stemming from incompressibility, viz., $\phi^{(\omega uu)}_{nnL} = -\frac{1}{2}\phi^{(\omega uu)}_{LLL}$. Taking now a first integral of the resulting equation leads to

$$\partial_t\tilde{R}_{LL}(\mathbf{r}) = \frac{1}{r}[2\phi^{(uu\omega)}_{Lnn} + \phi^{(uu\omega)}_{LLL} + 2\phi^{(\omega uu)}_{nLn} - 2\phi^{(\omega uu)}_{Lnn}] + 2\nu\frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\frac{\partial\tilde{R}_{LL}(\mathbf{r})}{\partial r}\right]. \quad (24)$$

Equation (24) agrees with the equation derived in [13]: indeed, one has to note that the nonlinear terms in the right-hand side (rhs) of (24) can also be written as $(4/r^4)\partial_r(r^3\phi^{(u)}_{L23})$. In the inertial range, this becomes

$$\frac{2}{r^4}\frac{\partial}{\partial r}[r^3\phi^{(u)}_{L32}(\mathbf{r})] = -\frac{1}{3}\tilde{\epsilon}, \quad (25)$$

which leads to

$$\phi^{(u)}_{L32}(\mathbf{r}) = \langle u_L(\mathbf{x})u_3(\mathbf{x})u_2(\mathbf{x}')\rangle = -\tilde{\epsilon}r^2/30. \quad (26)$$

It is thus shown that the above equations already derived in [13] and Eq. (20), obtained in this paper, are equivalent [21]; however, Eq. (26) does not write directly in terms of structure functions as opposed to Eq. (20).

Finally, note that it is unlikely that a law in terms *only* of longitudinal components, as in the 4/5 law of Kolmogorov, would arise for helicity since helicity involves all indices of tensors, dissociating the two normal components, as can be seen for example from the defining function $\tilde{\alpha}(r)$ needed in order to specify the third-order velocity tensor.

V. CONCLUSION

In this paper we have derived an exact law for mixed velocity-vorticity third–order *structure* functions, based on the conservation of helicity in turbulent fluids, and which is compatible with the derivation by Chketiani [13] of a law for helical fluids in terms of velocity third-order *correlation* functions. This law may help explain the role of helicity in turbulent flows as can be studied in experiments, both in the laboratory and with numerical computations, the latter perhaps providing an easier access to vorticity, although there are now proven techniques to probe the vorticity field, using for example the acoustic scattering by vorticity [22] or four-wire probe configurations (see [23] and references therein). This law also complements that derived for enstrophy and involving as well the vorticity field, experimentally measured by a vorticity probe comprising four X wires [24].

The exact laws (20, derived here) and (26, derived in [13]) put dynamical constraints on energy transfer to small scales, since it states that the mixed correlators between velocity and vorticity at third order (in fact, their difference) must scale as the distance r. Note now that if we assume that, dimensionally, the vorticity were to scale as u/r, Eq. (20) would give for the helical part of the flow a r^2 scaling for the third-order *structure* function; this is of course dimensionnally consistent with the 4/5 scaling law since $\tilde{\epsilon}\sim\epsilon/r$. With $\omega\sim u/r$, Eq. (20) does corroborate the scaling of Eq. (26). The derivation of this law has involved several new functions [see relations (9), (12), (13)], the scaling of which might be worth studying separately to see whether any subdominant behavior arises.

The Batchelor analogy in MHD between vorticity and magnetic induction—based on the similarity of the equation for vorticity with Ohm's law—seems to break down for structure functions built only either on $\delta\mathbf{b}$ or on $\delta\boldsymbol{\omega}$: we know already that at the level of spectra, the vorticity spectrum is singular—were it not for viscosity, whereas the magnetic energy spectrum is not singular [25,26]. The scaling of the magnetic structure functions in statistically steady flows has already been evaluated with direct numerical simulations in two space dimensions [27]. It may be of interest to see whether the scaling of velocity structure functions $\langle[\delta u_L(\mathbf{r})]^p\rangle$ for $p>3$ is sensitive or not to helicity, using $\tilde{Y}(\mathbf{r})$ [defined as the lhs of Eq. (20)] as a length scale to extend the range of power-law scaling. Futhermore, one may analyze—either with experimental or with numerical data—the scaling of the pth-order helical fluxes $\tilde{Y}^{p/3}(\mathbf{r})$. Indeed, both for the passive scalar [28], and for MHD [29], it was found that the fields built as powers of the *flux* of increments of conserved quantities—as defined here for helicity—may be scaling as for the velocity itself in neutral fluids, although the basic physical fields—temperature, or velocity and magnetic field—are more intermittent.

Thus, the law derived here, involving the helicity transfer rate in an exact relationship for structure functions, can be of some use in finding empirically the scaling laws that arise for turbulent fluids [30] and in studying the different regimes that may arise in helical fluids.

ACKNOWLEDGMENTS

We are thankful for the partial financial support we received from Programme National, CNRS PCMI, and PNST.

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[30] The derivation of the equivalent relationship for helical flows but in MHD will be reported elsewhere.