Quantum integrable system with two color components in two dimensions

Mu-Lin Yan
Center for Fundamental Physics, University of Science and Technology of China, Hefei Anhui 230026, People's Republic of China
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The Davey-Stewartson 1 (DS1) system [Proc. R. Soc. London, Ser. A **338**, 101 (1974)] is an integrable model in two dimensions. A quantum DS1 system with two color components in two dimensions has been reformulated. This two-dimensional problem has been reduced to two one-dimensional many-body problems with two color components. The solutions to the two-dimensional problem under consideration can be constructed from the resulting problem in one dimensions. For the latter problem with δ-function interactions and solution by the Bethe ansatz, we introduce symmetrical and antisymmetrical Young operators of the permutation group and obtain the exact solutions for the quantum DS1 system.

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I. INTRODUCTION

The Davey-Stewartson 1 (DS1) system is an integrable model in space of two spatial and one temporal dimensions [$(2+1)$D]. The quantized DS1 system with scalar fields (1 component, or 1C) can be formulated in terms of the Hamiltonian of quantum many-body problem in two dimensions, and some of them can be solved exactly [1,2]. Particularly, it has been shown in Ref. [2] that these two-dimensional (2D) quantum N-body system with 1C fields can be reduced to the solvable 1D quantum N-body systems with 1C fields and with two-body potentials [3]. Thus through solving 1D quantum N-body problems with 1C fields we can get the solutions for 2D quantum N-body problems with 1C fields. Here, the key step is to separate the spatial variables of 2D quantum N-body problems with 1C fields by constructing an ansatz [1,2]

$$\Psi(\xi_1,\ldots,\xi_N,\eta_1,\ldots,\eta_N) = \prod_{i<j}\left[1-\frac{c}{4}\epsilon(\xi_{ij})\epsilon(\eta_{ij})\right] \times X(\xi_1,\ldots,\xi_N)Y(\eta_1,\ldots,\eta_N),$$

where $\xi_{ij}=\xi_i-\xi_j$ and $\eta_{ij}=\eta_i-\eta_j$. This ansatz will be called the N-body variable-separation ansatz. It is well known that the variable-separation methods are widely used in solving high-dimensional one-particle problems. For instance, for getting the wave functions of electron in hydrogen atom, the ansatz $\Psi(r,\theta,\phi)=R(r)P(\theta)\Phi(\phi)$ is used to reduce the 3D problem to the 1D problem (this ansatz is what we call the one-body variable-separation ansatz). The N-body variable-separation ansatz can be thought of as the extension of one-body variable-separation ansatz. Since the N-body problems are much more complicated than the one-body problems, it will be highly nontrivial to construct a N-body variable-separation ansatz. Reference [1] provided the first example for it and showed that the idea of variable separating works indeed for the N-body problems induced from the DS1 system.

In this paper, we intend to generalize the above idea to the multicomponent DS1 system, namely, to construct a N-body variable-separation ansatz for the multicomponent case and to solve a specific model of 2D quantum DS1 system with multicomponents.

The 1D N-body model with two components has been investigated for long time [4,5]. The most famous one is the model with delta-function interaction between 2C fermions [4]. It was solved by the Bethe ansatz [6] and leads to the Yang-Baxter equation and its thermodynamics studies [7,8] because of the completeness of the Bethe ansatz solutions. In this paper, for definiteness, we shall study specific 2D quantum N-body system with 2C fields associated with the DS1 system. This quantum N-body problem under consideration can be reduced to two 1D quantum N-body problems with 2C fields of Ref. [4] and then be exactly solved by using an appropriate N-body variable-separation ansatz and the Bethe ansatz.

II. QUANTUM DS1 SYSTEM WITH TWO COMPONENTS IN TWO DIMENSIONS

Following the usual DS1 equation [1,9], the equation for the DS1 system with two components reads

$$i\dot{\mathbf{q}}=-\tfrac{1}{2}(\partial_x^2+\partial_y^2)\mathbf{q}+iA_1\mathbf{q}+iA_2\mathbf{q}, \tag{1}$$

where $\mathbf{q}$ has two color components,

$$\mathbf{q}=\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{2}$$

and

$$(\partial_x-\partial_y)A_1=-ic(\partial_x+\partial_y)(\mathbf{q}^\dagger\cdot\mathbf{q}),$$

$$(\partial_x+\partial_y)A_2=ic(\partial_x-\partial_y)(\mathbf{q}^\dagger\cdot\mathbf{q}),$$

where † means the Hermitian transposition, and c is the coupling constant. Introducing the coordinates $\xi=x+y, \eta=x-y$, we have

$$A_1=-ic\partial_\xi\partial_\eta^{-1}(\mathbf{q}^\dagger\cdot\mathbf{q})-iu_1(\xi), \tag{3}$$

$$A_2=ic\partial_\eta\partial_\xi^{-1}(\mathbf{q}^\dagger\cdot\mathbf{q})+iu_2(\eta), \tag{4}$$

where

$$\partial_\eta^{-1}(\mathbf{q}^\dagger\cdot\mathbf{q})=\frac{1}{2}\left(\int_{-\infty}^{\eta}d\eta'-\int_{\eta}^{\infty}d\eta'\right)\mathbf{q}^\dagger(\xi,\eta',t)\cdot\mathbf{q}(\xi,\eta',t), \tag{5}$$

and u_1 and u_2 are constants of integration. According to Ref. [2], we choose them as

$$u_1(\xi)=\tfrac{1}{2}\int d\xi' d\eta' U_1(\xi-\xi')\mathbf{q}^\dagger(\xi',\eta',t)\cdot\mathbf{q}(\xi',\eta',t), \tag{6}$$

$$u_2(\eta)=\tfrac{1}{2}\int d\xi' d\eta' U_2(\eta-\eta')\mathbf{q}^\dagger(\xi',\eta',t)\cdot\mathbf{q}(\xi',\eta',t). \tag{7}$$

Thus, Eq. (1) can be written as

$$\begin{aligned}i\dot{\mathbf{q}}=&-(\partial_\xi^2+\partial_\eta^2)\mathbf{q}+c[\partial_\xi\partial_\eta^{-1}(\mathbf{q}^\dagger\cdot\mathbf{q})+\partial_\eta\partial_\xi^{-1}(\mathbf{q}^\dagger\cdot\mathbf{q})]\mathbf{q}\\&+\tfrac{1}{2}\int d\xi' d\eta'[U_1(\xi-\xi')+U_2(\eta-\eta')](\mathbf{q}^{\dagger'}\cdot\mathbf{q}')\mathbf{q},\end{aligned} \tag{8}$$

where $\mathbf{q}'=\mathbf{q}(\xi',\eta',t)$. We quantize the system with the canonical commutation relations

$$[q_a(\xi,\eta,t),q_b^\dagger(\xi',\eta',t)]_\pm=2\delta_{ab}\delta(\xi-\xi')\delta(\eta-\eta'), \tag{9}$$

$$[q_a(\xi,\eta,t),q_b(\xi',\eta',t)]_\pm=0, \tag{10}$$

where $a,b=1$ or 2, $[,]_+$ and $[,]_-$ are anticommutator and commutator, respectively. Then Eq. (8) can be written in the form

$$\dot{\mathbf{q}}=i[H,\mathbf{q}], \tag{11}$$

where H is the Hamiltonian of the system

$$\begin{aligned}H=\tfrac{1}{2}\int d\xi d\eta\Big\{&-\mathbf{q}^\dagger(\partial_\xi^2+\partial_\eta^2)\cdot\mathbf{q}\\&+\frac{c}{2}\mathbf{q}^\dagger[(\partial_\xi\partial_\eta^{-1}+\partial_\eta\partial_\xi^{-1})(\mathbf{q}^\dagger\cdot\mathbf{q})]\cdot\mathbf{q}\\&+\tfrac{1}{4}\int d\xi' d\eta'\mathbf{q}^\dagger[U_1(\xi-\xi')+U_2(\eta-\eta')]\\&\times(\mathbf{q}'^\dagger\cdot\mathbf{q}')\cdot\mathbf{q}\Big\}.\end{aligned} \tag{12}$$

The N-particle eigenvalue problem is

$$H|\Psi\rangle=E|\Psi\rangle, \tag{13}$$

where

$$\begin{aligned}|\Psi\rangle=&\int d\xi_1 d\eta_1\cdots d\xi_N d\eta_N\\&\times\sum_{a_1\cdots a_N}\Psi_{a_1\cdots a_N}(\xi_1\eta_1\cdots\xi_N\eta_N)\\&\times\mathbf{q}_{a_1}^\dagger(\xi_1\eta_1)\cdots\mathbf{q}_{a_N}^\dagger(\xi_N\eta_N)|0\rangle.\end{aligned} \tag{14}$$

The N-particle wave function $\Psi_{a_1\ldots a_N}$ is defined by Eq. (14), which satisfies the N-body Schrödinger equation

$$\begin{aligned}-\sum_i(\partial_{\xi_i}^2+\partial_{\eta_i}^2)\Psi_{a_1\cdots a_N}&+c\sum_{i<j}[\epsilon(\xi_{ij})\delta'(\eta_{ij})\\&+\epsilon(\eta_{ij})\delta'(\xi_{ij})]\Psi_{a_1\cdots a_N}\\&+\sum_{i<j}[U_1(\xi_{ij})+U_2(\eta_{ij})]\Psi_{a_1\cdots a_N}\\=E\Psi_{a_1\cdots a_N},&\end{aligned} \tag{15}$$

where $\xi_{ij}=\xi_i-\xi_j$, $\delta'(\xi_{ij})=\partial_{\xi_i}\delta(\xi_{ij})$, and $\epsilon(\xi_{ij})=1$ for $\xi_{ij}>0$,0 for $\xi_{ij}=0$,-1 for $\xi_{ij}<0$. Since there are products of distributions in Eq. (15), an appropriate regularization for avoiding uncertainty is necessary. This issue has been discussed in Ref. [10].

III. VARIABLE SEPARATION OF QUANTUM DS1 WITH TWO COMPONENTS AND BETHE ANSATZ

Our purpose is to solve the N-body Schrödinger Eq. (15). The results in Ref. [2] remind us that we can make the following ansatz:

$$\begin{aligned}\Psi_{a_1\cdots a_N}=&\sum_{\substack{a_1'\cdots a_N'\\b_1'\cdots b_N'}}\prod_{i<j}\left[1-\frac{c}{4}\epsilon(\xi_{ij})\epsilon(\eta_{ij})\right]\\&\times\mathcal{M}_{a_1\cdots a_N,a_1'\cdots a_N'}\mathcal{N}_{a_1\cdots a_N,b_1'\cdots b_N'}\\&\times X_{a_1'\cdots a_N'}(\xi_1\cdots\xi_N)Y_{b_1'\cdots b_N'}(\eta_1\cdots\eta_N),\end{aligned} \tag{16}$$

where $\mathcal{M}$ and $\mathcal{N}$ are matrices being independent of ξ and η, and both $X_{a_1\cdots a_N}(\xi_1\cdots\xi_N)$ and $Y_{b_1\cdots b_N}(\eta_1\cdots\eta_N)$ are one-dimensional wave functions of N bodies. Substituting Eq. (16) into Eq. (15), we obtain

$$-\sum_i\partial_{\xi_i}^2X_{a_1\cdots a_N}+\sum_{i<j}U_1(\xi_{ij})X_{a_1\cdots a_N}=E_1X_{a_1\cdots a_N}, \tag{17}$$

$$-\sum_i\partial_{\eta_i}^2Y_{b_1\cdots b_N}+\sum_{i<j}U_2(\eta_{ij})Y_{b_1\cdots b_N}=E_2Y_{b_1\cdots b_N}, \tag{18}$$

where $U_1(\xi_{ij})$ and $U_2(\eta_{ij})$ are two-body potentials, Eqs. (17) and (18) are one-dimensional N-body Schrödinger equations and $E_1+E_2=E$. Above derivation indicates that the two-dimensional N-body Schrödinger Eq. (15) has been reduced into two one-dimensional N-body Schrödinger equations. Namely, the variables in the two-dimensional N-body wave function $\Psi_{a_1\cdots a_N}$ have been separated.

At this stage $\mathcal{M}$ and $\mathcal{N}$ are unknown temporarily. It is expected that for any given pair of exactly solvable 1D N-body problems and the correspondent solutions, we could construct the solutions $\Psi_{A_1\cdots A_N}$ for 2D N-body problems Eq. (15) through constructing an appropriate $\mathcal{M}\times\mathcal{N}$ matrix. It has been known that the 1D N-body problem in the form of Eq. (17) or (18) can be solved exactly for a class of potentials [4,5,11]. To illustrate the construction of $\mathcal{M}\times\mathcal{N}$ matrix, we take both potentials in Eqs. (17) and (18) to be the delta functions, i.e., $U_1(\xi_{ij})=2g\,\delta(\xi_{ij})$ and $U_2(\eta_{ij})=2g\,\delta(\eta_{ij})$ ($g>0$, the coupling constant). Then Eqs. (17) and (18) become

$$-\sum_i \partial^2_{\xi_i}X_{a_1\cdots a_N}+2g\sum_{i<j}\delta(\xi_{ij})X_{a_1\cdots a_N}=E_1X_{a_1\cdots a_N} \tag{19}$$

$$-\sum_i \partial^2_{\eta_i}Y_{b_1\cdots b_N}+2g\sum_{i<j}\delta(\eta_{ij})Y_{b_1\cdots b_N}=E_2Y_{b_1\cdots b_N}. \tag{20}$$

As X and Y are wave functions of Fermions with two components, denoted by X^F and Y^F, the problem has been solved by Yang long ago [4] (more explicitly, see Refs. [12] and [13]). According to the Bethe ansatz, the continual solution of Eq. (19) in the region of $0<\xi_{Q_1}<\xi_{Q_2}<\cdots<\xi_{Q_N}<L$ reads

$$\begin{aligned}X^F&=\sum_P \alpha_P^{(Q)}\exp\{i[k_{P_1}\xi_{Q_1}+\cdots+k_{P_N}\xi_{Q_N}]\}\\&=\alpha^{(Q)}_{12\cdots N}e^{i(k_1\xi_{Q_1}+k_2\xi_{Q_2}+\cdots+k_N\xi_{Q_N})}\\&\quad+\alpha^{(Q)}_{21\cdots N}e^{i(k_2\xi_{Q_1}+k_1\xi_{Q_2}+\cdots+k_N\xi_{Q_N})}\\&\quad+(N!-2)\ \text{other terms},\end{aligned} \tag{21}$$

where $X^F\in\{X^F_{a_1\cdots a_N}\}$, $P=[P_1,P_2,\ldots,P_N]$ and $Q=[Q_1,Q_2,\ldots,Q_N]$ are two permutations of the integers $1,2,\ldots,N$, and

$$\alpha^{(Q)}_{\cdots ij\cdots}=Y^{lm}_{ji}\alpha^{(Q)}_{\cdots ji\cdots}, \tag{22}$$

$$Y^{lm}_{ji}=\frac{-i(k_j-k_i)P^{lm}+g}{i(k_j-k_i)-g}. \tag{23}$$

The eigenvalue is given by

$$E_1=k_1^2+k_2^2+\cdots+k_N^2, \tag{24}$$

where $\{k_i\}$ are determined by the Bethe ansatz equations,

$$e^{ik_jL}=\prod_{\beta=1}^{M}\frac{i(k_j-\Lambda_\beta)-g/2}{i(k_j-\Lambda_\beta)+g/2} \tag{25}$$

$$\prod_{j=1}^{N}\frac{i(k_j-\Lambda_\alpha)-g/2}{i(k_j-\Lambda_\alpha)+g/2}=-\prod_{\beta=1}^{M}\frac{i(\Lambda_\alpha-\Lambda_\beta)+g}{i(\Lambda_\alpha-\Lambda_\beta)-g}, \tag{26}$$

with $\alpha=1,\cdots,M$, $j=1,\cdots,N$. Through exactly the same procedures we can get the solution Y^F and E_2 to Eq. (20).

As X and Y are boson wave functions, denoted by X^B and Y^B, it is easy to show that

$$X^B=\sum_P \beta_P^{(Q)}\exp\{i[k_{P_1}\xi_{Q_1}+\cdots+k_{P_N}\xi_{Q_N}]\}, \tag{27}$$

$$\beta^{(Q)}_{\cdots ij\cdots}=Z^{lm}_{ji}\beta^{(Q)}_{\cdots ji\cdots}, \tag{28}$$

$$Z^{lm}_{ji}=\frac{i(k_j-k_i)P^{lm}+g}{i(k_j-k_i)-g} \tag{29}$$

and the Bethe ansatz equations are as follows [13]:

$$e^{ik_jL}=(-1)^{N+1}\prod_{i=1}^{N}\frac{k_j-k_i+ig}{k_j-k_i-ig}\prod_{\beta=1}^{M}\frac{\Lambda_\beta-k_j+ig/2}{\Lambda_\beta-k_j-ig/2}, \tag{30}$$

$$\prod_{\alpha=1}^{M}\frac{\Lambda_\beta-\Lambda_\alpha+ig}{\Lambda_\beta-\Lambda_\alpha-ig}=(-1)^{M+1}\prod_{j=1}^{N}\frac{\Lambda_\beta-k_j+ig/2}{\Lambda_\beta-k_j-ig/2}. \tag{31}$$

Y^B is the same as X^B. It is well known that X^F and Y^F (X^B and Y^B) are antisymmetrical (symmetrical) as the coordinates and the color indices of the particles interchange simultaneously, instead of the coordinates merely interchanging.

IV. YOUNG OPERATOR OF PERMUTATION GROUP

For permutation group S_N: $\{e_i, i=1,\cdots,N!\}$, the totally symmetrical Young operator is

$$\mathcal{O}_N=\sum_{i=1}^{N!}e_i, \tag{32}$$

and the totally antisymmetrical Young operator is

$$\mathcal{A}_N=\sum_{i=1}^{N!}(-1)^{P_i}e_i. \tag{33}$$

The Young diagram for $\mathcal{O}_N$ is

1	2	3	···	N

,

and for $\mathcal{A}_N$, it is

1
2
⋮
N

.

For S_3, for example, we have

$$\mathcal{O}_3=1+P^{12}+P^{13}+P^{23}+P^{12}P^{23}+P^{23}P^{12}, \tag{34}$$

$$\mathcal{A}_3=1-P^{12}-P^{13}-P^{23}+P^{12}P^{23}+P^{23}P^{12}. \tag{35}$$

Lemma 1: $(\mathcal{O}_N X_F)(\xi_1,\xi_2,\cdots,\xi_N)$ is antisymmetrical with respect to the coordinate's interchanges of $(\xi_i \leftrightarrow \xi_j)$.

Proof: From the definition of $\mathcal{O}_N$ [Eq. (32)], we have

$$\mathcal{O}_N P^{ab} = P^{ab}\mathcal{O}_N = \mathcal{O}_N . \tag{36}$$

For the $N=3$ case, for example, the direct calculations show $\mathcal{O}_3 P^{12} = P^{12}\mathcal{O}_3 = \mathcal{O}_3, \mathcal{O}_3 P^{23} = P^{23}\mathcal{O}_3 = \mathcal{O}_3$, and so on. Using Eqs. (36) and (23), we have

$$\mathcal{O}_N Y_{ij}^{lm} = (-1)\mathcal{O}_N . \tag{37}$$

From Eqs. (21) and (23), X^F can be written as

$$\begin{aligned} X^F = \{ & e^{i(k_1\xi_{Q_1}+k_2\xi_{Q_2}+\cdots+k_N\xi_{Q_N})} + Y_{12}^{12} e^{i(k_2\xi_{Q_1}+k_1\xi_{Q_2}+\cdots+k_N\xi_{Q_N})} \\ & + Y_{13}^{23} Y_{12}^{12} e^{i(k_2\xi_{Q_1}+k_3\xi_{Q_2}+k_1\xi_{Q_3}+\cdots+k_N\xi_{Q_N})} \\ & + (N!-3) \ \text{ other terms}\} \alpha^{(Q)}_{12\cdots N}. \end{aligned} \tag{38}$$

Using Eqs. (37) and (38), we obtain

$$\begin{aligned} & (\mathcal{O}_N X^F)(\xi_1,\cdots,\xi_N) \\ & \quad = \{ e^{i(k_1\xi_{Q_1}+k_2\xi_{Q_2}+\cdots+k_N\xi_{Q_N})} - e^{i(k_2\xi_{Q_1}+k_1\xi_{Q_2}+\cdots+k_N\xi_{Q_N})} \\ & \qquad + e^{i(k_2\xi_{Q_1}+k_3\xi_{Q_2}+k_1\xi_{Q_3}+\cdots+k_N\xi_{Q_N})} \\ & \qquad + (N!-3) \ \text{ other terms}\} \mathcal{O}_N \alpha^{(Q)}_{12\cdots N} \\ & \quad = \sum_P (-1)^P \exp\{ i[k_{P_1}\xi_{Q_1} + \cdots + k_{P_N}\xi_{Q_N}]\} \\ & \qquad \times (\mathcal{O}_N \alpha^{(Q)}_{12\cdots N}). \end{aligned} \tag{39}$$

Therefore, we conclude that $(\mathcal{O}_N X^F)(\xi_1,\cdots,\xi_N)$ is antisymmetrical with respect to $(\xi_i \leftrightarrow \xi_j)$.

Lemma 2: $(\mathcal{A}_N X^B)(\xi_1,\xi_2,\cdots,\xi_N)$ is antisymmetrical with respect to the coordinate's interchanges of $(\xi_i \leftrightarrow \xi_j)$.

Proof: Noting [see Eqs. (33), (29), (27)]

$$\mathcal{A}_N P^{ab} = P^{ab}\mathcal{A}_N = -\mathcal{A}_N , \tag{40}$$

$$\mathcal{A}_N Z_{ij}^{lm} = (-1)\mathcal{A}_N , \tag{41}$$

we then have

$$\begin{aligned} & (\mathcal{A}_N X^B)(\xi_1,\cdots,\xi_N) \\ & \quad = \sum_P (-1)^P \exp\{ i[k_{P_1}\xi_{Q_1} + \cdots + k_{P_N}\xi_{Q_N}]\} \\ & \quad \times (\mathcal{A}_N \beta^{(Q)}_{12\cdots N}). \end{aligned} \tag{42}$$

Then the lemma is proved.

V. SOLUTIONS OF THE PROBLEM

The ansatz of Eq. (16) can be compactly written as

$$\Psi = \prod_{i<j} \left(1 - \frac{c}{4}\epsilon(\xi_{ij})\epsilon(\eta_{ij}) \right)(\mathcal{M}X)(\mathcal{N}Y), \tag{43}$$

where $(\mathcal{M}X)$ and $(\mathcal{N}Y)$ are required to be antisymmetrical under the interchanges of the coordinate variables. According to Lemmas 1 and 2, we see that

$$\mathcal{M},\mathcal{N} = \begin{cases} \mathcal{O}_N & \text{for 1D fermions} \\ \mathcal{A}_N & \text{for 1D bosons.} \end{cases} \tag{44}$$

As the DS1 fields $q_a(\xi\eta)$ in Eq. (1) are $(2+1)$D Bose fields, the commutators $\{[,]_-$, see Eqs. (9) and (10)$\}$ are used to quantized the system and the 2D N-body wave functions denoted in Ψ^B must be symmetrical under the color-interchange $(a_i \leftrightarrow a_j)$ and the coordinate interchange $[(\xi_i\eta_i) \leftrightarrow (\xi_j\eta_i)]$. Namely, the 2D Bose wave functions Ψ^B must satisfy

$$P^{a_i a_j}\Psi^B|_{\xi_i\eta_i \leftrightarrow \xi_j\eta_j} = \Psi^B. \tag{45}$$

As q_a are $(2+1)$D Fermi fields, the anticommutators should be used, and Ψ^F must be antisymmetrical under $(a_i \leftrightarrow a_j)$ and $[(\xi_i\eta_i) \leftrightarrow (\xi_j\eta_i)]$. Namely,

$$P^{a_i a_j}\Psi^F|_{\xi_i\eta_i \leftrightarrow \xi_j\eta_j} = -\Psi^F. \tag{46}$$

Thus, for the 2D boson case, two solutions of Ψ^B can be constructed as following

$$\begin{aligned} \Psi_1^B = \prod_{i<j} \left[1 - \frac{c}{4}\epsilon(\xi_{ij})\epsilon(\eta_{ij}) \right] & [\mathcal{O}_N X^F(\xi_1\cdots\xi_N)] \\ & \times [\mathcal{O}_N Y^F(\eta_1\cdots\eta_N)], \end{aligned} \tag{47}$$

$$\begin{aligned} \Psi_2^B = \prod_{i<j} \left[1 - \frac{c}{4}\epsilon(\xi_{ij})\epsilon(\eta_{ij}) \right] & [\mathcal{A}_N X^B(\xi_1\cdots\xi_N)] \\ & \times [\mathcal{A}_N Y^B(\eta_1\cdots\eta_N)]. \end{aligned} \tag{48}$$

Using Eqs. (36), (39), (40), and (42), we can check Eq. (45) directly. In addition, from the Bethe ansatz Eqs. (25), (26), (30), and (31) and $E=E_1+E_2$, we can see that the eigenvalues of Ψ_1^B and Ψ_2^B are different from each other generally, i.e., the states corresponding to Ψ_1^B and Ψ_2^B are nondegenerate.

For the 2D fermion case, the desired results are

$$\begin{aligned} \Psi_1^F = \prod_{i<j} \left[1 - \frac{c}{4}\epsilon(\xi_{ij})\epsilon(\eta_{ij}) \right] & [\mathcal{O}_N X^F(\xi_1\cdots\xi_N)] \\ & \times [\mathcal{A}_N Y^B(\eta_1\cdots\eta_N)], \end{aligned} \tag{49}$$

$$\begin{aligned} \Psi_2^F = \prod_{i<j} \left[1 - \frac{c}{4}\epsilon(\xi_{ij})\epsilon(\eta_{ij}) \right] & [\mathcal{A}_N X^B(\xi_1\cdots\xi_N)] \\ & \times [\mathcal{O}_N Y^F(\eta_1\cdots\eta_N)]. \end{aligned} \tag{50}$$

Equation (46) can also be checked directly. The eigenvalues corresponding to Ψ^F are also determined by the Bethe equations and $E=E_1+E_2$.

It is similar to Ref. [2] that we can prove $\Psi_{1,2}^B$ and $\Psi_{1,2}^F$ shown in above are of the exact solutions of the Eq. (15). Thus we conclude that the 2D quantum many-body problem induced from the quantum DS1 system with two component has been solved exactly.

VI. GROUND-STATE ENERGIES OF THE SYSTEM

In this section, we discuss the ground-state energies of the DS1 system solved in the previous section by using the Bethe ansatz Eqs. (25), (26) and (30), (31). Let the 2D N-body problem reduced from 2D DS1 system with two color (or spin) components has M colors down and $N-M$ colors up. Therefore both $X^{F,B}(\xi_1,\xi_2,\ldots\xi_N)$ and $Y^{F,B}(\eta_1,\eta_2,\ldots\eta_N)$ in Eqs. (47)–(50) are one dimensional N-body wave functions with M colors down and $N-M$ colors up. We are interested in the limit that N, M, and the length L of the box go to infinity proportionately, i.e., both $N/L=D$ and $M/L=D_m$ are finite.

For the one-dimensional N-fermion problem, by the nested Bethe ansatz (or Bethe-Yang ansatz) Eqs. (25) and (26), the corresponding integration equations for the ground state read [4]

$$2\pi\sigma_1=-\int_{-B_1}^{B_1}\frac{2g\sigma_1(\Lambda')d\Lambda'}{g^2+(\Lambda-\Lambda')^2}+\int_{-Q_1}^{Q_1}\frac{4g\rho_1(k)dk}{g^2+4(k-\Lambda)^2}, \tag{51}$$

$$2\pi\rho_1=1+\int_{-B_1}^{B_1}\frac{4g\sigma_1(\Lambda)d\Lambda}{g^2+4(k-\Lambda)^2}, \tag{52}$$

where $\rho_1(k)$ is particle (i.e., 1D fermion) density distribution function of k, and $\sigma_1(\Lambda)$ is color-down particle density distribution function of Λ. Namely, we have

$$D=\int_{-Q_1}^{Q_1}\rho_1(k)dk,\quad D_m=\int_{-B_1}^{B_1}\sigma_1(\Lambda)d\Lambda,$$
$$E_1/N=D^{-1}\int_{-Q_1}^{Q_1}k^2\rho_1(k)dk. \tag{53}$$

For 1D N-boson case, starting from the nested Bethe ansatz Eqs. (30) and (31), similar integration equations for ground state of bosons can be derived (see the Appendix). The results are as follows:

$$2\pi\sigma_2=\int_{-B_2}^{B_2}\frac{2g\sigma_2(\Lambda')d\Lambda'}{g^2+(\Lambda-\Lambda')^2}-\int_{-Q_2}^{Q_2}\frac{4g\rho_2(k)dk}{g^2+4(k-\Lambda)^2}, \tag{54}$$

$$2\pi\rho_2=1-\int_{-B_2}^{B_2}\frac{4g\sigma_2(\Lambda)d\Lambda}{g^2+4(\Lambda-k)^2}+\int_{-Q_2}^{Q_2}\frac{2g\rho_2(k')dk'}{g^2+(k-k')^2}, \tag{55}$$

where $\rho_2(k)$ and $\sigma_2(\Lambda)$ are bosonic particle density distribution function of k and its color-down particle density distribution function of Λ respectively, i.e.,

$$D=\int_{-Q_2}^{Q_2}\rho_2(k)dk,\quad D_m=\int_{-B_2}^{B_2}\sigma_2(\Lambda)d\Lambda,$$
$$E_2/N=D^{-1}\int_{-Q_2}^{Q_2}k^2\rho_2(k)dk. \tag{56}$$

The average energies of the 2D DS1 ground states described by Ψ_1^B, Ψ_2^B, Ψ_1^F, and Ψ_2^F [see Eqs. (47)–(50)] are denoted by $E(\Psi_1^B)$, $E(\Psi_2^B)$, $E(\Psi_1^F)$, and $E(\Psi_2^F)$, respectively. Then, the average energies per particle for the ground-states are as follows:

$$E(\Psi_1^B)/N=2E_1/N=2D^{-1}\int_{Q_1}^{Q_1}k^2\rho_1(k)dk, \tag{57}$$

$$E(\Psi_2^B)/N=2E_2/N=2D^{-1}\int_{Q_2}^{Q_2}k^2\rho_2(k)dk, \tag{58}$$

$$\begin{aligned}E(\Psi_1^F)/N&=\frac{1}{N}(E_1+E_2)\\&=D^{-1}\left[\int_{Q_1}^{Q_1}k^2\rho_1(k)dk+\int_{Q_2}^{Q_2}k^2\rho_2(k)dk\right]\\&=\frac{1}{2}[E(\Psi_1^B)+E(\Psi_2^B)],\end{aligned} \tag{59}$$

$$E(\Psi_2^F)/N=E(\Psi_1^F)/N. \tag{60}$$

From these equations, the following can been seen: (1) The average energies per particle for the ground states of this two-dimensional DS1 problem are reduced into the average energies per particle of one-dimensional many-body problems. As D and D_m are given, by solving the integration Eqs. (51)–(56), we obtain the $\rho_1(k)$ and $\rho_2(k)$, and then get the desired results of $E(\Psi_1^B)/N$, $E(\Psi_2^B)/N$, $E(\Psi_1^F)/N$, and $E(\Psi_2^F)/N$. (2) For the two bosonic solutions of the 2D DS1 system with two colors [Eqs. (47) and (48)], the average ground state energies per particle are twice as large as one of the 1D fermions or 1D bosons. (3) For the fermion solutions of this 2D DS1 system, $E(\Psi_1^F)/N$ and $E(\Psi_2^F)/N$ are the sum of 1D fermion average energy per particle and the 1D bosons. (4) In general, $E(\Psi_1^B)\neq E(\Psi_2^B)\neq E(\Psi_{1,\text{ or }2}^F)$. Namely, for the same DS1 system, if the statistics of the wave functions (or particles) is different, the corresponding ground-state energies are different. This is remarkable and reflects the statistical effects in the 2D DS1 system.

VII. DISCUSSION AND SUMMARY

Finally, we would like to speculate some further applications of the results presented in this paper to the mathematical physics. Our results may be useful in the following two respects. First, the Bethe ansatz Eqs. (25) and (26) for fermion wave functions and Eqs. (30) and (31) for boson wave functions can be solved, respectively, even though the equations are systems of transcendental equations for which the roots are not easy to locate. The so-called string hypothesis is used for the analysis and classification of the roots for the Bethe ansatz equations [7,8]. Thus, we could study their ground state, the excitation, and the thermodynamics based on it [7,8]. Then, the thermodynamical properties of the 1D Bose or Fermi gas with δ-function interaction and with two components can be explored. The Eqs. (47)–(50) indicate that under the thermodynamical limit the 2D DS1 gases (with two color components) are classified into 2D Bose gases and 2D Fermi gases. By Eqs. (47) and (48), the 2D Bose gases are composed of two 1D Fermi gases or 1D Bose gases, and by Eqs. (49) and (50), the 2D Fermi gases are composed of 1D Fermi gas and 1D Bose gas. Hence, the thermodynamics of 2D DS1 gases with two color components can be derived exactly. It would be interesting in physics, because this is an interesting and nontrivial example to illustrate coupling (or fusing) of two 1D 2-component gases with δ-function interacting and with different or the same statistics. Second, the colorless DS1 equation originated in studies of nonlinear phenomena [9]. Five years ago, Pang, Pu, and Zhao [14] showed an example that the solutions of the initial-boundary-value problem for the related classical DS1 equation in Ref. [15] are consistent with the solutions for the quantum DS1 system with time-dependent applied forces. This indicates that the classical solutions of the DS1 equation correspond to the classical limit of the solutions for the quantum DS1 system. This method reveals the solutions of the colorless DS1 equation. To the quantum DS1 system with color indices studied in this present paper, similar correspondences are expected. Hence, the structure of the solutions of the quantum DS1-system with color indices revealed in this paper would be helpful to understand the corresponding classical solutions of DS1 systems with color. The specific studies on the above speculations would be meaningful; however, they are beyond the scope of this present paper.

To summarize, we formulated the quantum multicomponent DS1 system in terms of the quantum multicomponent many-body Hamiltonian in 2D space. Then we reduced this 2D Hamiltonian to two 1D multicomponent many-body problems. As the potential between two particles with two components in one dimension is a δ function, the Bethe ansatz was used to solve these 1D problems. By using the ansatz of Ref. [1] and introducing some useful Young operators, we presented a N-body variable-separation ansatz for fusing two 1D solutions to construct 2D wave functions of the quantum many-body problem, which is induced from the quantum two-component DS1 system. There are two types of wave functions: bosons and fermions. Both of them satisfy the 2D many-body Schrödinger equation of the DS1 system exactly. The results have been used to study the ground states of the system. Some further applications of the results presented in this paper are speculated and discussed.

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APPENDIX

Let us derive Eqs. (54) and (55) in the text. We start from the Bethe ansatz Eqs. (30) and (31) of 1D bosons with two color components. Taking the logarithm of Eqs. (30) and (31) respectively, we have

$$k_j L = 2\pi I_k - 2\sum_{i=1}^{N} \tan^{-1}\frac{k_j - k_i}{g} - 2\sum_{\beta=1}^{M} \tan^{-1}\frac{2(\Lambda_\beta - k_j)}{g}, \tag{A1}$$

$$2\sum_{\alpha=1}^{M} \tan^{-1}\frac{\Lambda_\beta - \Lambda_\alpha}{g} = 2\pi J_\Lambda + 2\sum_{j=1}^{N} \tan^{-1}\frac{2(\Lambda_\beta - k_j)}{g}, \tag{A2}$$

where (for the case of $N=$even, $M=$odd)

$$\frac{1}{2} + I_k = \text{successive integers from } \ 1-\frac{1}{2}N \quad \text{to} \ +\frac{1}{2}N,$$

$$J_\Lambda = \text{successive integers from}$$

$$-\frac{1}{2}(M-1) \ \text{ to } \ +\frac{1}{2}(M-1).$$

We can now approach the limit $N\to\infty$, $M\to\infty$, $L\to\infty$ proportionally, obtaining

$$k = 2\pi f_2 - 2\int_{-Q_2}^{Q_2} dk' \rho_2(k') \tan^{-1}\frac{(k-k')}{g} - 2\int_{-B_2}^{B_2} d\Lambda\, \sigma_2(\Lambda) \tan^{-1}\frac{2(\Lambda - k)}{g}, \tag{A3}$$

$$2\int_{-B_2}^{B_2} d\Lambda' \sigma_2(\Lambda') \tan^{-1}\frac{\Lambda - \Lambda'}{g} = 2\pi h_2 + 2\int_{-Q_2}^{Q_2} dk \rho_2(k) \tan^{-1}\frac{2(\Lambda - k)}{g}, \tag{A4}$$

$$\frac{dh_2}{d\Lambda} = \sigma_2, \quad \frac{df_2}{dk} = \rho_2, \tag{A5}$$

$$D = \frac{N}{L} = \int_{-Q_2}^{Q_2} \rho_2(k) dk, \quad D_m = \frac{M}{L} = \int_{-B_2}^{B_2} \sigma_2(\Lambda) d\Lambda. \tag{A6}$$

Or, after differentiation,

$$2\pi\sigma_2 = \int_{-B_2}^{B_2} \frac{2g\sigma_2(\Lambda') d\Lambda'}{g^2 + (\Lambda - \Lambda')^2} - \int_{-Q_2}^{Q_2} \frac{4g\rho_2(k) dk}{g^2 + 4(k-\Lambda)^2}, \tag{A7}$$

$$2\pi\rho_2 = 1 - \int_{-B_2}^{B_2} \frac{4g\sigma_2(\Lambda) d\Lambda}{g^2 + 4(\Lambda - k)^2} + \int_{-Q_2}^{Q_2} \frac{2g\rho_2(k') dk'}{g^2 + (k-k')^2}, \tag{A8}$$

which are just Eqs. (54) and (55).

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