Plane-wave superpositions defined by orthonormal scalar functions on two- and three-dimensional manifolds

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Vector plane-wave superpositions defined by a given set of orthonormal scalar functions on a two- or three-dimensional manifold—beam manifold—are treated. We present a technique for composing orthonormal beams and some other specific types of fields such as three-dimensional standing waves, moving and evolving whirls. It can be used for any linear fields, in particular, electromagnetic fields in complex media and elastic fields in crystals. For electromagnetic waves in an isotropic medium or free space, unique families of exact solutions of Maxwell's equations are obtained. The solutions are illustrated by calculating fields, energy densities, and energy fluxes of beams defined by the spherical harmonics. It is shown that the obtained results can be used for a transition from the plane-wave approximation to more accurate models of real incident beams in free-space techniques for characterizing complex media. A mathematical formalism convenient for the treatment of various beams defined by the spherical harmonics is presented.

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I. INTRODUCTION

Natural and artificial complex media (anisotropic, chiral, bianisotropic) are of considerable current interest to both theorists and experimentalists. Bianisotropic media are the most general linear media in electromagnetics, in which the electric displacement **D** and the magnetic field strength **H** depend on both the electric field strength **E** and the magnetic displacement **B** [1–3]. In the case of motionless bianisotropic media, it is convenient to use also the constitutive relations, where **D** and **B** are expressed in terms of **E** and **H**, since the tangential components of **E** and **H** are continuous across the interfaces.

In the 1970s, the concept of bianisotropic medium was mostly used in electrodynamics of moving media [1,2] and optics of gyrotropic crystals [3]. Beginning in the mid-1980s, the field of applications has expanded considerably [4]. Huge advances in material sciences have come up with constructing new chiral composite materials with technological promise at microwave frequencies [5,6]. Recently, helicoidal bianisotropic media have been fabricated and the sculptured thin film concept for use in many areas of science and technology has been proposed [5,7]. Magnetostatically controlled bianisotropic materials [8] is another class of promising particulate composites. This provides new impetus for theoretical studies concerning calculations of effective medium properties of composite materials [9–13] and the development of new techniques for measuring electromagnetic parameters of complex media [14–17].

There exists a variety of techniques for the analysis of wave propagation in complex media and for solving direct and inverse scattering problems for such media [18–34], in particular, the characteristic matrix method [19], covariant impedance methods [20–25], the vector circuit theory [26], Green's functions techniques [27–33], and invariant embedding and wave-splitting approaches [25,31–34]. In recent years, the conception of refractive index tensor [35], Beltrami-Maxwell formalism [36], and fractional calculus [37,38] provided new promising tools for investigating wave propagation in isotropic, chiral, and anisotropic media. Extensive lists of references on research in the field of bianisotropic and chiral media and their applications can be found elsewhere [4].

In Refs. [23,24] the Lorentz-covariant impedance methods in electrodynamics of motionless and uniformly moving linear media are developed, and the exact solutions of the direct and the inverse scattering problems for such media are found, which can form a basis for the development of free-space techniques for characterizing complex media. Two such techniques, with different ways to extract the whole set of constitutive parameters, as well as the results of their computer modeling are presented in Refs. [16,17]. Computer modeling of them, which included the simulation of measurement errors, has shown [16,17] that both techniques make it possible to calculate all constitutive parameters of an anisotropic, chiral, or general bianisotropic medium, provided that the reflection and the transmission coefficients of planar samples under normal and oblique incidence of plane harmonic waves are measured with sufficient accuracy.

In recent years, considerable progress has been made in the development of measurement facilities to describe amplitude, phase, and polarization properties of microwave signals, and to measure the reflection and the transmission coefficients of planar samples [14,15]. This forms a groundwork for practical implementation of the free-space techniques presented in Refs. [16,17]. However, in many cases the plane-wave approximation of beams, used in the measurement setups, proves to be inadequate, especially for thick samples.

In the last decade, some new types of time-harmonic waves with degenerate evolution operators and linear, quadratic, and cubic dependence of amplitude on coordinates, which can be excited in complex media, have been found

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and studied [22,39]. As in the case of conic refraction, in investigating possible physical phenomena caused by such degeneracy, the plane-wave model of the incident beam is also inadequate.

On the basis of various techniques, electromagnetic fields in free space (i.e., vacuum), isotropic media, and some special cases of anisotropic media have been studied extensively in recent decades and many interesting solutions of the Helmholtz equation such as fractional solutions [37], nondiffracting—Bessel and Bessel-Gauss—beams [40], focus wave modes, localized wave transmission, and electromagnetic missiles [41], have been suggested.

In contrast to various beams in isotropic media, Green functions and plane harmonic waves—eigenwaves—in chiral, anisotropic, and bianisotropic media, investigated in many details, electromagnetic beams in complex media has been insufficiently investigated. Among the techniques which are the most general and effective tools in the analysis of linear fields, Green's functions and angular-spectrum representations seems to hold the lead. Works [27,42] provide prominent examples of the versatility of these approaches. Angular-spectrum representations become especially useful in the case of complex media, since eigenwaves are the only waves in these media, which have relatively simple and well understood properties. Since any superposition of eigenwaves in a linear medium is an exact solution of the corresponding wave equation, two questions naturally arise: (1) Which superpositions should be considered? (2) How can the corresponding integral representations be transformed to quickly converging or analytic expressions for the field? Of course, there are no unambiguous answers to them. We propose just one version from the whole host of possible answers.

The purpose of this paper is as follows.

(1) We present a technique for composing a set of orthonormal beams and some other specific types of fields in a general linear medium or free space, defined by a set of orthonormal scalar functions on a two- or three-dimensional manifold.

(2) We also present the relations for the calculation of eigenwaves parameters necessary to apply the proposed technique to electromagnetic waves in bianisotropic media.

(3) We illustrate this technique by calculating fields, energy densities, and energy fluxes of electromagnetic beams with wide angular spectrum (with solid angles $\Omega=2\pi$ and $\Omega=4\pi$), defined by the spherical harmonics.

(4) We show that the proposed approach provides a means to generalize the free-space technique for characterizing complex media [16,17], the covariant impedance methods [20–25], and the wave-splitting technique [25], formulated for the plane incident wave, to the case of incident beams with finite angular spectrum.

The outline of the paper is as follows. In the next section, basic equations for orthonormal beams and some other specific linear fields, defined by a given set of scalar orthonormal functions, are presented. In Sec. III, relations for the calculation of the parameters of eigenwaves, required for beam composition, are presented. Some details of beam parametrization and representation are discussed in Sec. IV. By way of illustration of the general theory, electromagnetic beams, defined by the spherical harmonics, are presented in Sec. V. In Sec. VI, the solutions, describing moving electromagnetic whirls, are presented. Two examples of fields with three-dimensional beam manifold are presented in Sec. VII. In Sec. VIII, we treat the general beam, which can be expanded into a series of orhonormal beams, and suggest a procedure to find the coefficients of this series, providing a means to generalize the techniques, developed in Refs. [16,17,20–25], to the case of incident beams. In the Appendix, some scalar and vector functions, defined by the spherical harmonics and extensively used in this paper, are presented.

II. BASIC EQUATIONS

A. Eigenwaves

The plane harmonic vector wave (eigenwave)

$$\mathbf{W}(\mathbf{r},t)=\mathbf{W}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{2.1}$$

is one of the primary and extremely fruitful notions in electrodynamics and elastodynamics of homogeneous anisotropic media, and many other branches of field theory. Since the phase $\mathbf{k}\cdot\mathbf{r}-\omega t$ is Lorentz invariant, it is convenient to rewrite Eq. (2.1) in terms of the four-dimensional vectors $\mathbf{x}=\mathbf{r}+ct\mathbf{e}_4$ and $\mathbf{K}=\mathbf{k}+(\omega/c)\mathbf{e}_4$, where c is the velocity of light in vacuum, $(\mathbf{e}_i)$ is an orthonormal basis in Minkowski vector space $\mathcal{V}$ $(\mathbf{e}_i^2=1, i=1,2,3, \mathbf{e}_4^2=-1)$, i.e.,

$$\mathbf{W}(\mathbf{x})=\mathbf{W}_0 e^{i\mathbf{x}\cdot\mathbf{K}}. \tag{2.2}$$

Electromagnetic, elastic, and other types of plane waves in linear media can be treated in the frame of similar mathematical techniques [27]. Therefore, we shall specify below the physical meaning of the oscillating quantity $\mathbf{W}$ only in those cases where it is essential. In particular, $\mathbf{W}$ can be any of the following quantities: the electric (magnetic) field strength $\mathbf{E}$ $(\mathbf{H})$, the electric (magnetic) displacement $\mathbf{D}$ $(\mathbf{B})$, the six-dimensional vectors col$(\mathbf{E},\mathbf{B})$ and col$(\mathbf{D},\mathbf{H})$, and the four-dimensional field (induction) tensor F (G)—for electromagnetic waves; the displacement vector $\mathbf{u}$—for elastic waves, and so on. Let $\mathcal{W}$ be the corresponding complex vector space $(\mathbf{W}\in\mathcal{W})$.

In a homogeneous linear medium substitution of $\mathbf{W}$ [Eq. (2.2)] into the appropriate field equations results in an eigenvalue equation of the form

$$C(\mathbf{K})\mathbf{W}_0=0, \tag{2.3}$$

where $C(\mathbf{K})$ is a linear operator depending on the constitutive parameters and the four-dimensional wave vector $\mathbf{K}$ or, in other terms, the three-dimensional wave vector $\mathbf{k}$ and the frequency ω. If the determinant of $C(\mathbf{K})$ vanishes, i.e.,

$$\det C(\mathbf{K})=0, \tag{2.4}$$

Eq. (2.3) has a nonzero solution $\mathbf{W}_0$. The scalar dispersion equation (2.4), relating $\mathbf{k}$ and ω, specifies a three-dimensional hyperspace $\mathcal{K}_C$ in the four-dimensional wave vector space $\mathcal{K}$.

The kernel $\mathcal{W}_K=\ker C(\mathbf{K})\in\mathcal{W}$, i.e., the set of solutions $\mathbf{W}_0$ of Eq. (2.3), where $\mathbf{K}$ is an arbitrary solution of Eq. (2.4), defines the amplitude subspace of the eigenwave $\mathbf{W}$

[Eq. (2.2)]. If this kernel is one-dimensional, Eq. (2.3) specifies $\mathbf{W}_0$ up to a complex scalar factor, in other words, the eigenwave polarization is uniquely defined. Otherwise, $\mathcal{W}_K$ is two dimensional, and the polarization is defined by an arbitrary complex vector $\mathbf{W}_0 \in \mathcal{W}_K$, i.e., the eigenwave propagates along an optic (or acoustic) axis and may have any polarization. In particular, this is the case for electromagnetic waves in an isotropic medium.

B. Beam manifold, beam base, and beam state

Let $u:\mathcal{B}\to C^1$ be a complex scalar function on a real manifold $\mathcal{B}$ (beam manifold). Let us consider an eigenwave superposition (termed below the ''beam'' for the sake of brevity)

$$\mathbf{W}(\mathbf{x}) = \int_{\mathcal{B}} e^{i\mathbf{x}\cdot\mathbf{K}(b)} u(b)\mathbf{W}(b)d\mathcal{B}, \tag{2.5}$$

where $d\mathcal{B}$ is the infinitesimal element of $\mathcal{B}$, $\mathbf{K}(b)\in\mathcal{K}_C$, and $\mathbf{W}(b)\in\mathcal{W}_{\mathbf{K}(b)} \equiv \ker C(\mathbf{K}(b))$ for every $b\in\mathcal{B}$. The mapping $\beta:\mathcal{B}\to\mathcal{K}\times\mathcal{W}$ by $b\mapsto(\mathbf{K}(b),\mathbf{W}(b))$ specifies the set of the eigenwaves involved in the beam (beam base), whereas the function $u:\mathcal{B}\to C^1$ specifies the beam state. Owing to the linearity of the medium, $\mathbf{W}(\mathbf{x})$ [Eq. (2.5)] is an exact solution of the field equations, provided that the integral on the right side of Eq. (2.5) exists.

In analysis of electromagnetic fields radiated by given sources (dipoles, line, and surface currents, moving point charges) [27,42], the amplitude function u is dictated by the source properties such as a given distribution of surface current density flowing in a plane [42]. In this article, we treat a different problem. We search for amplitude functions u which yield exact solutions of the homogeneous Maxwell equations with some prescribed property, for example, the orthonormality.

C. Beam types

Let us assume that there exists a set (u_n) of complex scalar functions $u_n:\mathcal{B}\to C^1$, satisfying the orthogonality conditions

$$\langle u_m|u_n\rangle \equiv \int_{\mathcal{B}} u_m^*(b)u_n(b)d\mathcal{B} = \delta_{mn}, \tag{2.6}$$

where u_m^* is the complex conjugate function to u_m, and δ_{mn} is the Kronecker δ function. In this paper, emphasis is given to the case of the two-dimensional manifold $\mathcal{B}$ and time-harmonic beams $\mathbf{W}$ [Eq. (2.5)]. However, two special cases of beams with three-dimensional manifold $\mathcal{B}=\mathcal{B}_3$ are treated in Sec. VII.

Let us consider a set of beams $\mathbf{W}_n$, which in a Lorentz frame L with the basis $(\mathbf{e}_i)$ can be written as

$$\mathbf{W}_n(\mathbf{r},t) = e^{-i\omega t}\int_{\mathcal{B}} e^{i\mathbf{r}\cdot\mathbf{k}(b)} \nu(b)u_n(b)\mathbf{W}(b)d\mathcal{B}, \tag{2.7}$$

where $\nu:\mathcal{B}\to C^1$ is some complex function on $\mathcal{B}$, which we shall use for normalization or orthonormalization of these beams. The mapping $\beta_3:\mathcal{B}\to\mathcal{K}_3\times\mathcal{W}$ specifies the beam base by $b\mapsto(\mathbf{k}(b),\mathbf{W}(b))$, where $\mathbf{k}(b)\in\mathcal{K}_3\subset\mathcal{K}_C$, $\mathbf{W}(b)\in\mathcal{W}_{\mathbf{K}(b)}$, and $\mathcal{K}_3$ is the wave vector surface, i.e., $\mathbf{K}(b)=\mathbf{k}(b)+(\omega/c)\mathbf{e}_4\in\mathbf{K}_C$.

1. Scalar product s_{mn}

Since the beams $\mathbf{W}_n$ [Eq. (2.7)] are composed by integrating on the two-dimensional manifold $\mathcal{B}$, let us introduce a scalar product

$$s_{mn} \equiv \langle \mathbf{W}_m|Q|\mathbf{W}_n\rangle = \int_{\sigma_0} \mathbf{W}_m^\dagger(\mathbf{r},t)Q\mathbf{W}_n(\mathbf{r},t)d\sigma_0, \tag{2.8}$$

where Q is some Hermitian operator in $\mathcal{W}$, $\mathbf{W}_m^\dagger(\mathbf{r},t)$ is the Hermitian conjugate of $\mathbf{W}_m(\mathbf{r},t)$, σ_0 is the plane with unit normal $\mathbf{q}$, passing through the point $\mathbf{r}=0$. We assume here that the tangential component

$$\mathbf{t}(b) = I\mathbf{k}(b) = \mathbf{k}(b) - \mathbf{q}[\mathbf{q}\cdot\mathbf{k}(b)] \tag{2.9}$$

of the wave vector $\mathbf{k}(b)$ is real for all $b\in\mathcal{B}$. Here, $I=\mathbb{1}-\mathbf{q}\otimes\mathbf{q}$ is the projection operator onto the plane σ_0, $\mathbb{1}$ is the unit dyadic, and $\otimes$ is the tensor product.

Substituting $\mathbf{W}_n$ [Eq. (2.7)] into Eq. (2.8) results in

$$s_{mn} = (2\pi)^2\int_{\mathcal{B}} \nu^*(b)u_m^*(b)\mathbf{W}^\dagger(b)d\mathcal{B} \times Q\int_{\mathcal{B}'} \nu(b')u_n(b')\mathbf{W}(b')\delta[\mathbf{t}(b')-\mathbf{t}(b)]d\mathcal{B}', \tag{2.10}$$

where δ is the Dirac δ function. In the general case, for each $b\in\mathcal{B}$, there exists a set $[b_\alpha, \alpha=1,2,\ldots,N(b); b_1\equiv b]$ of points $b_\alpha\in\mathcal{B}$, such that the wave vectors $\mathbf{k}(b_\alpha)$ have the same tangential components

$$\mathbf{t}(b_\alpha) = \mathbf{t}(b)\ [\alpha=1,2,\ldots,N(b); b_1\equiv b]. \tag{2.11}$$

Therefore, calculating the integral on $\mathcal{B}'$ by the change of variables $b'\to\mathbf{t}$, we obtain

$$s_{mn} = \int_{\mathcal{B}} u_m^*(b)\sum_{\alpha=1}^{N(b)} u_n(b_\alpha)T_\alpha(b)d\mathcal{B}, \tag{2.12}$$

where

$$T_\alpha(b) = (2\pi)^2\nu^*(b)\nu(b_\alpha)\frac{g(b_\alpha)}{J(b_\alpha)}\mathbf{W}^\dagger(b)Q\mathbf{W}(b_\alpha), \tag{2.13}$$

and $J(b)=D(t^j)/D(\xi^i)$ is the Jacobian determinant of the mapping $b\to\mathbf{t}$, calculated in terms of the local coordinate systems $(\xi^i, i=1,2)$ on $\mathcal{B}$ and $(t^j, j=1,2)$ on the $\mathbf{t}$ plane, preserving the orientation $[J(b)>0]$, and $d\mathcal{B}=g(b)d\xi^1 d\xi^2$.

2. Normalized beams (beams I)

To normalize the beams $\mathbf{W}$ [Eq. (2.7)] to some constant N_Q, i.e., to provide the fulfilment of the condition $s_{nn}\equiv\langle\mathbf{W}_n|Q|\mathbf{W}_n\rangle = N_Q$, let us assume that the function ν reduces to a normalizing constant factor $[\nu(b)\equiv\nu_n=\nu_n^*>0]$. Then, from Eqs. (2.12) and (2.13) follows

$$\nu_n=\frac{1}{2\pi}\left[\frac{1}{N_Q}\int_{\mathcal{B}}u_n^*(b)\sum_{\alpha=1}^{N(b)}u_n(b_\alpha)\times\frac{g(b_\alpha)}{J(b_\alpha)}\mathbf{W}^\dagger(b)Q\mathbf{W}(b_\alpha)d\mathcal{B}\right]^{-1/2}. \quad (2.14)$$

In this case, one can use in Eq. (2.8) any Hermitian operator Q (for example, the unit operator in $\mathcal{W}$) and an arbitrary orientation of the plane σ_0, for which an integral on $\mathcal{B}$ in Eq. (2.14) is real and positive. Instead of Eq. (2.8), one can use any other convenient normalization of $\mathbf{W}_n$ [Eq. (2.7)] with the corresponding normalizing constant $\nu(b)=\nu_n$. The normalized beams $\mathbf{W}_n$ [Eq. (2.7)] with $\nu(b)=\nu_n$ [Eq. (2.14)] (beams I for brevity sake) are closely related to the orthonormal functions (u_n) [Eq. (2.6)], but they are not orthogonal themselves, i.e., in the general case $s_{mn}\neq 0$ for $m\neq n$.

3. Orthonormal beams (beams II)

From Eqs. (2.6) and (2.12) follows that the beams $\mathbf{W}_n$ [Eq. (2.7)] become orthonormal (let us denote them beams II), i.e.,

$$s_{mn}=\langle \mathbf{W}_m|Q|\mathbf{W}_n\rangle=N_Q\delta_{mn}, \quad (2.15)$$

if

$$\sum_{\alpha=1}^{N(b)}u_n(b_\alpha)T_\alpha(b)=N_Q u_n(b) \quad (2.16)$$

for all $b\in\mathcal{B}$. In particular, this condition is satisfied, if

$$T_\alpha(b)=N_Q\delta_{1\alpha}. \quad (2.17)$$

It is evident from Eq. (2.13) that the corresponding orthonormalizing function $\nu(b)$ is real and is given by

$$\nu(b)=\frac{1}{2\pi}\sqrt{\frac{N_Q J(b)}{g(b)\mathbf{W}^\dagger(b)Q\mathbf{W}(b)}}. \quad (2.18)$$

The expression under the square root in Eq. (2.18) has to be finite and positive almost everywhere, i.e., for all $b\in\mathcal{B}$ with the allowable exception of a set of measure zero in $\mathcal{B}$. This necessary condition is imposed on the mapping β_3, the operator Q, and the normal $\mathbf{q}$ to the plane σ_0 [Eq. (2.8)]. Assuming that it is met, there are two basic ways to compose a set of orthonormal beams $\mathbf{W}_n$ [Eq. (2.7)]. The sufficient condition (2.17) is met, when either

$$N(b)=1 \quad (2.19)$$

or

$$N(b)>1, \quad \mathbf{W}^\dagger(b)Q\mathbf{W}(b_\alpha)=0, \quad 1<\alpha\leqslant N(b) \quad (2.20)$$

for all $b\in\mathcal{B}$ and b_α given by Eq. (2.11).

To compose the orthonormal beams $\mathbf{W}_n$ [Eq. (2.7)] satisfying the condition (2.19) (beams IIa), it is necessary to set the mapping β_3 and the normal $\mathbf{q}$ such that the maping $b\mapsto\mathbf{t}(b)$ is one-one (injective). In other words, the beam IIa base consists of eigenwaves with different tangential components $\mathbf{t}=I\mathbf{k}(b)$. This condition can easily be fulfiled for various types of fields and media.

For some sets of eigenwaves (see Sec. III C), there exists a Hermitian operator Q, depending on the normal $\mathbf{q}$, such that the amplitudes $\mathbf{W}(b_\alpha)$ satisfy the condition (2.20), provided that the wave vectors $\mathbf{k}(b_\alpha)$, $\alpha=1,2,\ldots,N(b)$ have the same tangential component $\mathbf{t}(b)$ [Eq. (2.11)]. In this case, in addition to beams IIa, one can compose the orthonormal beams with a noninjective mapping $b\mapsto\mathbf{t}(b)$ (beams IIb). This is the main reason why $\langle\mathbf{W}_m|Q|\mathbf{W}_n\rangle$ is used above instead of $\langle\mathbf{W}_m|\mathbf{W}_n\rangle$.

For brevity, when $\langle\mathbf{W}_m|Q|\mathbf{W}_n\rangle=N_Q\delta_{mn}$, we designate $\mathbf{W}_n$ as orthonormal functions, whereas this term is more suitable for functions $\mathbf{V}_n=Q^{1/2}\mathbf{W}_n$ $(\langle\mathbf{V}_m|\mathbf{V}_n\rangle=N_Q\delta_{mn})$, where $Q^{1/2}$ is a square root of the Hermitian operator Q. To eliminate the need for calculating $Q^{1/2}$, we use the functions $\mathbf{W}_n$ which have usually a more pronounced physical meaning than $\mathbf{V}_n$. A similar situation exists with regard to the amplitude orthogonality condition (2.20), which can be rewritten as $\mathbf{W}^\dagger(b)Q\mathbf{W}(b_\alpha)\equiv\mathbf{V}^\dagger(b)\mathbf{V}(b_\alpha)=0$, where $\mathbf{V}(b)=Q^{1/2}\mathbf{W}(b)$.

Beams II remain orthonormal under the transformation

$$\mathbf{W}(b)\mapsto\mathbf{W}'(b)=a(b)e^{i\psi(b)}\mathbf{W}(b), \quad (2.21)$$

where ψ and a are some real functions on $\mathcal{B}$, and $a(b)>0$ for all $b\in\mathcal{B}$. Replacing $\mathbf{W}(b)$ by $\mathbf{W}'(b)$ [Eq. (2.21)] in Eqs. (2.7) and (2.18), we obtain the set of orthonormal beams

$$\mathbf{W}'_n(\mathbf{r},t)=e^{-i\omega t}\int_{\mathcal{B}}e^{i[\mathbf{r}\cdot\mathbf{k}(b)+\psi(b)]}\nu(b)u_n(b)\mathbf{W}(b)d\mathcal{B}, \quad (2.22)$$

i.e., $\langle\mathbf{W}'_m|Q|\mathbf{W}'_n\rangle=N_Q\delta_{mn}$. The beams $\mathbf{W}_n$ [Eq. (2.7)] and $\mathbf{W}'_n$ [Eq. (2.22)] may be treated as two different phase states of the same beam. The function ψ specifies the phase change. Naturally, all types of orthonormal beams are invariant under the eigenwaves amplitude transformation $\mathbf{W}(b)\mapsto\mathbf{W}'(b)=a(b)\mathbf{W}(b)$. In particular, this makes it possible to set a beam base using dimensionless vectors.

4. Beams III

Let us now consider a beam for which the necessary condition for orthonormalization is not met, i.e., there exist domains $\mathcal{B}_+$, $\mathcal{B}_0$, and $\mathcal{B}_-$ of the manifold $\mathcal{B}=\mathcal{B}_+\cup\mathcal{B}_0\cup\mathcal{B}_-$, where the expression under the square root in Eq. (2.18) is positive, zero, and negative, respectively. We assume here that $\mathcal{B}_0$ is a set of measure zero in $\mathcal{B}$. Let either the condition (2.19) or (2.20) be met. Setting again the function $\nu(b)$ in Eq. (2.7) by the formula (2.18), from Eqs. (2.12) and (2.13) we obtain

$$\nu^*(b)=\pm\nu(b), \quad t_\alpha(b)=\pm N_Q\delta_{1\alpha}, \quad b\in\mathcal{B}_\pm, \quad (2.23)$$

$$s_{mn}=N_Q\left(\int_{\mathcal{B}_+}u_m^*(b)u_n(b)d\mathcal{B}-\int_{\mathcal{B}_-}u_m^*(b)u_n(b)d\mathcal{B}\right). \quad (2.24)$$

The beams $\mathbf{W}_n$ [Eq. (2.7)], described by Eqs. (2.18), (2.23), (2.24) and satisfying the conditions (2.19) or (2.20) (beams IIIa and beams IIIb) are not orthogonal, i.e., in the general case, $s_{mn}\neq N_Q\delta_{mn}$. However, their scalar products

$s_{mn} = \langle \mathbf{W}_m | Q | \mathbf{W}_n \rangle$ [Eq. (2.24)] are also invariant under the transformation (2.21), since the latter does not change the sign of the expression under the square root in Eq. (2.18) and, hence, the domains $\mathcal{B}_+$, $\mathcal{B}_0$, and $\mathcal{B}_-$. As for beams II, this transformation changes only the phase states of beams III. The set of phase states for all these beams, specified by various phase functions ψ, is infinite. One can normalize beams III to s_{mn} [Eq. (2.24)], by using the function

$$\nu(b) = \frac{1}{2\pi}\sqrt{\left|\frac{N_Q J(b)}{g(b)\mathbf{W}^\dagger(b) Q \mathbf{W}(b)}\right|}. \tag{2.25}$$

However, beams III with $\nu(b)$ [Eq. (2.18)] and $\nu(b)$ [Eq. (2.25)] differ only in phase. The corresponding phase change is specified by

$$\psi(b) = \begin{cases} 0, & b \in \mathcal{B}_+ \cup \mathcal{B}_0 \quad (2.26\text{a}) \\ \pi/2, & b \in \mathcal{B}_- . \quad (2.26\text{b}) \end{cases}$$

The orthonormal beams II can be treated as the special case ($\mathcal{B}_- = \varnothing$) of beams III. It is essential that some beams of type III can be composed from eigenwaves of all possible propagation directions.

In addition to the parameters of eigenwaves themselves in the medium under study, there are three key elements defining the properties of the presented beams: the manifold $\mathcal{B}$, the orthonormal base (u_n) of complex scalar functions on $\mathcal{B}$, and the beam base, i.e., the mapping $\beta_3 : \mathcal{B} \to \mathcal{K}_3 \times \mathcal{W}$. By setting these elements in various ways, one can compose a multitude of normalized and orthonormal beams with very interesting properties, some of which are presented in the subsequent sections. To compose the beams, it is necessary first to calculate parameters of eigenwaves. In the next section, we present the corresponding relations for electromagnetic waves.

III. EIGENWAVES PROPERTIES

Let us consider a linear medium which, at frequency ω in its rest frame L_0, is characterized by the constitutive equations [1–3]

$$\mathbf{D} = \epsilon \mathbf{E} + \alpha \mathbf{H}, \quad \mathbf{B} = \beta \mathbf{E} + \mu \mathbf{H}. \tag{3.1}$$

In the general case, the permittivity tensor ϵ, the permeability tensor μ, and the magnetoelectric pseudotensors α and β are assumed to be complex nonsymmetric and frequency dependent.

A. Wave vectors and amplitudes

For an eigenwave with wave vector $\mathbf{k}$ and frequency ω, the Maxwell equations reduce to

$$\mathbf{D} = -\mathbf{m} \times \mathbf{H}, \quad \mathbf{B} = \mathbf{m} \times \mathbf{E}, \tag{3.2}$$

where $\mathbf{m} = \mathbf{k}/k_0$ is the refraction vector [3], i.e., the dimensionless ‘‘relative wave vector,’’ $k_0 = \omega/c$ is the wave number in vacuum. By using Eqs. (3.1) and (3.2), we obtain Eq. (2.3) with

$$C(\mathbf{K}) = \begin{pmatrix} \epsilon & \alpha + \mathbf{m}^\times \\ \beta - \mathbf{m}^\times & \mu \end{pmatrix}, \tag{3.3}$$

$$\mathbf{K} = k_0(\mathbf{m} + \mathbf{e}_4), \quad \mathbf{W}_0 = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \tag{3.4}$$

where $\mathbf{m}^\times$ is the antisymmetric tensor dual to $\mathbf{m}$ ($\mathbf{m}^\times \mathbf{E} = \mathbf{m} \times \mathbf{E}$). The equivalent equation can be written in terms of $\mathbf{E}$ field:

$$\chi \mathbf{E} = 0, \ \chi = \epsilon + (\mathbf{m}^\times + \alpha)\mu^{-1}(\mathbf{m}^\times - \beta). \tag{3.5}$$

Hence, the dispersion equation (2.4) becomes

$$|\chi| \equiv |\epsilon + (\mathbf{m}^\times + \alpha)\mu^{-1}(\mathbf{m}^\times - \beta)| = 0, \tag{3.6}$$

where $|\chi|$ is the determinant of χ. The set of solutions $\mathbf{m}$ of Eq. (3.6) defines the wave vector surface $\mathcal{K}_3$ by $\mathbf{k} = k_0 \mathbf{m}$ and, by virtue of Eq. (3.4), the section of the hyperspace $\mathcal{K}_C$ (see Sec. II A), corresponding to the frequency ω in the frame L_0. To find the whole $\mathcal{K}_C$, one has to solve Eq. (3.6) at various frequencies with taking into account the frequency dependence of ϵ, μ, α, and β.

Let $\mathbf{m}$ be an arbitrary solution of Eq. (3.6). Then, the amplitude $\mathbf{W}_0$ [Eq. (3.4)] of the corresponding eigenwave is given by

$$\mathbf{E} = \overline{\chi} \mathbf{p}, \quad \mathbf{H} = \mu^{-1}(\mathbf{m}^\times - \beta)\mathbf{E}, \tag{3.7}$$

where $\overline{\chi}$ is the adjoint tensor ($\overline{\chi}\chi = \chi\overline{\chi} = |\chi| \mathbb{1}$), and $\mathbf{p}$ is an arbitrary vector. If χ is a dyad, i.e., $\overline{\chi} = 0$ and $\chi = \mathbf{c}_E \otimes \mathbf{n}_E$, the amplitude subspace $\mathcal{W}_K = \ker C(\mathbf{K})$ becomes two dimensional, $\mathbf{E}$ is an arbitrary vector normal to $\mathbf{n}_E = \mathbf{p}\chi$, and $\mathbf{H}$ is given by Eq. (3.7) as before.

B. Wave vector surface parametrization by the tangential component t of k

Let $\mathbf{b} = \mathbf{t}/k_0$ and η be the tangential and normal components of $\mathbf{m}$ ($\mathbf{b} \cdot \mathbf{q} = 0$). Substituting $\mathbf{m} = \mathbf{b} + \eta \mathbf{q}$ in Eq. (3.6), we obtain the quartic equation [22]

$$|\eta^2 A + \eta B + C| \equiv \sum_{n=1}^{4} a_n \eta^n + |C| = 0, \tag{3.8}$$

where

$$a_1 = (\overline{C}B)_t, \quad a_2 = (\overline{B}C + \overline{C}A)_t, \tag{3.9a}$$

$$a_3 = |B| + (ABC + CBA + AB_tC_t - A_tBC - B_tCA - C_tAB)_t, \tag{3.9b}$$

$$a_4 = (\overline{A}C + \overline{B}A)_t, \tag{3.9c}$$

$$A = \mathbf{q}^\times \mu^{-1} \mathbf{q}^\times, \tag{3.10a}$$

$$B = (\mathbf{b}^\times + \alpha)\mu^{-1}\mathbf{q}^\times + \mathbf{q}^\times \mu^{-1}(\mathbf{b}^\times - \beta), \tag{3.10b}$$

$$C = \epsilon + (\mathbf{b}^\times + \alpha)\mu^{-1}(\mathbf{b}^\times - \beta), \tag{3.10c}$$

and A_t is the trace of A. The roots ($\eta_j, j=1,2,3,4$) of this equation specify all four wave vectors $\mathbf{k}_j=\mathbf{t}+k_0\eta_j\mathbf{q}$, which have the same given tangential component $\mathbf{t}=k_0\mathbf{b}$.

C. Amplitude orthogonality in a nondissipative medium

In a nondissipative medium, the constitutive parameters satisfy the condition [1–3]

$$\epsilon^\dagger=\epsilon, \quad \mu^\dagger=\mu, \quad \alpha^\dagger=\beta. \tag{3.11}$$

For an eigenwave with the refraction vector $\mathbf{m}_j=\mathbf{k}_j/k_0=\mathbf{b}+\eta_j\mathbf{q}$ and the amplitude $\mathbf{W}_j$, Eqs. (2.3) and (3.3) result in

$$R\mathbf{W}_j=\eta_j Q\mathbf{W}_j, \tag{3.12}$$

where

$$R=\begin{pmatrix}\epsilon & \alpha+\mathbf{b}^\times\\ \beta-\mathbf{b}^\times & \mu\end{pmatrix}, \quad Q_0=\begin{pmatrix}0 & -\mathbf{q}^\times\\ \mathbf{q}^\times & 0\end{pmatrix}. \tag{3.13}$$

Since $\mathbf{q}$ is real, the matrix Q_0 is Hermitian ($Q_0^\dagger=Q_0$). As a consequence of Eqs. (3.11), the matrix R becomes Hermitian ($R^\dagger=R$) at real values of $\mathbf{b}$. Therefore, from Eq. (3.12) immediately follows

$$(\eta_j-\eta_i^*)\mathbf{W}_i^\dagger Q_0\mathbf{W}_j=0. \tag{3.14}$$

If $\eta_j-\eta_i^*\neq 0$, Eq. (3.14) reduces to the well-known orthogonality relation [27]

$$\mathbf{W}_i^\dagger Q_0\mathbf{W}_j\equiv\mathbf{q}\cdot(\mathbf{E}_i^*\times\mathbf{H}_j+\mathbf{E}_j\times\mathbf{H}_i^*)=0, \tag{3.15}$$

which relates the amplitudes of eigenwaves with wave vectors $\mathbf{k}_i$ and $\mathbf{k}_j$, having the same real tangential component $\mathbf{t}=k_0\mathbf{b}=I\mathbf{k}_i=I\mathbf{k}_j$, $\mathbf{t}^*=\mathbf{t}$. Hence, the electromagnetic beams of types II and III can propagate in nondissipative linear media and free space.

For a time-harmonic field, the normal component $S_q=\mathbf{q}\cdot\mathbf{S}$ of the time average Poynting vector $\mathbf{S}$ can be written as

$$S_q=\frac{c}{16\pi}\mathbf{q}\cdot(\mathbf{E}^*\times\mathbf{H}+\mathbf{E}\times\mathbf{H}^*)=\mathbf{W}^\dagger Q\mathbf{W}, \tag{3.16}$$

where $Q=(c/16\pi)Q_0$. Therefore, for electromagnetic beams $\mathbf{W}_n$ [Eq. (2.7)], the condition $\langle\mathbf{W}_n|Q|\mathbf{W}_n\rangle=N_Q$ [see Eq. (2.8)] is in fact the normalization to the beam energy flux N_Q through the plane σ_0:

$$\langle\mathbf{W}_n|Q|\mathbf{W}_n\rangle=\int_{\sigma_0}S_q d\sigma_0=N_Q. \tag{3.17}$$

In particular, such normalization is used in analysis of waveguide problems [27,43].

In the composition of electromagnetic beams I (see Sec. II C) the amplitude orthogonality is not warranted. Hence, these beams can propagate in any linear medium.

IV. BEAM PARAMETRIZATION AND REPRESENTATION

A. Two main ways to set the beam base

There are two main ways to set the beam base $\beta_3:\mathcal{B}\to\mathcal{K}_3\times\mathcal{W}$, i.e., to specify the wave vectors $\mathbf{k}$ and amplitudes $\mathbf{W}$ of eigenwaves composing the beam as functions $\mathbf{k}=\mathbf{k}(b)$ and $\mathbf{W}=\mathbf{W}(b)$ on the manifold $\mathcal{B}$.

One can set first the unit wave normals of these eigenwaves by a function $\hat{\mathbf{k}}=\hat{\mathbf{k}}(b)$. Then, in the case of electromagnetic waves, one has to calculate the refractive indices $n_j(b)=n_j(\hat{\mathbf{k}}(b))$ of all isonormal waves from Eq. (3.8) ($\eta_j=n_j$ at $\mathbf{t}=k_0\mathbf{b}=0$) and, by choosing some branch $n_j(b)$, to specify the wave vector function $\mathbf{k}(b)=k_0\mathbf{m}(b)=k_0 n_j(b)\hat{\mathbf{k}}(b)$ and the amplitude function $\mathbf{W}(b)=\mathrm{col}(\mathbf{E}(b),\mathbf{H}(b))$ in Eq. (3.4) as well.

The alternative is to set first the tangenial components of wave vectors by a real vector function $\mathbf{t}=\mathbf{t}(b)$ [$\mathbf{q}\cdot\mathbf{t}(b)=0$ for all $b\in\mathcal{B}$]. Then, the normal component $\xi_j(b)=\xi_j(\mathbf{t}(b))$ of $\mathbf{k}(b)=\mathbf{t}(b)+\xi_j(b)\mathbf{q}$ is chosen from the roots of Eq. (3.8); $\xi_j=k_0\eta_j$. The amplitude function $\mathbf{W}=\mathbf{W}(b)$ is calculated from $\mathbf{k}=\mathbf{k}(b)$ as described above.

Both the normal $\mathbf{k}(b)=k(b)\hat{\mathbf{k}}(b)$ and tangential $\mathbf{k}(b)=\mathbf{t}(b)+\xi(b)\mathbf{q}$ parametrizations have advantages and disadvantages. The wave numbers k_j are determined by more readily solved equations than the normal components ξ_j, such as, for example, a bicubic equation and a full sixth order equation in the case of elastic waves in crystals. Therefore, in nondissipative media, the normal parametrization is more convenient than the tangential one. However, in absorbing media, when a beam is composed from inhomogeneous eigenwaves with complex normal $\xi(b)=\mathbf{q}\cdot\mathbf{k}(b)$ and real tangential components $\mathbf{t}(b)=I\mathbf{k}(b)$ of wave vectors $\mathbf{k}(b)$, the tangential parametrization is more appropriate. This parametrization is also very useful in the analysis of fields radiated by a given point, line, or surface source, which are composed of both homogeneous and inhomogeneous plane waves [42].

B. Beam expansion into series

If the beam $\mathbf{W}$ in Eq. (2.5) consists of homogeneous eigenwaves of frequency ω, i.e., $\hat{\mathbf{k}}^*(b)=\hat{\mathbf{k}}(b)$ for all $b\in\mathcal{B}$, it may be of advantage to expand it into a series by using the formula [44]

$$e^{i\mathbf{k}\cdot\mathbf{r}}=4\pi\sum_{l=0}^{+\infty} i^l j_l(kr)\sum_{m=-l}^{l} Y_l^{m*}(\hat{\mathbf{k}})Y_l^m(\hat{\mathbf{r}}), \tag{4.1}$$

where

$$\hat{\mathbf{k}}=\mathbf{k}/k=\sin\theta_1(\mathbf{e}_1\cos\varphi_1+\mathbf{e}_2\sin\varphi_1)+\mathbf{e}_3\cos\theta_1, \tag{4.2}$$

$$\hat{\mathbf{r}}=\mathbf{r}/r=\sin\gamma(\mathbf{e}_1\cos\psi+\mathbf{e}_2\sin\psi)+\mathbf{e}_3\cos\gamma, \tag{4.3}$$

$$Y_l^m(\hat{\mathbf{k}})\equiv Y_l^m(\theta_1,\varphi_1),\ Y_l^m(\hat{\mathbf{r}})\equiv Y_l^m(\gamma,\psi), \tag{4.4}$$

$$Y_l^m(\theta,\varphi)=N_{lm}P_l^{|m|}(\cos\theta)e^{im\varphi}, \tag{4.5}$$

$$N_{lm}=\sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}, \tag{4.6}$$

$Y_l^m(\theta,\varphi)$ are the spherical harmonics, $P_l^m(\cos\theta)$ and $j_l(kr)$ are the spherical Legendre and Bessel functions [44,45].

Substituting the expansion (4.1) into Eq. (2.5), we obtain

$$\mathbf{W}(\mathbf{r},t)=e^{-i\omega t}\sum_{l=0}^{+\infty} i^l \sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{r}})\mathbf{W}_l^m(r), \tag{4.7}$$

where

$$\mathbf{W}_l^m(r)=4\pi\int_{\mathcal{B}} j_l(k(b)r)Y_l^{m*}(\hat{\mathbf{k}}(b))\nu(b)u(b)\mathbf{W}(b)d\mathcal{B}. \tag{4.8}$$

Within the framework of this description, the beam is characterized by a set of radial vector functions $\mathbf{W}_l^m=\mathbf{W}_l^m(r)$. In an isotropic medium, these relations become

$$\mathbf{W}(\mathbf{r},t)=e^{-i\omega t}\sum_{l=0}^{+\infty} i^l j_l(kr)\sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{r}})\mathbf{W}_l^m, \tag{4.9}$$

where the coordinate independent vector coefficients

$$\mathbf{W}_l^m=4\pi\int_{\mathcal{B}} Y_l^{m*}(\hat{\mathbf{k}}(b))\nu(b)u(b)\mathbf{W}(b)d\mathcal{B} \tag{4.10}$$

completely characterize the beam. Equation (4.9) illustrates in effect the well-known and fruitfully used [44,45] fact that the functions $j_l(kr)Y_l^m(\gamma,\psi)\exp(-iwt)$ are particular solutions of the scalar wave equation.

V. BEAMS DEFINED BY SPHERICAL HARMONICS

The general relations presented in Sec. II make it possible to compose beams related with various sets of orthonormal functions, in particular, orthogonal polynomials and spherical harmonics. As an illustration let us consider the latter. In this case, the manifold $\mathcal{B}$ (see Sec. II) is a unit sphere ($\mathcal{B}=S^2$), and the spherical harmonics $Y_l^m(\theta,\varphi)$ [Eq. (4.5)] satisfy the relations

$$\langle Y_l^m|Y_{l'}^{m'}\rangle\equiv\int_0^{2\pi}d\varphi\int_0^{\pi}Y_l^{m*}(\theta,\varphi)Y_{l'}^{m'}(\theta,\varphi)\sin\theta\,d\theta$$
$$=\delta_{ll'}\delta_{mm'}, \tag{5.1}$$

i.e., $d\mathcal{B}=\sin\theta\,d\theta d\varphi$ and $g=\sin\theta$ [see Eq. (2.18)]. Hence, Eq. (2.7) becomes

$$\mathbf{W}_j^s(\mathbf{r},t)=e^{-i\omega t}\int_0^{2\pi}d\varphi\int_0^{\pi}e^{i\mathbf{r}\cdot\mathbf{k}(\theta,\varphi)}\nu(\theta,\varphi)Y_j^s(\theta,\varphi)$$
$$\times\mathbf{W}(\theta,\varphi)\sin\theta\,d\theta. \tag{5.2}$$

It is essential that, in the general case, the coordinates θ and φ on $\mathcal{B}=S^2$ do not coincide with the spherical coordinates θ_1 and φ_1 of $\hat{\mathbf{k}}$ [Eq. (4.2)]. In particular, using the normal parametrization, one can set the angular spectrum of eigenwaves by $\hat{\mathbf{k}}=\hat{\mathbf{k}}(\theta,\varphi)\equiv\hat{\mathbf{k}}(\theta_1(\theta,\varphi),\varphi_1(\theta,\varphi))$ [Eq. (4.2)]. Alternatively, one can set the function $\mathbf{t}=\mathbf{t}(\theta,\varphi)$ in the framework of tangential parametrization (see Sec. IV A).

The beam $\mathbf{W}_j^s$ [Eq. (5.2)] can be expanded into the series $\mathbf{W}_j^s(\mathbf{r},t)=\mathbf{W}(\mathbf{r},t)$ [Eq. (4.7)], where the radial functions are given by

$$\mathbf{W}_l^m(r)=4\pi\int_0^{2\pi}d\varphi\int_0^{\pi}j_l(k(\theta,\varphi)r)Y_l^{m*}(\hat{\mathbf{k}}(\theta,\varphi))$$
$$\times Y_j^s(\theta,\varphi)\nu(\theta,\varphi)\mathbf{W}(\theta,\varphi)\sin\theta\,d\theta. \tag{5.3}$$

In an isotropic medium, Eqs. (4.7) and (5.3) result in $\mathbf{W}_j^s(\mathbf{r},t)=\mathbf{W}(\mathbf{r},t)$ [Eq. (4.9)] with the coefficients

$$\mathbf{W}_l^m=4\pi\int_0^{2\pi}d\varphi\int_0^{\pi}Y_l^{m*}(\hat{\mathbf{k}}(\theta,\varphi))Y_j^s(\theta,\varphi)\nu(\theta,\varphi)$$
$$\times\mathbf{W}(\theta,\varphi)\sin\theta\,d\theta. \tag{5.4}$$

In this article, we shall restrict our further consideration to some specific types of electromagnetic beams, defined by the spherical harmonics, in isotropic media and free space. The applications of the suggested approach to beams in complex media will be presented separately. Let us consider two types of beams composed of eigenwaves with wide angular spectrum Ω, specified by

$$\varphi_1=\varphi\in[0,2\pi],\quad \theta_1=\theta\in[0,\pi/2], \tag{5.5}$$

$$\varphi_1=\varphi\in[0,2\pi],\quad \theta_1=\theta\in[0,\pi], \tag{5.6}$$

i.e., with the solid angle $\Omega=2\pi$ and $\Omega=4\pi$, respectively. These beams can be expressed in terms of scalar function U_j^s and vector functions $\mathbf{R}_j^s$, $\mathbf{M}_j^s$, and $\mathbf{A}_j^s$, defined by the spherical harmonic Y_j^s, the radial, the meridional, and the azimuthal basis vectors

$$\mathbf{e}_r(\theta,\varphi)=\sin\theta(\mathbf{e}_1\cos\varphi+\mathbf{e}_2\sin\varphi)+\mathbf{e}_3\cos\theta, \tag{5.7a}$$

$$\mathbf{e}_\theta(\theta,\varphi)=\cos\theta(\mathbf{e}_1\cos\varphi+\mathbf{e}_2\sin\varphi)-\mathbf{e}_3\sin\theta, \tag{5.7b}$$

$$\mathbf{e}_\varphi(\varphi)=-\mathbf{e}_1\sin\varphi+\mathbf{e}_2\cos\varphi. \tag{5.7c}$$

The definitions and the properties of these functions are presented in the Appendix.

A. Orthonormal beams with $\Omega=2\pi$

Let us consider a beam in an isotropic medium or free space, composed of eigenwaves with wave vectors given by Eqs. (4.2) and (5.5) and defined by the spherical harmonic Y_j^s. In this case, the beam manifold $\mathcal{B}$ is the northern hemisphere $\mathcal{N}$ given by Eq. (5.5), and the mapping $b\equiv(\theta,\varphi)\mapsto\mathbf{t}(b)=I\mathbf{k}(b)$ is injective (one—one), i.e., $N(b)=1$ for all $b\in\mathcal{N}$. Since $Y_j^s(\pi-\theta,\varphi)=(-1)^{j+|s|}Y_j^s(\theta,\varphi)$, the spherical harmonics Y_j^s and $Y_{j'}^s$ are orthogonal on $\mathcal{N}$, i.e.,

$$\int_0^{2\pi}d\varphi\int_0^{\pi/2}Y_j^{s*}(\theta,\varphi)Y_{j'}^s(\theta,\varphi)\sin\theta\,d\theta=0, \tag{5.8}$$

if $j+j'$ is even. Hence, using the beam manifold $\mathcal{N}$, we can compose two different sets of orthonormal beams (beams IIa) defined by the spherical harmonics Y_j^s with even j ($j=0,2,\ldots;s=0,\pm1,\ldots,\pm j$) and odd j ($j=1,3,\ldots;s=0,\pm1,\ldots,\pm j$), respectively. The corresponding orthonormalizing function is given by

$$\nu = \frac{1}{2\pi}\sqrt{\frac{2N_Q J}{g\mathbf{W}^\dagger Q\mathbf{W}}} = \frac{1}{\lambda}\sqrt{\frac{2N_Q \cos\theta}{\mathbf{W}^\dagger Q\mathbf{W}}}, \tag{5.9}$$

where $\theta\in[0,\pi/2], \varphi\in[0,2\pi]$, and $\lambda=2\pi/k$ is the wave length. Here, we have taken into account that $g=\sin\theta$, and

$$\mathbf{t}=k\sin\theta(\mathbf{e}_1\cos\varphi+\mathbf{e}_2\sin\varphi). \tag{5.10}$$

We shall show below that, for all beams treated in this section, the orthonormalizing function ν [Eq. (5.9)] reduces to a constant. Therefore, the integral (2.7) becomes

$$\mathbf{W}_j^s(\mathbf{r},t) = \nu e^{-i\omega t}\int_0^{2\pi} d\varphi \int_0^{\pi/2} e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)} Y_j^s(\theta,\varphi) \times\mathbf{W}(\theta,\varphi)\sin\theta\, d\theta. \tag{5.11}$$

It should be emphasized that these beams are exact solutions of homogeneous Maxwell's equations, which differ fundamentally from the well-known approximate solutions (under the paraxial approximation)—the Hermite-Gaussian and Laguerre-Gaussian beams [46].

Let us consider now a nondissipative isotropic medium with refractive index $n=\sqrt{\epsilon\mu}$ and set two amplitude functions by

$$\mathbf{W}(\theta,\varphi)\equiv\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}Z_0\mathbf{e}_\theta\\ \mathbf{e}_\varphi\end{pmatrix}, \tag{5.12a}$$

$$= \begin{pmatrix}Z_0\mathbf{e}_\varphi\\ -\mathbf{e}_\theta\end{pmatrix}, \tag{5.12b}$$

where $Z_0=\sqrt{\mu/\epsilon}$. By setting $\mathbf{q}=\mathbf{e}_3$, from Eqs. (3.13), (3.16), and (5.7)–(5.12) we obtain two types of beams defined by the spherical harmonic Y_j^s:

$$\mathbf{E}=\nu_0 Z_0 e^{-i\omega t}\mathbf{M}_{jN}^{s0}[1] = \nu_0 Z_0 e^{i(s\psi-\omega t)}\{\mathbf{e}I_j^{ss-1}[\cos] + \mathbf{e}^* I_j^{ss+1}[\cos] - \mathbf{e}_3 I_j^{ss}[\sin]\}, \tag{5.13a}$$

$$\mathbf{H}=\nu_0 e^{-i\omega t}\mathbf{A}_{jN}^{s0}[1] = i\nu_0 e^{i(s\psi-\omega t)}\{\mathbf{e}^* I_j^{ss+1}[1] - \mathbf{e}I_j^{ss-1}[1]\}, \tag{5.13b}$$

and

$$\mathbf{E}=\nu_0 Z_0 e^{-i\omega t}\mathbf{A}_{jN}^{s0}[1], \tag{5.14a}$$

$$\mathbf{H}=-\nu_0 e^{-i\omega t}\mathbf{M}_{jN}^{s0}[1], \tag{5.14b}$$

where

$$\nu_0=\frac{4}{\lambda}\sqrt{\frac{\pi N_Q}{cZ_0}}, \tag{5.15}$$

$\mathbf{e}$ is given by Eq. (A14a), and $\lambda=2\pi/k=2\pi v/\omega$. The amplitude functions $\mathbf{M}_{jN}^{s0}[1]$ and $\mathbf{A}_{jN}^{s0}[1]$ are given by Eqs. (A12) and (A13) with $f=1$. These two beam types, E_M beam or H_A beam [Eq. (5.13)], and E_A beam or H_M beam [Eq. (5.14)], correspond to $\mathbf{W}$ [Eq. (5.12a)] and $\mathbf{W}$ [Eq. (5.12b)], respectively. At $s=0$, these fields are described by $\mathbf{M}_{jN}^{00}$ [Eq. (A23c)] and $\mathbf{A}_{jN}^{00}$ [Eq. (A23d)] with $f=1$.

The time average energy densities w_e and w_m of electric and magnetic fields and Poynting's vector $\mathbf{S}$ for E_M and E_A beams are given by

$$w_e=\frac{1}{16\pi}\epsilon|\mathbf{E}|^2=\begin{cases} w_0 w_M & \text{for } E_M-\text{beam}\\ w_0 w_A & \text{for } E_A-\text{beam,}\end{cases} \tag{5.16a}$$

$$w_m=\frac{1}{16\pi}\mu|\mathbf{H}|^2=\begin{cases} w_0 w_A & \text{for } E_M-\text{beam}\\ w_0 w_M & \text{for } E_A-\text{beam.}\end{cases} \tag{5.16b}$$

$$\mathbf{S}=\frac{c}{8\pi}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) = S_0(S_R'\mathbf{e}_R + S_A'\mathbf{e}_A + S_N'\mathbf{e}_3), \tag{5.17}$$

where

$$w_M=|\mathbf{M}_{jN}^{s0}[1]|^2=\sum_{p=0}^{1}\left\{\frac{1}{2}(J_{jp}^{ss-1}[\cos])^2+\frac{1}{2}(J_{jp}^{ss+1}[\cos])^2 + (J_{jp}^{ss}[\sin])^2\right\}, \tag{5.18a}$$

$$w_A=|\mathbf{A}_{jN}^{s0}[1]|^2=\frac{1}{2}\sum_{p=0}^{1}\{(J_{jp}^{ss-1}[1])^2+(J_{jp}^{ss+1}[1])^2\}, \tag{5.18b}$$

$$S_R'=\sum_{p=0}^{1}(-1)^p J_{j1-p}^{ss}[\sin]\{\beta(-s)J_{jp}^{ss-1}[1] + \beta(s)J_{jp}^{ss+1}[1]\}, \tag{5.19a}$$

$$S_A'=\sum_{p=0}^{1}J_{jp}^{ss}[\sin]\{\beta(s)J_{jp}^{ss+1}[1]-\beta(-s)J_{jp}^{ss-1}[1]\}, \tag{5.19b}$$

$$S_N'=\sum_{p=0}^{1}\{J_{jp}^{ss-1}[\cos]J_{jp}^{ss-1}[1]+J_{jp}^{ss+1}[\cos]J_{jp}^{ss+1}[1]\}, \tag{5.19c}$$

$$w_0=S_0/v,\ S_0=N_Q/\lambda^2, \tag{5.20}$$

$$\beta(s)=\begin{cases}-1 & (s=-1,-2,\dots)\\ 1 & (s=0,1,2,\dots),\end{cases} \tag{5.21}$$

and the functions J_{jp}^{sm} are given by Eq. (A5). Both energy densities w_e [Eq. (5.16a)] and w_m [Eq. (5.16b)] as well as the components S_R', S_A', and S_N' are independent of the azimuthal angle ψ [see Eqs. (A14)], for the beams defined by the zonal spherical harmonics $(s=0)$, $S_A'\equiv 0$. It is evident from Eqs. (3.17) and (5.2) that the total energy flux through any plane $z=r\cos\gamma=$const is the same for all beams and is equal to N_Q.

The energy characteristics of some beams are presented in Figs. 1–5. They show that all these beams are well focused in a very small core region with waist radius about 1.5λ. Only in this region are there high values of energy densities of meridional (Fig. 1) and azimuthal (Fig. 2) fields [see also Eqs. (5.16a) and (5.16b)], as well as high values of normal

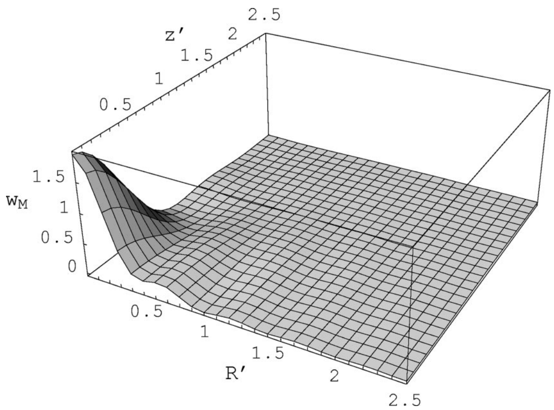

FIG. 1. Normalized energy density w_M [Eq. (5.18a)] as a function of cylindrical coordinates $R' = R/\lambda$ and $z' = z/\lambda$; $j = s = 0$.

S'_N (Figs. 3 and 4) and azimuthal S'_A (for $s \neq 0$, Fig. 5) components of energy flux vector $\mathbf{S}$ [Eq. (5.17)]. All these values rapidly decrease outside it. In the waist plane $z = 0$, there are both domains with positive and negative values of normal component S'_N ($s = 0$, Fig. 3). Similarly, for some beams (see, for example, curve *D* on Fig. 5), there are domains with positive and negative values of azimuthal component S'_A. For the beams defined by the zonal spherical harmonics ($s = 0$, Fig. 3), $S'_N = 0$ at the beam center ($z = 0, R = 0$), whereas, for the beams illustrated by curves *A* and *B* on Fig. 4, S'_N has a maximum at this point.

B. Standing waves and whirls with $\Omega = 4\pi$

Let now the wave vectors $\mathbf{k} = \mathbf{k}(\theta, \varphi)$ be given by Eqs. (4.2) and (5.6), and the amplitude functions $\mathbf{W}(\theta, \varphi)$ be determined by expressions (5.12). Let us set $\mathbf{q} = \mathbf{e}_3$ and define ν by Eq. (2.25) (beams III with the angular spectrum $\Omega = 4\pi$). In this case, $\mathcal{B}_+$ and $\mathcal{B}_-$ are the northern ($0 \leqslant \theta < \pi/2$) and southern ($\pi/2 < \theta \leqslant \pi$) hemispheres, and $J/(g\mathbf{W}^\dagger Q \mathbf{W})$ is uncertain on the equator $\mathcal{B}_0$ ($\theta = \pi/2$). For all these types of waves the function $\nu = \nu(\theta, \varphi)$ [Eq. (2.25)] reduces to a constant, and condition (2.20) is met [$b \equiv b_1 = (\theta, \varphi)$, $b_2 = (\pi - \theta, \varphi)$, $N(b) = 2$]. Hence, Eq. (2.24) becomes

$$\langle \mathbf{W}_j^s | Q | \mathbf{W}_{j'}^{s'} \rangle = [1 - (-1)^{j+j'}] \frac{\delta_{ss'} N_Q}{4\pi N_{j's}} \mathcal{P}_{jj'}^{ss}[1], \tag{5.22}$$

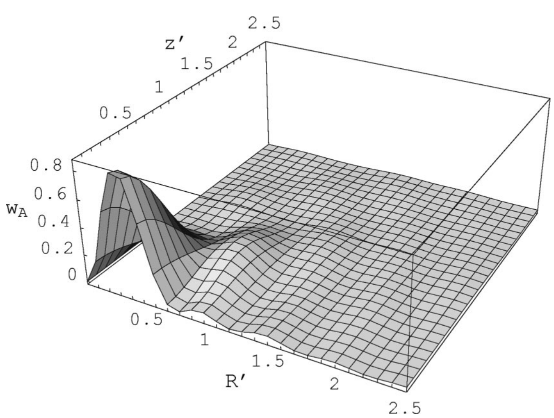

FIG. 2. Normalized energy density w_A [Eq. (5.18b)]; $j = s = 0$.

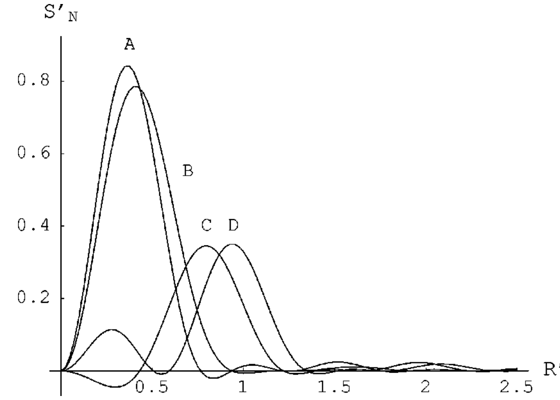

FIG. 3. Normal component S'_N [Eq. (5.19c)] of the normalized energy flux vector as a function of R'; $z = 0$; $s = 0$; (*A*) $j = 0$; (*B*) $j = 1$; (*C*) $j = 2$; (*D*) $j = 3$.

where $N_{j's}$ and $\mathcal{P}_{jj'}^{ss}[1]$ are given in Eqs. (4.6) and (A3). If $s' \neq s$ or $j + j'$ is even, the beams $\mathbf{W}_j^s$ and $\mathbf{W}_{j'}^{s'}$ are orthogonal, i.e., $\langle \mathbf{W}_j^s | Q | \mathbf{W}_{j'}^{s'} \rangle = 0$. For each beam $\langle \mathbf{W}_j^s | Q | \mathbf{W}_j^s \rangle = 0$, i.e., the total time average energy flux through the plane σ_0 is zero. That is why such beams are essentially standing waves.

Substituting $\mathbf{W}$ [Eq. (5.12)] in Eqs. (2.25) and (5.2), we obtain two types of standing waves (E_M wave or H_A wave and E_A wave or H_M wave):

$$\mathbf{E} = Z_0 \frac{\nu_0}{\sqrt{2}} e^{-i\omega t} \mathbf{M}_j^{s0}[1], \tag{5.23a}$$

$$\mathbf{H} = \frac{\nu_0}{\sqrt{2}} e^{-i\omega t} \mathbf{A}_j^{s0}[1], \tag{5.23b}$$

$$\mathbf{E} = Z_0 \frac{\nu_0}{\sqrt{2}} e^{-i\omega t} \mathbf{A}_j^{s0}[1], \tag{5.24a}$$

$$\mathbf{H} = -\frac{\nu_0}{\sqrt{2}} e^{-i\omega t} \mathbf{M}_j^{s0}[1], \tag{5.24b}$$

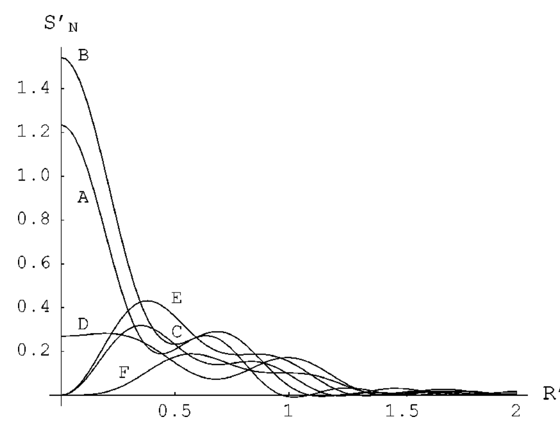

FIG. 4. Normal component S'_N [Eq. (5.19c)] of the normalized energy flux vector as a function of R'; $z = 0$; (*A*) $j = s = 1$; (*B*) $j = 2$, $s = 1$; (*C*) $j = s = 2$; (*D*) $j = 3$, $s = 1$; (*E*) $j = 3$, $s = 2$; (*F*) $j = s = 3$.

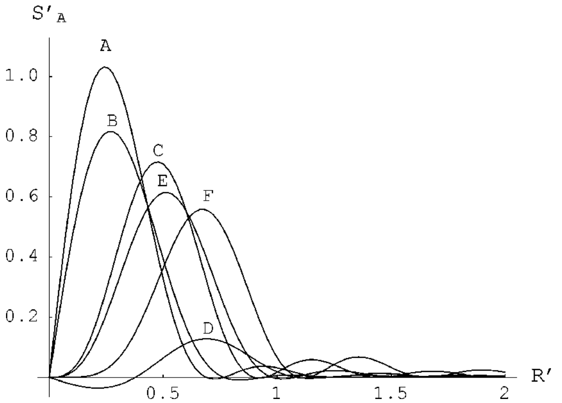

FIG. 5. Azimuthal component S_A' [Eq. (5.19b)] of the normalized energy flux vector as a function of R'; $z=0$; (A) $j=s=1$; (B) $j=2$, $s=1$; $(C) j=s=2$; (D) $j=3$, $s=1$; (E) $j=3$, $s=2$; $(F) j=s=3$.

where

$$\mathbf{M}_j^{s0}[1]=\mathbf{M}_{jN}^{s0}[1]+\mathbf{M}_{jS}^{s0}[1]$$
$$=2e^{is\psi}\{\mathbf{e}i^{|s-1|+q}J_{jq}^{ss-1}[\cos]$$
$$+\mathbf{e}^*i^{|s+1|+q}J_{jq}^{ss+1}[\cos]-\mathbf{e}_3 i^{|s|+p}J_{jp}^{ss}[\sin]\}, \quad (5.25)$$

$$\mathbf{A}_j^{s0}[1]=\mathbf{A}_{jN}^{s0}[1]+\mathbf{A}_{jS}^{s0}[1]$$
$$=2ie^{is\psi}\{\mathbf{e}^*i^{|s+1|+p}J_{jp}^{ss+1}[1]-\mathbf{e}i^{|s-1|+p}J_{jp}^{ss-1}[1]\}. \quad (5.26)$$

Here and in the following sections $p=1-q=0$ if $j+|s|$ is even, $p=1-q=1$ if $j+|s|$ is odd.

The beams, defined by the zonal spherical harmonics ($s=0$), are described by

$$\mathbf{M}_j^{00}[1]=2(\mathbf{e}_R i^{q+1}J_{jq}^{01}[\cos]-\mathbf{e}_3 i^p J_{jp}^{00}[\sin]), \quad (5.27a)$$

$$\mathbf{A}_j^{00}[1]=2\mathbf{e}_A i^{p+1}J_{jp}^{01}[1]. \quad (5.27b)$$

The time average energy densities w_e and w_m of electric and magnetic fields of E_M wave [Eq. (5.23)] and E_A wave [Eq. (5.24)] are given by Eqs. (5.16) and (5.20) with

$$w_M=\frac{1}{2}|\mathbf{M}_j^{s0}[1]|^2$$
$$=(J_{jq}^{ss-1}[\cos])^2+(J_{jq}^{ss+1}[\cos])^2+2(J_{jp}^{ss}[\sin])^2, \quad (5.28a)$$

$$w_A=\frac{1}{2}|\mathbf{A}_j^{s0}[1]|^2=(J_{jp}^{ss-1}[1])^2+(J_{jp}^{ss+1}[1])^2. \quad (5.28b)$$

The time average Poynting's vector for the both standing waves has the form

$$\mathbf{S}=S_0 S_A' \mathbf{e}_A, \quad (5.29)$$

where

$$S_A'=2J_{jp}^{ss}[\sin]\{\beta(s)J_{jp}^{ss+1}[1]-\beta(-s)J_{jp}^{ss-1}[1]\}. \quad (5.30)$$

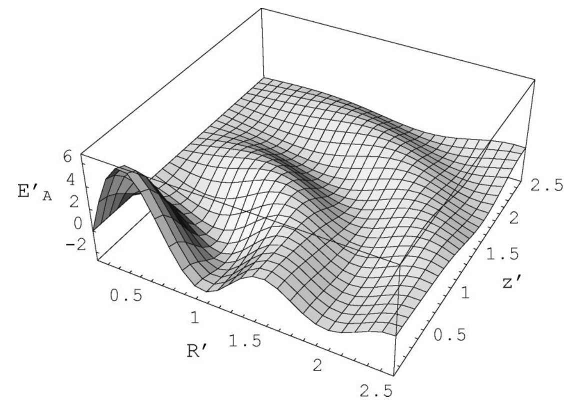

FIG. 6. Normalized azimuthal component E_A' of the instantaneous electric field of E_A-wave as a function of $R'=R/\lambda$ and $z'=z/\lambda$; $j=s=0$; $\omega t=\pi/4$.

The energy densities w_e and w_m as well as the only nonvanishing component S_A' of $\mathbf{S}$ are independent of the azimuthal angle ψ. For the beams defined by the zonal spherical harmonics, $\mathbf{S}(\mathbf{r})=0$ for all $\mathbf{r}$.

Since the beams under consideration are composed from eigenwaves of all possible propagation directions, they are in effect the three-dimensional standing waves with rather involved structures of interrelated electric and magnetic fields specified by functions $\mathbf{M}_j^{s0}[1]$ [Eq. (5.25)] and $\mathbf{A}_j^{s0}[1]$ [Eq. (5.26)] (see also the Appendix and Figs. 6 and 7). Beams with $s\neq 0$ are essentially electromagnetic whirls with azimuthal energy fluxes. For any of these waves, i.e., at any values $j=0,1,\ldots;s=0,\pm 1,\ldots,\pm j$, the time average outgoing energy fluxes are vanishing everywhere: $\mathbf{e}_3\cdot\mathbf{S}(\mathbf{r})=0,\mathbf{e}_r\cdot\mathbf{S}(\mathbf{r})=0$. Since $\mathbf{e}_3\cdot\mathbf{S}(\mathbf{r})=0$ for all $\mathbf{r}$, the normalization of the form (3.17) is inapplicable in this case. The normalization parameter N_Q specifies the azimuthal energy flux and the energy density by virtue of S_0 and w_0 [Eq. (5.20)], which are the same for all standing waves under consideration.

The electromagnetic beams, considered in this section, are time harmonic in the Lorentz reference frame L with the basis $(\mathbf{e}_i, i=1,2,3,4)$. In this frame, the intensity of field oscillations is time independent. It depends only on r and γ and tends to zero as r approaches infinity. In other Lorentz frames such waves will be observed as a moving localized field with a rather involved dependence of its components on

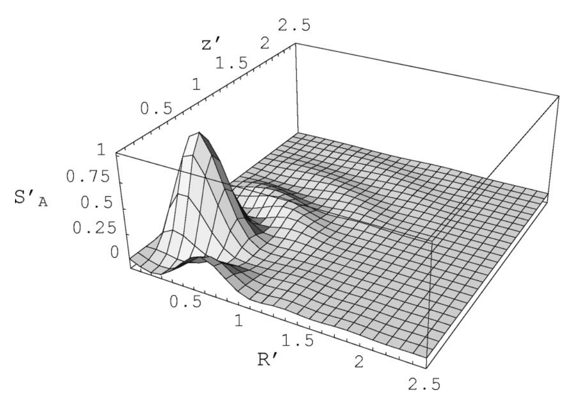

FIG. 7. Azimuthal component S_A' [Eq. (5.30)] of the normalized energy flux vector as a function of R' and z'; $j=3$, $s=1$.

spatial and temporal coordinates. Since the time-average energy flux of an electromagnetic three-dimensional standing wave with $s\neq 0$ is azimuthal in the rest frame L [see Eq. (5.29)], in other frames it will be observed as a kind of moving electromagnetic whirl or electromagnetic "missile." If $s=0$, its Poynting vector is identically zero in L, but the term "whirl" still can be used to emphasize the peculiarities of the field structure (see Fig. 6).

VI. MOVING ELECTROMAGNETIC WHIRLS

The parameters of electromagnetic whirls (see Sec. V B) are calculated in the Lorentz frame L with the basis $(\mathbf{e}_i)$ and the space-time coordinates $(\mathbf{r},t)$. Let L' be a Lorentz frame with the basis $(\mathbf{e}_{i'})$ and the space-time coordinates $(\mathbf{r}',t')$, in which a whirl and, hence, the frame L are uniformly moving at velocity $\mathbf{V}$. The three-dimensional localization of the field is a characteristic feature for both the whirls presented here and the focus wave modes [41]. However, they differ fundamentally: all moving whirls, treated in this paper, satisfy the condition $V<c$, whereas the focus wave modes [41] are moving at light velocity.

The electromagnetic whirls, presented in Sec. V B, are described by the twice-contravariant antisymmetric field tensors

$$F(\mathbf{r},t)=\frac{\nu_0}{\sqrt{2}}e^{-i\omega t}\{\mathcal{E}\cdot\mathbf{A}_j^{s0}[1](\mathbf{r})+\mathbf{e}_4\wedge\mathbf{M}_j^{s0}[1](\mathbf{r})\}, \quad (6.1a)$$

$$F(\mathbf{r},t)=\frac{\nu_0}{\sqrt{2}}e^{-i\omega t}\{-\mathcal{E}\cdot\mathbf{M}_j^{s0}[1](\mathbf{r})+\mathbf{e}_4\wedge\mathbf{A}_j^{s0}[1](\mathbf{r})\}, \quad (6.1b)$$

where

$$\mathcal{E}=\mathbf{e}_1\wedge\mathbf{e}_2\otimes\mathbf{e}_3+\mathbf{e}_3\wedge\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\wedge\mathbf{e}_3\otimes\mathbf{e}_1, \quad (6.2)$$

and $F\equiv\mathcal{E}\cdot\mathbf{B}+\mathbf{e}_4\wedge\mathbf{E}$, the exterior and dot products [23,47] are given by $\mathbf{e}_i\wedge\mathbf{e}_j=\mathbf{e}_i\otimes\mathbf{e}_j-\mathbf{e}_j\otimes\mathbf{e}_i$ and $(\mathbf{e}_i\wedge\mathbf{e}_j\otimes\mathbf{e}_k)\cdot\mathbf{B}=\mathbf{e}_i\wedge\mathbf{e}_j(\mathbf{e}_k\cdot\mathbf{B})$. In the analytical investigating of moving whirls, it is convenient to use the intrinsic tensor techniques, developed in Ref. [23] on the basis of the exterior algebra [47]. Since

$$\mathbf{x}=\mathbf{r}+ct\mathbf{e}_4=\mathbf{r}'+ct'\mathbf{e}_{4'}, \quad (6.3a)$$

$$\mathbf{e}_4=\gamma(\boldsymbol{\beta}+\mathbf{e}_{4'}), \quad (6.3b)$$

where $\mathbf{r}\cdot\mathbf{e}_4=\mathbf{r}'\cdot\mathbf{e}_{4'}=\boldsymbol{\beta}\cdot\mathbf{e}_{4'}=0$, $\gamma=(1-\boldsymbol{\beta}^2)^{-1/2}$, $\boldsymbol{\beta}=\mathbf{V}/c$, the tensor fields $F^{i'j'}(\mathbf{r}',t')$ of the moving electromagnetic whirls can be readily calculated from Eq. (6.1) by the Lorentz transformaton. By way of illustration, the normalized energy density of the electromagnetic whirl F [Eq. (6.1a)], moving with the velocity $\mathbf{V}=0.95c\mathbf{e}_{1'}$ with respect to L', is presented in Fig. 8.

It seems that similar moving electromagnetic whirls can also propagate in dispersive linear media. The corresponding solutions of wave equations can be found as follows. Let L' and L be the rest frame of the medium and the uniformly moving frame in which the whirl to be found is time harmonic with frequency ω. By substituting $\mathbf{K}=k\hat{\mathbf{k}}(\theta,\varphi)+\mathbf{e}_4\omega/c$ [$\hat{\mathbf{k}}(\theta,\varphi)$ is given by Eq. (4.2) with $\theta_1=\theta$ and $\varphi_1=\varphi$] in the four-dimensional dispersive equation (2.4) and solving it for the unknown wave number k at all $\theta\in[0,\pi]$ and $\varphi\in[0,2\pi]$, one can obtain the three-dimensional wave vector $\mathbf{k}=k(\theta,\varphi)\hat{\mathbf{k}}(\theta,\varphi)$ in the frame L as well as the four-dimensional one $\mathbf{K}=\mathbf{k}(\theta,\varphi)+\mathbf{e}_4\omega/c$ as functions of θ and φ. Then, by solving Eq. (2.3) with $\mathbf{K}=\mathbf{K}(\theta,\varphi)$, one can find the amplitude function $\mathbf{W}(\theta,\varphi)$. Finally, substitution of $\mathbf{k}(\theta,\varphi)$ and $\mathbf{W}(\theta,\varphi)$ in Eqs. (2.25) and (5.2) results in a beam defined by the spherical harmonic Y_j^s, which will be observed as a time-harmonic standing wave in the frame L and as a moving whirl in the frame L' related with the medium. Naturally, to perform all this, the dispersive properties of the medium should be specified in an explicit form.

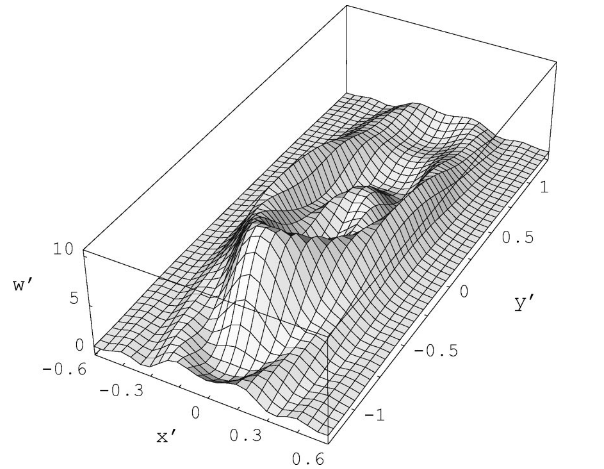

FIG. 8. Normalized energy density w' of an electromagnetic whirl moving with the velocity $\mathbf{V}=0.95c\mathbf{e}_{1'}$ with respect to the frame L'; $j=3$, $s=1$; $x'=x^{1'}/\lambda$; $y'=x^{2'}/\lambda$; $x^{3'}=0$.

VII. BEAMS WITH THREE-DIMENSIONAL BASE

A. Orthonormal beams

The presented above technique can be readily extended to the case of fields with three-dimensional beam manifold $\mathcal{B}_3$. One can replace Eqs. (2.7) and (2.8) by

$$\breve{\mathbf{W}}_n(\mathbf{x})\equiv\breve{\mathbf{W}}_n(\mathbf{r},t)=\int_{\mathcal{B}_3}e^{i\mathbf{x}\cdot\mathbf{K}(b)}\nu(b)u_n(b)\mathbf{W}(b)d\mathcal{B}_3, \quad (7.1)$$

$$\breve{s}_{mn}\equiv\langle\breve{\mathbf{W}}_m|Q|\breve{\mathbf{W}}_n\rangle =\int_{-\infty}^{+\infty}dt\int_{\sigma_0}\breve{\mathbf{W}}_m^{\dagger}(\mathbf{r},t)Q\breve{\mathbf{W}}_n(\mathbf{r},t)d\sigma_0. \quad (7.2)$$

This provides a natural way to generalize the time-harmonic beams, presented in Sec. V A, to beams with more involved time dependence:

$$\breve{\mathbf{W}}_n(\mathbf{r},t)=\int_{\omega_1}^{\omega_2}f_n(\omega)\sqrt{w(\omega)}\mathbf{W}_n(\mathbf{r},t)d\omega, \quad (7.3)$$

where $\mathbf{W}_n(\mathbf{r},t)$ is given by Eq. (2.7) with the two-dimensional manifold $\mathcal{B}$, i.e., $\mathcal{B}_3=\mathcal{B}\times[\omega_1,\omega_2]$, and (f_n) are real orthonormal functions, for example, orthogonal polynomials [45], with the weight function $w>0$, i.e.,

$$\int_{\omega_1}^{\omega_2} f_m(\omega) f_n(\omega) w(\omega) d\omega = h_m \delta_{mn}, \tag{7.4}$$

and (h_m) are the normalizing coefficients. In this case,

$$\breve{s}_{mn} = 2\pi \int_{\omega_1}^{\omega_2} f_m(\omega) f_n(\omega) w(\omega) s_{mn} d\omega \tag{7.5}$$

with s_{mn} [Eq. (2.8)]. For all beams $\mathbf{W}_n(\mathbf{r},t)$ treated in Sec. V A, s_{mn} is frequency independent. Hence Eqs. (7.4) and (2.8) yield $\breve{s}_{mn} = 2\pi h_m s_{mn} \delta_{mn}$. This enables the results, obtained in this paper, to be expanded to cover the beams $\breve{\mathbf{W}}_n(\mathbf{r},t)$ [Eq. (7.3)]. Since, for all orthonormal beams $\mathbf{W}_n(\mathbf{r},t)$ with the two-dimensional base $\mathcal{B}$, s_{mn} specifies the energy flux through the plane σ_0, for the corresponding orthonormal beams $\breve{\mathbf{W}}_n(\mathbf{r},t)$ [Eq. (7.3)], $\breve{s}_{mn}$ [Eq. (7.2)] specifies the total energy transmission through this plane. To find in an explicit form the beams $\breve{\mathbf{W}}_n(\mathbf{r},t)$ [Eq. (7.3)] in a dispersive medium, the frequency dependence of constitutive parameters must be taken into account.

B. Quasimonochromatic beams

Quasimonochromatic beams

$$\breve{\mathbf{W}}_j^s(\mathbf{r},t) = \frac{1}{\omega_0} \int_{\omega_-}^{\omega_+} \mathbf{W}_j^s(\mathbf{r},t) d\omega \tag{7.6}$$

with $\mathbf{W}_j^s(\mathbf{r},t)$ [Eq. (5.2)], defined by the spherical harmonics, is another interesting special case of beams with three-dimensional base $\mathcal{B}_3 = S^2 \times [\omega_-, \omega_+]$. By way of illustration, let us consider electromagnetic beams in free space given by

$$\breve{F}(\mathbf{r},t) = \frac{1}{\omega_0} \int_{\omega_-}^{\omega_+} F(\mathbf{r},t) d\omega \tag{7.7}$$

with $F(\mathbf{r},t)$ [Eq. (6.1)], and $\omega_\pm = \omega_0 \pm \Delta\omega$ ($\Delta\omega \ll \omega_0$). This yields two types of beams uniquely defined by the vector functions

$$\begin{aligned}\breve{\mathbf{M}}_j^{s0}[1] &= \frac{1}{\omega_0} \int_{\omega_-}^{\omega_+} e^{-i\omega t} \mathbf{M}_j^{s0}[1] d\omega \\ &= e^{is\psi}\{(\mathbf{e}_R + i\mathbf{e}_A) i^{|s-1|+q} L_{jq}^{ss-1}[\cos] \\ &\quad + (\mathbf{e}_R - i\mathbf{e}_A) i^{|s+1|+q} L_{jq}^{ss+1}[\cos] \\ &\quad - 2\mathbf{e}_3 i^{|s|+p} L_{jp}^{ss}[\sin]\},\end{aligned} \tag{7.8}$$

$$\begin{aligned}\breve{\mathbf{A}}_j^{s0}[1] &= \frac{1}{\omega_0} \int_{\omega_-}^{\omega_+} e^{-i\omega t} \mathbf{A}_j^{s0}[1] d\omega \\ &= e^{is\psi}\{(\mathbf{e}_A - i\mathbf{e}_R) i^{|s-1|+p} L_{jp}^{ss-1}[1] \\ &\quad + (\mathbf{e}_A + i\mathbf{e}_R) i^{|s+1|+p} L_{jp}^{ss+1}[1]\},\end{aligned} \tag{7.9}$$

where

$$\begin{aligned}L_{jp}^{sm}[f] &= \frac{1}{\omega_0} \int_{\omega_-}^{\omega_+} e^{-i\omega t} J_{jp}^{sm}[f] d\omega \\ &= \sum_{\nu=0}^{+\infty} (-1)^\nu g_{|m|+2\nu+p}(r,t) \\ &\quad \times P_{|m|+2\nu+p}^{|m|}(\cos\gamma) \mathcal{P}_{j|m|+2\nu+p}^{s|m|}[f],\end{aligned} \tag{7.10}$$

$$g_l(r,t) = \frac{1}{\omega_0} \int_{\omega_-}^{\omega_+} j_l(k(\omega) r) e^{-i\omega t} d\omega. \tag{7.11}$$

The functions $\breve{\mathbf{M}}_j^{s0}[1]$ [Eq. (7.8)] and $\breve{\mathbf{A}}_j^{s0}[1]$ [Eq. (7.9)] can be obtained from the amplitude functions $\mathbf{M}_j^{s0}[1]$ [Eq. (5.25)] and $\mathbf{A}_j^{s0}[1]$ [Eq. (5.26)] of electromagnetic standing waves by the replacement of the time-independent functions $J_{jp}^{sm}[f](r,\gamma)$ [Eq. (A5)] by the time-dependent functions $L_{jp}^{sm}[f](r,\gamma,t)$ [Eq. (7.10)]. To obtain the field tensors $\breve{F}$ for the beams under consideration, it is sufficient to replace $e^{-i\omega t}\mathbf{M}_j^{s0}[1]$ and $e^{-i\omega t}\mathbf{A}_j^{s0}[1]$ in Eq. (6.1) by $\breve{\mathbf{M}}_j^{s0}[1]$ [Eq. (7.8)] and $\breve{\mathbf{A}}_j^{s0}[1]$ [Eq. (7.9)]. These beams are composed of eigenwave packets radially moving in all possible directions with the group velocity $v_g = \partial\omega/\partial k$. In the case of electromagnetic beams in vacuum, $k = \omega/c$ and $v_g = c$.

The evolution of such beams is specified by functions g_l [Eq. (7.11)], which can be approximated as follows. The spherical Bessel function $j_l(z)$ can be written [45]

$$j_l(z) = f_l(z)\sin z + (-1)^{l+1} f_{-l-1}(z)\cos z, \tag{7.12}$$

where $l = 0, \pm 1, \pm 2, \ldots$, and the functions $f_l(z)$ are given by the recurrence formula

$$f_{l-1}(z) + f_{l+1}(z) = (2l+1) z^{-1} f_l(z) \tag{7.13}$$

with $f_0(z) = z^{-1}$ and $f_1(z) = z^{-2}$. Using the condition $\Delta\omega \ll \omega_0$, from Eqs. (7.11) and (7.12) we obtain

$$\begin{aligned}g_l(r,t) \approx \frac{\Delta\omega}{\omega_0} e^{-i\omega_0 t}\{ & e^{ik_0 r}[f_l(k_0 r) \\ & + i(-1)^{l+1} f_{-l-1}(k_0 r)] j_0[\Delta\omega(r/v_g - t)] \\ & + e^{-ik_0 r}[-f_l(k_0 r) \\ & + i(-1)^{l+1} f_{-l-1}(k_0 r)] j_0[\Delta\omega(r/v_g + t)]\},\end{aligned} \tag{7.14}$$

where $k_0 = 2\pi/\lambda_0 = k(\omega_0)$. The function $g_l(r,t)$ [Eq. (7.14)] tends to zero as $t \to \pm\infty$. Hence, for the beams under consideration, when $t \to \pm\infty$, $\breve{\mathbf{W}}_j^s(\mathbf{r},t)$ tends to zero at all points $\mathbf{r}$. The obtained solutions describe initiation and evolution of a whirl, which originates at the infinity at $t = -\infty$ as infinitely small converging wave propagating with the group velocity v_g. At $\Delta\omega t \ll -1$, its amplitude profile can be roughly approximated by the function $j_0[\Delta\omega(r/v_g + t)]$. It has an infinite series of peaks, the highest of which is at the distance $r = -v_g t$. As $t \to 0$, this wave, growing in amplitude, approches the origin $\mathbf{r} = 0$ and forms a whirl which varies in intensity as different ''peaks and valleys'' reach the neighborhood of the point $\mathbf{r} = 0$. At $t = 0$, the whirl reaches its

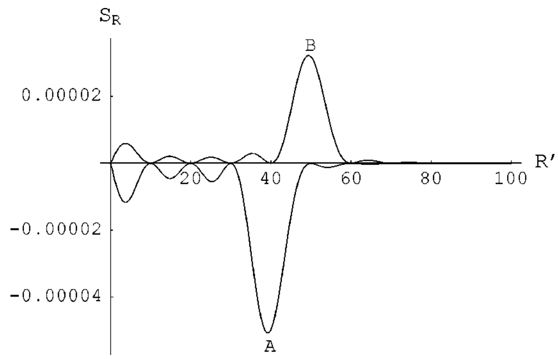

FIG. 9. Radial component S_R' of the normalized energy flux vector as a function of $R'=R/\lambda_0$; $z=0$; $j=s=0$; $\Delta\omega/\omega_0=0.05$; (A) $\omega_0 t=-40\lambda_0$; (B) $\omega_0 t=50\lambda_0$.

maximum intensity. The total field can be described as the superposition of converging and expanding waves with ever changing proportion, given by the functions $j_0[\Delta\omega(r/v_g+t)]$ and $j_0[\Delta\omega(r/v_g-t)]$, respectively. At $t>0$, the whirl, still passing through maximums and minimums of activity, gradually transforms into an expanding wave which vanishes in the infinity as $t\rightarrow+\infty$. Resuming, at $\Delta\omega t\ll-1$ and $\Delta\omega t\gg 1$, there are converging and expanding waves (see Fig. 9). In between, there is the oscillating whirl in the neighborhood of the origin.

VIII. GENERAL BEAM

The general time-harmonic beam with two-dimensional manifold $\mathcal{B}$ and the general beam with more involved time dependence and three-dimensional manifold $\mathcal{B}$ can be written as

$$\mathbf{W}(\mathbf{r},t)=e^{-i\omega t}\int_{\mathcal{B}}e^{i\mathbf{r}\cdot\mathbf{k}(b)}\nu(b)u(b)\mathbf{W}(b)d\mathcal{B}, \quad (8.1a)$$

$$\mathbf{W}(\mathbf{x})=\int_{\mathcal{B}}e^{i\mathbf{x}\cdot\mathbf{K}(b)}\nu(b)u(b)\mathbf{W}(b)d\mathcal{B}, \quad (8.1b)$$

where $u:\mathcal{B}\rightarrow C^1$ is a complex scalar function on $\mathcal{B}$. Let (u_n) be an orthonormal base of complex functions on $\mathcal{B}$. Then, the function u can be expanded into a series as

$$u(b)=\sum_n c_n u_n(b), \quad (8.2)$$

where $c_n=\langle u_m|u\rangle$. Hence, from Eqs. (2.7), (8.1), and (8.2) we obtain an expansion of $\mathbf{W}$ [Eq. (8.1)] into a series of beams $\mathbf{W}_n$ [Eq. (2.7)] or $\mathbf{W}_n$ [Eq. (7.3)] as

$$\mathbf{W}=\sum_n c_n\mathbf{W}_n, \quad (8.3)$$

which is valid for all types of beams. It is essential that, for the orthonormal beams (beams II), the coefficients c_n can be extracted from the beam $\mathbf{W}$ as follows:

$$c_n=\frac{1}{N_Q}\langle\mathbf{W}_n|Q|\mathbf{W}\rangle. \quad (8.4)$$

What is even more important, they are measurable values, provided that there exists a source of orthonormal beams $\mathbf{W}_n$. As shown in Secs. V A and VII A, $\mathcal{I}=\langle\mathbf{W}|Q|\mathbf{W}\rangle$ is the energy flux through the plane σ_0 in the case of time-harmonic beams with two-dimensional manifold $\mathcal{B}$, and it is the total energy transmitted through this plane in the case of beams $\mathbf{W}_n$ [Eq. (7.3)] with three-dimensional $\mathcal{B}$. In both cases, $\mathcal{I}$ can be measured.

Each of the complex coefficients c_n of the beam $\mathbf{W}$ [Eq. (8.3)] can be calculated from the results of three measurements as

$$c_n=\frac{\mathcal{I}_2-\mathcal{I}_1+i(\mathcal{I}_3-\mathcal{I}_1)}{2\alpha N_Q}, \quad (8.5)$$

where

$$\mathcal{I}_1=\langle\mathbf{W}|Q|\mathbf{W}\rangle, \quad (8.6a)$$

$$\mathcal{I}_2=\langle(\mathbf{W}+\alpha\mathbf{W}_n)|Q|(\mathbf{W}+\alpha\mathbf{W}_n)\rangle, \quad (8.6b)$$

$$\mathcal{I}_3=\langle(\mathbf{W}+i\alpha\mathbf{W}_n)|Q|(\mathbf{W}+i\alpha\mathbf{W}_n)\rangle, \quad (8.6c)$$

and $\alpha^*=\alpha$ is a real parameter specifying the amplitude of an auxiliary beam $\mathbf{W}_n$. In the second and third measurements, the energy flux or the total energy is measured for superpositions of $\mathbf{W}$ with the auxiliary beams $\alpha\mathbf{W}_n$ and $e^{i\pi/2}\alpha\mathbf{W}_n$, respectively, i.e., in the third measurement the additional phase shift $\pi/2$ or $\pi/2+2m\pi$, $m=1,2,\ldots$ is inserted.

This provides a means to generalize the solutions of the direct and inverse scattering problems, obtained in Refs. [16,17,20–25] for the case of plane harmonic incident waves, to the case of time-harmonic beams obliquely incident onto a general bianisotropic slab. To this end, the fields of reflected and transmitted waves can be expanded into a series of orthonormal beams. The presented techniques make it possible to calculate the complex scalar coefficients of these series. Assuming that they are given (or measured), it is possible to reconstruct the reflection and the transmission coefficients of the slab for partial incident plane waves and then, using the techniques developed in Refs. [16,17,23], to extract the whole set of constitutive parameters of the medium under study.

IX. CONCLUSION

In this paper, linear fields defined by a set of orthonormal scalar functions on a two-dimensional or three-dimensional beam manifold $\mathcal{B}$ are treated. The presented technique makes it possible to compose a set of orthonormal beams, normalized to either the energy flux through a given plane σ_0 (beams with two-dimensional $\mathcal{B}$) or to the total energy transmitted through this plane (beams with three-dimensional $\mathcal{B}$), as well as some other specific exact solutions of wave equations such as three-dimensional standing waves, moving and evolving whirls.

By applying this approach to electromagnetic waves in isotropic media, unique families of exact solutions of Maxwell's equations are obtained. Each family consists of normalized or orthonormal vector, functions which have integral expansions in eigenwaves with wave normals lying in the same given solid angle Ω.

The families of orthonormal beams with $\Omega = 2\pi$, i.e., the superpositions of eigenwaves propagating into a given half space, and the families of three-dimensional standing waves with $\Omega = 4\pi$, i.e., the superpositions of eigenwaves propagating in all directions, are of particular interest. The former, owing to the orthonormality conditions, forms convenient functional bases for more complex fields and provides a helpful technique for modeling the beams now in use and invesigating their scattering and propagation in various media. The later provides a unique global description of the complex medium under study, which is supplementary to the eigenwave description. Whereas each eigenwave specifies the properties of the medium for one particular direction of propagation, the field value of a three-dimensional standing wave in any point is defined by all eigenwaves. Moreover, even in free space or/and isotropic media they possess very interesting properties and rather involved field structure. The high energy density in a very small core region of beam (about several wavelengths of composing eigenwaves) is an inherent feature of all beams treated in the paper by way of illustration.

A mathematical formalism, facilitating analytical and numerical analysis of beams, defined by the spherical harmonics, is developed and illustrated by calculating fields, energy densities, and energy fluxes of various electromagnetic beams.

The obtained results provide a means to generalize the free-space techniques for characterizing complex media as well as the covariant wave-splitting technique to the case of beams.

The presented approach was also applied to elastic beams, sound waves in isotropic media, and weak gravitational waves. The obtained results for elastic, sound, and weak gravitational orthonormal beams, three-dimensional standing waves, and moving and evolving whirls will be presented elsewhere.

The solutions are found, which describe electromagnetic and weak gravitational whirls moving without dispersion with speed $0<V<c$. The solutions are also found, which describe evolving electromagnetic and gravitational whirls with finite time of observable activity. This brings up the question of whether such whirls exist in nature and, if so, how do they enter in physics and, in particular, astrophysics.

APPENDIX: SOME FUNCTIONS DEFINED BY THE SPHERICAL HARMONICS

In solving the scalar and vector inhomogeneous Helmholtz equations as well as various problems with spherical symmetry, the mathematical techniques based on the use of the vector spherical functions and Hansen's multipole functions [44] play a very important role. In particular, by using these functions, the radiated electromagnetic fields outside of a source region can be found by calculating the multipole moments [44]. In this paper, we treat the specific solutions of the homogeneous Helmholtz equations—the beams defined by the spherical harmonics. In this particular case, instead of the vector spherical functions and Hansen's multipole functions, it is more convenient to use another set of scalar and vector functions which are differently, but also closely, related with the scalar spherical harmonics. In this paper, the potentialities of the proposed technique are illustrated by calculating fields, energy densities, and energy fluxes of various electromagnetic beams. However, it is also very useful in the analysis of sound, elastic, and weak gravitational fields defined by the spherical harmonics.

1. Function U_{jN}^{sn}

Let $f=f(\theta)$ be a scalar, vector, or tensor function of the polar angle θ. Let us introduce a function U_{jN}^{sn} of $\mathbf{r}$, defined by f and the spherical harmonic Y_j^s through the integration over the northern hemisphere as follows:

$$\begin{aligned} U_{jN}^{sn} &= U_{jN}^{sn}[f](\mathbf{r}) \\ &\equiv U_{jN}^{sn}(r,\gamma,\psi) \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi/2} e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)} Y_j^s(\theta,\varphi) f(\theta) e^{in\varphi} \sin\theta\, d\theta \\ &= e^{i(s+n)\psi} I_j^{ss+n}[f](r,\gamma), \end{aligned} \tag{A1}$$

where n is an integer, $\mathbf{e}_r(\theta,\varphi)$ is given by Eq. (5.7a), r,γ,ψ are the spherical coordinates $[\mathbf{r}=\mathbf{r}(r,\gamma,\psi)$, Eq. (4.3)] and

$$\begin{aligned} I_j^{sm} &= I_j^{sm}[f](r,\gamma) \\ &= \sum_{l=|m|}^{+\infty} i^l j_l(kr) P_l^{|m|}(\cos\gamma) \mathcal{P}_{jl}^{sm}[f], \end{aligned} \tag{A2}$$

$$\begin{aligned} \mathcal{P}_{jl}^{sm} &= \mathcal{P}_{jl}^{sm}[f] \\ &= 8\pi^2 N_{js} N_{lm}^2 \int_0^{\pi/2} P_j^{|s|}(\cos\theta) P_l^{|m|}(\cos\theta) f(\theta) \sin\theta\, d\theta. \end{aligned} \tag{A3}$$

The above notations emphasize the fact that $\mathcal{P}_{jl}^{sm}$ [Eq. (A3)] and U_{jN}^{sn} [Eq. (A1)], at fixed $\mathbf{r}=\mathbf{r}(r,\gamma,\psi)$, and I_j^{sm} [Eq. (A2)], at fixed r and γ, are functionals regarding f. For any given f, on the other hand, $\mathcal{P}_{jl}^{sm}$ is a constant, whereas U_{jN}^{sn} and I_j^{sm} are the functions of $\mathbf{r}$ and r,γ, respectively. If it cannot cause a misunderstanding, we shall omit the arguments $(\mathbf{r})$, (r,γ), or $[f]$, in particular, when all terms of equations refer to the same arbitrary point $(\mathbf{r})$ or (r,γ), or the same arbitrary function $[f]$ [see Eqs. (A4), (A20), and (A21)].

The real and imaginary parts of I_j^{sm} [Eq. (A2)] can be separated as

$$I_j^{sm} = i^{|m|}(J_{j0}^{sm} + iJ_{j1}^{sm}), \tag{A4}$$

where

$$\begin{aligned} J_{jp}^{sm} &= J_{jp}^{sm}[f](r,\gamma) \\ &= \sum_{\nu=0}^{+\infty} (-1)^\nu j_{|m|+2\nu+p}(kr) \\ &\quad \times P_{|m|+2\nu+p}^{|m|}(\cos\gamma) \mathcal{P}_{j|m|+2\nu+p}^{s|m|}[f]. \end{aligned} \tag{A5}$$

2. Function U_{jS}^{sn}

Since $Y_j^s(\pi-\theta,\varphi)=(-1)^{j+|s|}Y_j^s(\theta,\varphi)$, the function U_{jS}^{sn}, defined by integrating over the southern hemisphere, can be expressed in terms of U_{jN}^{sn} [Eq. (A1)] as

$$\begin{aligned}U_{jS}^{sn}&=U_{jS}^{sn}[f](\mathbf{r})\\&\equiv U_{jS}^{sn}(r,\gamma,\psi)\\&=\int_0^{2\pi}d\varphi\int_{\pi/2}^{\pi}e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)}Y_j^s(\theta,\varphi)f(\theta)e^{in\varphi}\sin\theta\,d\theta\\&=(-1)^{j+|s|}U_{jN}^{sn}[f_-](R_3\mathbf{r})\\&\equiv(-1)^{j+|s|}U_{jN}^{sn}[f_-](r,\pi-\gamma,\psi),\end{aligned}\tag{A6}$$

where $f_-=f(\pi-\theta)$ and

$$\begin{aligned}R_m&=\mathbb{1}-2\mathbf{e}_m\otimes\mathbf{e}_m\\&=\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_3-2\mathbf{e}_m\otimes\mathbf{e}_m\end{aligned}\tag{A7}$$

is the operator of mirror reflection in the plane normal to $\mathbf{e}_m[m=1,2,3;R_3\mathbf{r}(r,\gamma,\psi)=\mathbf{r}(r,\pi-\gamma,\psi)]$.

If the function f satisfies the condition

$$f_-(\theta)\equiv f(\pi-\theta)=(-1)^q f(\theta),\tag{A8}$$

the functions U_{jN}^{sn} [Eq. (A1)] and U_{jS}^{sn} [Eq. (A6)] have even greater structural similarity

$$U_{jN}^{sn}=i^{|s+n|}e^{i(s+n)\psi}(J_{j0}^{ss+n}+iJ_{j1}^{ss+n}),\tag{A9}$$

$$U_{jS}^{sn}=(-1)^{j+|s|+q}i^{|s+n|}e^{i(s+n)\psi}(J_{j0}^{ss+n}-iJ_{j1}^{ss+n}).\tag{A10}$$

3. Functions $\mathbf{R}_{jN}^{sn}$, $\mathbf{M}_{jN}^{sn}$, and $\mathbf{A}_{jN}^{sn}$

In this paper, we widely use vector functions defined by a scalar function $f=f(\theta)$, the spherical harmonic Y_j^s, and the radial, the meridional, and the azimuthal basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\varphi$ [Eq. (5.7)] as

$$\begin{aligned}\mathbf{R}_{jN}^{sn}&=\mathbf{R}_{jN}^{sn}[f](\mathbf{r})\\&\equiv\mathbf{R}_{jN}^{sn}[f](r,\gamma,\psi)\\&=\int_0^{2\pi}d\varphi\int_0^{\pi/2}e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)}Y_j^s(\theta,\varphi)\\&\quad\times f(\theta)e^{in\varphi}\mathbf{e}_r(\theta,\varphi)\sin\theta\,d\theta\\&=\frac{1}{2}(\mathbf{e}_1+i\mathbf{e}_2)U_{jN}^{sn-1}[f\circ\sin]\\&\quad+\frac{1}{2}(\mathbf{e}_1-i\mathbf{e}_2)U_{jN}^{sn+1}[f\circ\sin]+\mathbf{e}_3U_{jN}^{sn}[f\circ\cos]\\&=e^{i(s+n)\psi}\{\mathbf{e}I_j^{ss+n-1}[f\circ\sin]+\mathbf{e}^*I_j^{ss+n+1}[f\circ\sin]\\&\quad+\mathbf{e}_3I_j^{ss+n}[f\circ\cos]\},\end{aligned}\tag{A11}$$

$$\begin{aligned}\mathbf{M}_{jN}^{sn}&=\mathbf{M}_{jN}^{sn}[f](\mathbf{r})\\&\equiv\mathbf{M}_{jN}^{sn}[f](r,\gamma,\psi)\\&=\int_0^{2\pi}d\varphi\int_0^{\pi/2}e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)}Y_j^s(\theta,\varphi)\\&\quad\times f(\theta)e^{in\varphi}\mathbf{e}_\theta(\theta,\varphi)\sin\theta\,d\theta\\&=\frac{1}{2}(\mathbf{e}_1+i\mathbf{e}_2)U_{jN}^{sn-1}[f\circ\cos]\\&\quad+\frac{1}{2}(\mathbf{e}_1-i\mathbf{e}_2)U_{jN}^{sn+1}[f\circ\cos]-\mathbf{e}_3U_{jN}^{sn}[f\circ\sin]\\&=e^{i(s+n)\psi}\{\mathbf{e}I_j^{ss+n-1}[f\circ\cos]+\mathbf{e}^*I_j^{ss+n+1}[f\circ\cos]\\&\quad-\mathbf{e}_3I_j^{ss+n}[f\circ\sin]\},\end{aligned}\tag{A12}$$

$$\begin{aligned}\mathbf{A}_{jN}^{sn}&=\mathbf{A}_{jN}^{sn}[f](\mathbf{r})\\&\equiv\mathbf{A}_{jN}^{sn}[f](r,\gamma,\psi)\\&=\int_0^{2\pi}d\varphi\int_0^{\pi/2}e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)}Y_j^s(\theta,\varphi)\\&\quad\times f(\theta)e^{in\varphi}\mathbf{e}_\varphi(\varphi)\sin\theta\,d\theta\\&=\frac{1}{2}(\mathbf{e}_2-i\mathbf{e}_1)U_{jN}^{sn-1}[f]+\frac{1}{2}(\mathbf{e}_2+i\mathbf{e}_1)U_{jN}^{sn+1}[f]\\&=ie^{i(s+n)\psi}\{-\mathbf{e}I_j^{ss+n-1}[f]+\mathbf{e}^*I_j^{ss+n+1}[f]\},\end{aligned}\tag{A13}$$

where

$$\mathbf{e}=(\mathbf{e}_R+i\mathbf{e}_A)/2,\tag{A14a}$$

$$\mathbf{e}_R=\mathbf{e}_1\cos\psi+\mathbf{e}_2\sin\psi,\tag{A14b}$$

$$\mathbf{e}_A=-\mathbf{e}_1\sin\psi+\mathbf{e}_2\cos\psi,\tag{A14c}$$

$$\mathbf{r}=R\mathbf{e}_R+z\mathbf{e}_3,\ R=r\sin\gamma,\ z=r\cos\gamma,\tag{A14d}$$

and $f\circ g$ denotes the composition of functions f and g, i.e., $(f\circ g)(\theta)=f(g(\theta))$.

4. Functions $\mathbf{R}_{jS}^{sn}$, $\mathbf{M}_{jS}^{sn}$, and $\mathbf{A}_{jS}^{sn}$

The similar vector functions, related with the southern hemisphere $(\theta\in[\pi/2,\pi])$, are given

$$\begin{aligned}\mathbf{R}_{jS}^{sn}&=\mathbf{R}_{jS}^{sn}[f](\mathbf{r})\\&\equiv\mathbf{R}_{jS}^{sn}[f](r,\gamma,\psi)\\&=\int_0^{2\pi}d\varphi\int_{\pi/2}^{\pi}e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)}Y_j^s(\theta,\varphi)\\&\quad\times f(\theta)e^{in\varphi}\mathbf{e}_r(\theta,\varphi)\sin\theta\,d\theta\\&=\frac{1}{2}(\mathbf{e}_1+i\mathbf{e}_2)U_{jS}^{sn-1}[f\circ\sin]\\&\quad+\frac{1}{2}(\mathbf{e}_1-i\mathbf{e}_2)U_{jS}^{sn+1}[f\circ\sin]+\mathbf{e}_3U_{jS}^{sn}[f\circ\cos],\end{aligned}\tag{A15}$$

$$
\begin{aligned}
\mathbf{M}_{jS}^{sn} &= \mathbf{M}_{jS}^{sn}[f](\mathbf{r}) \\
&\equiv \mathbf{M}_{jS}^{sn}[f](r,\gamma,\psi) \\
&= \int_0^{2\pi} d\varphi \int_{\pi/2}^{\pi} e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)} Y_j^s(\theta,\varphi) \\
&\quad \times f(\theta) e^{in\varphi} \mathbf{e}_\theta(\theta,\varphi) \sin\theta\, d\theta \\
&= \frac{1}{2}(\mathbf{e}_1 + i\mathbf{e}_2) U_{jS}^{sn-1}[f\circ \cos] \\
&\quad + \frac{1}{2}(\mathbf{e}_1 - i\mathbf{e}_2) U_{jS}^{sn+1}[f\circ \cos] - \mathbf{e}_3 U_{jS}^{sn}[f\circ \sin],
\end{aligned} \tag{A16}
$$

$$
\begin{aligned}
\mathbf{A}_{jS}^{sn} &= \mathbf{A}_{jS}^{sn}[f](\mathbf{r}) \\
&\equiv \mathbf{A}_{jS}^{sn}[f](r,\gamma,\psi) \\
&= \int_0^{2\pi} d\varphi \int_{\pi/2}^{\pi} e^{ik\mathbf{r}\cdot\mathbf{e}_r(\theta,\varphi)} Y_j^s(\theta,\varphi) \\
&\quad \times f(\theta) e^{in\varphi} \mathbf{e}_\varphi(\varphi) \sin\theta\, d\theta \\
&= \frac{1}{2}(\mathbf{e}_2 - i\mathbf{e}_1) U_{jS}^{sn-1}[f] + \frac{1}{2}(\mathbf{e}_2 + i\mathbf{e}_1) U_{jS}^{sn+1}[f].
\end{aligned} \tag{A17}
$$

It is evident from Eqs. (A1), (A6), (A11), and (A15) that

$$\nabla U_{jN}^{sn} = ik\mathbf{R}_{jN}^{sn}, \; \nabla U_{jS}^{sn} = ik\mathbf{R}_{jS}^{sn}. \tag{A18}$$

If the function f satisfies condition (A8), the following relations are valid [see also Eqs. (A6) and (A7)]:

$$\mathbf{R}_{jS}^{sn}(\mathbf{r}) = (-1)^{j+|s|+q} R_3 \mathbf{R}_{jN}^{sn}(R_3\mathbf{r}), \tag{A19a}$$

$$\mathbf{M}_{jS}^{sn}(\mathbf{r}) = -(-1)^{j+|s|+q} R_3 \mathbf{M}_{jN}^{sn}(R_3\mathbf{r}), \tag{A19b}$$

$$\mathbf{A}_{jS}^{sn}(\mathbf{r}) = (-1)^{j+|s|+q} \mathbf{A}_{jN}^{sn}(R_3\mathbf{r}). \tag{A19c}$$

5. Symmetry properties

Let $f = f(\theta)$ be a scalar function. Then, the scalar and vector functions defined above through the integration over the northern and southern hemisphere, have the same symmetry properties (so that we omit the subscripts N and S):

$$U_j^{-s,-n}(\mathbf{r}) = U_j^{sn}(R_2\mathbf{r}), \tag{A20a}$$

$$\mathbf{R}_j^{-s,-n}(\mathbf{r}) = R_2 \mathbf{R}_j^{sn}(R_2\mathbf{r}), \tag{A20b}$$

$$\mathbf{M}_j^{-s,-n}(\mathbf{r}) = R_2 \mathbf{M}_j^{sn}(R_2\mathbf{r}), \tag{A20c}$$

$$\mathbf{A}_j^{-s,-n}(\mathbf{r}) = R_1 \mathbf{A}_j^{sn}(R_2\mathbf{r}), \tag{A20d}$$

$$U_j^{sn}(S(\psi_{s+n})\mathbf{r}) = U_j^{sn}(\mathbf{r}), \tag{A21a}$$

$$\mathbf{R}_j^{sn}(S(\psi_{s+n})\mathbf{r}) = S(\psi_{s+n}) \mathbf{R}_j^{sn}(\mathbf{r}), \tag{A21b}$$

$$\mathbf{M}_j^{sn}(S(\psi_{s+n})\mathbf{r}) = S(\psi_{s+n}) \mathbf{M}_j^{sn}(\mathbf{r}), \tag{A21c}$$

$$\mathbf{A}_j^{sn}(S(\psi_{s+n})\mathbf{r}) = S(\psi_{s+n}) \mathbf{A}_j^{sn}(\mathbf{r}), \tag{A21d}$$

where R_1 and R_2 are given by Eq. (A7) $[R_1\mathbf{r}(r,\gamma,\psi) = \mathbf{r}(r,\gamma,\pi-\psi),\; R_2\mathbf{r}(r,\gamma,\psi) = \mathbf{r}(r,\gamma,-\psi)]$, and

$$
\begin{aligned}
S(\psi_s) &= e^{\psi_s \mathbf{e}_3^{\times}} \\
&= \mathbf{e}_3 \otimes \mathbf{e}_3 + \cos\psi_s(\mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2) \\
&\quad + \sin\psi_s(\mathbf{e}_2\otimes\mathbf{e}_1 - \mathbf{e}_1\otimes\mathbf{e}_2)
\end{aligned} \tag{A22}
$$

is the operator of rotation by $\psi_s = 2\pi/s\,(s\neq 0)$ about $\mathbf{e}_3$, i.e., $S(\psi_s)\mathbf{r}(r,\gamma,\psi) = \mathbf{r}(r,\gamma,\psi+\psi_s)$. The functions $\mathbf{R}_j^{00}$, $\mathbf{M}_j^{00}$, and $\mathbf{A}_j^{00}$, defined by the zonal spherical harmonic Y_j^0 ($s = 0$), also satisfy Eqs. (A20) and (A21) with $\psi_{s+n} \equiv \psi_0$ being an arbitrary angle.

6. Functions defined by the zonal spherical harmonics

If $s = n = 0$, Eqs. (A1), and (A11)–(A13) reduce to

$$U_{jN}^{00} = I_j^{00}[f], \tag{A23a}$$

$$\mathbf{R}_{jN}^{00} = \mathbf{e}_R I_j^{01}[f\circ \sin] + \mathbf{e}_3 I_j^{00}[f\circ \cos], \tag{A23b}$$

$$\mathbf{M}_{jN}^{00} = \mathbf{e}_R I_j^{01}[f\circ \cos] - \mathbf{e}_3 I_j^{00}[f\circ \sin], \tag{A23c}$$

$$\mathbf{A}_{jN}^{00} = \mathbf{e}_A I_j^{01}[f]. \tag{A23d}$$

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