Global exponential stability and periodic solutions of cellular neural networks with delay

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In this paper, some sufficient conditions for the global exponential stability and the existence of periodic solutions of cellular neural networks with delay (DCNN) model are obtained by means of a Lyapunov functional approach. These conditions can be used to design globally stable DCNN's and periodic oscillatory DCNN's and thus have important significance in both theory and applications.

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I. INTRODUCTION

Consider the following cellular neural networks with delay (DCNN) model:

$$x_i'(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(\sigma_j x_j(t-\tau_{ij})) + I_i(t), \quad c_i > 0, \quad i = 1,2,\ldots,n \tag{1}$$

in which n corresponds the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the ith unit at time t, $f_j(x_j(t))$ denotes the output of the jth unit at time t, a_{ij}, b_{ij}, τ_{ij}, σ_i, c_i are constants, a_{ij} denotes the strength of the jth unit on the ith unit at time t, b_{ij} denotes the strength of the jth unit on the ith unit at time $t-\tau_{ij}$, $I_i(t)$ denotes the external bias on the ith unit at time t, τ_{ij} corresponds to the transmission delay in the communication from the ith unit to the jth unit and is non-negative, σ_i denotes the amplifier gain, and c_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs.

In the following, we assume that each of the relations between the output of the cell f_i $(i=1,2,\ldots,n)$ and the state of the cell possess the following properties.

(H1) f_i $(i=1,2,\ldots,n)$ is bounded on R.

(H2) There is a number $\mu_i > 0$ such that

$$|f_i(u) - f_i(v)| \leqslant \mu_i |u-v|$$

for any $u,v \in R$.

The initial conditions associated with Eq. (1) are of the form

$$x_i(s) = \phi_i(s), \quad s \in [-\tau, 0], \quad \tau = \max_{1 \leqslant i,j \leqslant n} \tau_{ij}, \tag{2}$$

where it is usually assumed that ϕ_i is continuous, ϕ_i: $[-\tau,0] \to R$, $i=1,2,\ldots,n$.

Recently, the study of the DCNN model (1) has received much attention. See, for instance, Refs. [1–8] and the references cited therein. However, most of these papers only obtained the sufficient conditions for the global asymptotic stability of the DCNN model (1) in some special cases [1–7]. The purpose of this paper is to establish some sufficient conditions for the global exponential stability and the existence of periodic solutions of the DCNN model (1). Our results extend and improve the corresponding results in the above works.

II. STABILITY ANALYSIS

Consider the special case of the DCNN model (1) as $I_i(t) = I_i$, i.e.,

$$x_i'(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(\sigma_j x_j(t-\tau_{ij})) + I_i, \quad c_i > 0, \quad i = 1,2,\ldots,n, \tag{3}$$

where I_i, $i=1,2,\ldots,n$ are constant numbers.

Lemma 1. Assume that the output of the cell function f_i $(i=1,2,\ldots,n)$ satisfies the hypotheses (H1) and (H2) above. Then there exists an equilibrium for the DCNN model (3).

The proof is similar to that of Lemma 1 in Ref. [2] and is omitted.

Remark 1. Lemma 1 does not guarantee the uniqueness of the equilibrium. However, in this paper, we derive some criteria on the DCNN model (3), which will guarantee not only the uniqueness of the equilibrium but also the global exponential stability. The uniqueness of the equilibrium will follow from the global exponential stability to be established below.

Lemma 2. The following inequality holds:

$$y \prod_{k=1}^{m} x_k^{p_k} \leqslant \frac{1}{\alpha} \sum_{k=1}^{m} p_k x_k^{\alpha} + \frac{1}{\alpha} y^{\alpha} \quad \text{for any}$$

$$\alpha > 1, \quad x_k \geqslant 0 \quad (k=1,2,\ldots,m), \quad y \geqslant 0;$$

where $p_k > 0$ $(k=1,2,\ldots,m)$ are constants, and $\sum_{k=1}^{m} p_k = \alpha - 1$.

Proof. For fixed $x_k \geqslant 0$ $(k=1,2,\ldots,m)$, let

$$g(y) = \frac{1}{\alpha} \sum_{k=1}^{m} p_k x_k^{\alpha} + \frac{1}{\alpha} y^{\alpha} - y \prod_{k=1}^{m} x_k^{p_k}.$$

Then, we have

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$$g'(y)=y^{\alpha-1}-\prod_{k=1}^{m} x_k^{p_k}, \quad g''(y)=(\alpha-1)y^{\alpha-2}>0$$

$$\text{for } y>0.$$

From Theorem 1 in Ref. [10], we get

$$\prod_{k=1}^{m} x_k^{\alpha p_k/(\alpha-1)} \leqslant \frac{1}{\alpha-1}\sum_{k=1}^{m} p_k x_k^{\alpha}.$$

Therefore,

$$\inf_{0\leqslant y<+\infty} g(y)=g\left(\prod_{k=1}^{m} x_k^{p_k/(\alpha-1)}\right)$$
$$=\frac{1}{\alpha}\left(\sum_{k=1}^{m} p_k x_k^{\alpha}-(\alpha-1)\prod_{k=1}^{m} x_k^{\alpha p_k/(\alpha-1)}\right)$$
$$\geqslant 0.$$

This leads to the conclusion of Lemma 2.

Theorem 1. Suppose that the output of the cell function f_i $(i=1,2,\ldots,n)$ satisfies the hypotheses (H1) and (H2) above. Assume, furthermore, that the system parameters satisfy the following condition.

(H3) There exist constants $\alpha>1$, $d_i>0$, $i=1,2,\ldots,n$, such that

$$c_i>\frac{1}{\alpha}\sum_{j=1}^{n}\left(|a_{ij}|\sum_{k=1}^{m} p_k\mu_j^{\alpha\alpha_{kj}/p_k}+|b_{ij}||\sigma_j|\sum_{k=1}^{m} p_k\mu_j^{\alpha\beta_{kj}/p_k}\right)$$
$$+\frac{1}{\alpha d_i}\sum_{j=1}^{n}(|a_{ji}|d_j\mu_i^{\alpha\alpha_{m+1,i}}+|b_{ji}||\sigma_i|d_j\mu_i^{\alpha\beta_{m+1,i}}),$$
$$i=1,2,\ldots,n,$$

where α_{ij},β_{ij} $(i=1,2,\ldots,m+1,\ j=1,2,\ldots,n)$ are constants, and $\Sigma_{i=1}^{m+1}\alpha_{ij}=1$, $\Sigma_{i=1}^{m+1}\beta_{ij}=1$ $(j=1,2,\ldots,n)$, $p_k>0$ $(k=1,2,\ldots,m)$ are constants, and $\Sigma_{k=1}^{m}p_k=\alpha-1$.

Then the equilibrium x^* of the DCNN model (3) is globally exponentially stable.

Proof. The existence of solutions of Eqs. (3) and (2) for all $t\geqslant 0$ is an immediate consequence of Corollary D in Ref. [9]. If $x^*=(x_1^*,x_2^*,\ldots,x_n^*)^T$ is an equilibrium of the DCNN model (3), one can derive from Eq. (3) that the deviations

$$y_i(t)=x_i(t)-x_i^* \quad (i=1,2,\ldots,n)$$

satisfy

$$y_i'(t)=-c_iy_i(t)+\sum_{j=1}^{n} a_{ij}[f_j(x_j^*+y_j(t))-f_j(x_j^*)]$$
$$+\sum_{j=1}^{n} b_{ij}\{f_j(\sigma_j[x_j^*+y_j(t-\tau_{ij})])-f_j(\sigma_jx_j^*)\}. \tag{4}$$

Clearly, $(0,0,\ldots,0)^T$ is an equilibrium of Eq. (4). Therefore, to prove the global exponential stability of the DCNN model (3), it is sufficient to prove the global exponential stability of Eq. (4).

From hypothesis (H3), we know that there exists a positive constant $\lambda<c_i$ such that

$$c_i-\lambda>\frac{1}{\alpha}\sum_{j=1}^{n}\left(|a_{ij}|\sum_{k=1}^{m} p_k\mu_j^{\alpha\alpha_{kj}/p_k}\right.$$
$$\left.+|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\sum_{k=1}^{m} p_k\mu_j^{\alpha\beta_{kj}/p_k}\right)$$
$$+\frac{1}{\alpha d_i}\sum_{j=1}^{n}(|a_{ji}|d_j\mu_i^{\alpha\alpha_{m+1,i}}$$
$$+|b_{ji}||\sigma_i|e^{\lambda\tau_{ji}}d_j\mu_i^{\alpha\beta_{m+1,i}}), \quad i=1,2,\ldots,n.$$

Let

$$z_i(t)=y_i(t)e^{\lambda t} \quad (i=1,2,\ldots,n).$$

Substituting them into Eq. (4) and simplifying, we get

$$z_i'(t)=(\lambda-c_i)z_i(t)+e^{\lambda t}\left\{\sum_{j=1}^{n} a_{ij}[f_j(x_j^*+e^{-\lambda t}z_j(t))\right.$$
$$-f_j(x_j^*)]+\sum_{j=1}^{n} b_{ij}\{f_j(\sigma_j[x_j^*+e^{-\lambda(t-\tau_{ij})}z_j(t-\tau_{ij})])$$
$$\left.-f_j(\sigma_jx_j^*)\}\right\}. \tag{5}$$

Consider the following Lyapunov functional defined by

$$V(t)=\frac{1}{\alpha}\sum_{i=1}^{n} d_i\left[|z_i(t)|^{\alpha}+\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\mu_j^{\alpha\beta_{m+1,j}}\right.$$
$$\left.\times\int_{t-\tau_{ij}}^{t}|z_j(s)|^{\alpha}ds\right]. \tag{6}$$

Calculating the upper right derivative D^+V of V along the solutions of Eq. (5), we get

$$D^+V=\sum_{i=1}^{n} d_i\Bigg\{|z_i(t)|^{\alpha-1}D^+|z_i(t)|+\frac{1}{\alpha}\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\mu_j^{\alpha\beta_{m+1,j}}[|z_j(t)|^{\alpha}-|z_j(t-\tau_{ij})|^{\alpha}]\Bigg\}$$
$$\leqslant\sum_{i=1}^{n} d_i\Bigg\{(\lambda-c_i)|z_i(t)|^{\alpha}+e^{\lambda t}|z_i(t)|^{\alpha-1}\Bigg[\sum_{j=1}^{n}|a_{ij}||f_j(x_j^*+e^{-\lambda t}z_j(t))-f_j(x_j^*)|$$
$$+\sum_{j=1}^{n}|b_{ij}||f_j(\sigma_j[x_j^*+e^{-\lambda(t-\tau_{ij})}z_j(t-\tau_{ij})])-f_j(\sigma_j x_j^*)|\Bigg]+\frac{1}{\alpha}\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\mu_j^{\alpha\beta_{m+1,j}}[|z_j(t)|^{\alpha}-|z_j(t-\tau_{ij})|^{\alpha}]\Bigg\}$$
$$\leqslant\sum_{i=1}^{n} d_i\Bigg\{(\lambda-c_i)|z_i(t)|^{\alpha}+\Bigg[\sum_{j=1}^{n}|a_{ij}||z_i(t)|^{\alpha-1}\mu_j|z_j(t)|+\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\mu_j|z_i(t)|^{\alpha-1}|z_j(t-\tau_{ij})|\Bigg]$$
$$+\frac{1}{\alpha}\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\mu_j^{\alpha\beta_{m+1,j}}[|z_j(t)|^{\alpha}-|z_j(t-\tau_{ij})|^{\alpha}]\Bigg\}$$
$$=\sum_{i=1}^{n} d_i\Bigg\{(\lambda-c_i)|z_i(t)|^{\alpha}+\Bigg[\sum_{j=1}^{n}|a_{ij}||\mu_j^{\alpha_{m+1,j}}z_j(t)|\prod_{k=1}^{m}|\mu_j^{\alpha_{kj}/p_k}z_i(t)|^{p_k}$$
$$+\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}|\mu_j^{\beta_{m+1,j}}z_j(t-\tau_{ij})|\prod_{k=1}^{m}|\mu_j^{\beta_{kj}/p_k}z_i(t)|^{p_k}\Bigg]+\frac{1}{\alpha}\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\mu_j^{\alpha\beta_{m+1,j}}[|z_j(t)|^{\alpha}-|z_j(t-\tau_{ij})|^{\alpha}]\Bigg\}.$$

By Lemma 2, we get

$$D^+V\leqslant\sum_{i=1}^{n} d_i\Bigg\{(\lambda-c_i)|z_i(t)|^{\alpha}+\Bigg[\sum_{j=1}^{n}|a_{ij}|\Bigg(\frac{1}{\alpha}\sum_{k=1}^{m}p_k\mu_j^{\alpha\alpha_{kj}/p_k}|z_i(t)|^{\alpha}+\frac{1}{\alpha}\mu_j^{\alpha\alpha_{m+1,j}}|z_j(t)|^{\alpha}\Bigg)$$
$$+\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\Bigg(\frac{1}{\alpha}\sum_{k=1}^{m}p_k\mu_j^{\alpha\beta_{kj}/p_k}|z_i(t)|^{\alpha}+\frac{1}{\alpha}\mu_j^{\alpha\beta_{m+1,j}}|z_j(t-\tau_{ij})|^{\alpha}\Bigg)\Bigg]$$
$$+\frac{1}{\alpha}\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\mu_j^{\alpha\beta_{m+1,j}}[|z_j(t)|^{\alpha}-|z_j(t-\tau_{ij})|^{\alpha}]\Bigg\}$$
$$\leqslant\sum_{i=1}^{n} d_i\Bigg\{\lambda-c_i+\frac{1}{\alpha}\sum_{j=1}^{n}\Bigg(|a_{ij}|\sum_{k=1}^{m}p_k\mu_j^{\alpha\alpha_{kj}/p_k}+|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\sum_{k=1}^{m}p_k\mu_j^{\alpha\beta_{kj}/p_k}\Bigg)$$
$$+\frac{1}{\alpha d_i}\sum_{j=1}^{n}(|a_{ji}|d_j\mu_i^{\alpha\alpha_{m+1,i}}+|b_{ji}||\sigma_i|e^{\lambda\tau_{ji}}d_j\mu_i^{\alpha\beta_{m+1,i}})\Bigg\}|z_i(t)|^{\alpha}$$
$$\leqslant 0.$$

Thus, we have

$$V(t)\leqslant V(0),\quad t\geqslant 0.$$

Therefore, we get

$$\sum_{i=1}^{n} d_i|z_i(t)|^{\alpha}\leqslant\sum_{i=1}^{n} d_i\Bigg[|z_i(0)|^{\alpha}+\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\mu_j^{\alpha\beta_{m+1,j}}\int_{-\tau_{ij}}^{0}|z_j(s)|^{\alpha}ds\Bigg]$$
$$\leqslant\sum_{i=1}^{n} d_i\Bigg[|\phi_i(0)-x_i^*|^{\alpha}+\sum_{j=1}^{n}|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\mu_j^{\alpha\beta_{m+1,j}}\tau_{ij}\sup_{-\tau_{ij}\leqslant s\leqslant 0}|\phi_j(s)-x_j^*|^{\alpha}\Bigg]$$
$$\leqslant\Bigg[\sum_{i=1}^{n} d_i+\sum_{i=1}^{n}\sum_{j=1}^{n} d_i|b_{ij}||\sigma_j|e^{\lambda\tau_{ij}}\mu_j^{\alpha\beta_{m+1,j}}\tau_{ij}\Bigg]\phi^{\alpha},$$

where $\phi = \max_{-\tau \leqslant s \leqslant 0} \max_{1 \leqslant i \leqslant n} |\phi_i(s) - x_i^*|$.

Hence, we have

$$|y_i(t)| \leqslant \left[\frac{1}{d_i} \left(\sum_{i=1}^{n} d_i + \sum_{i=1}^{n} \sum_{j=1}^{n} d_i |b_{ij}| |\sigma_j| \times e^{\lambda \tau_{ij}} \mu_j^{\alpha \beta_{m+1,j}} \tau_{ij} \right) \right]^{1/\alpha} \phi e^{-\lambda t},$$

$$t \geqslant 0, \quad i = 1,2, \ldots, n.$$

This implies that the equilibrium x^* of the DCNN model (3) is globally exponentially stable.

Corollary 1. Suppose that the output of the cell function f_i $(i=1,2,\ldots,n)$ satisfies the hypotheses (H1) and (H2) above. Assume, furthermore, that the system parameters satisfy one of the following conditions:

$$\text{(I)} \quad c_i > \frac{\mu_i}{d_i} \sum_{j=1}^{n} (|a_{ji}| d_j + |b_{ji}| |\sigma_i| d_j), \quad i=1,2,\ldots,n;$$

$$\text{(II)} \quad c_i > \sum_{j=1}^{n} (|a_{ij}| \mu_j + |b_{ij}| |\sigma_j| \mu_j), \quad i=1,2,\ldots,n;$$

$$\text{(III)} \quad 2c_i > \sum_{j=1}^{n} (|a_{ij}| \mu_j^{2\alpha_j^*} + |b_{ij}| |\sigma_j| \mu_j^{2\alpha_j}) + \sum_{j=1}^{n} (|a_{ji}| \mu_i^{2\beta_i^*} + |b_{ji}| |\sigma_i| \mu_i^{2\beta_i}),$$

where $\alpha_i, \beta_i, \alpha_i^*, \beta_i^*$ are constants, and $\alpha_i + \beta_i = 1$, $\alpha_i^* + \beta_i^* = 1$, $i=1,2,\ldots,n$;

$$\text{(IV)} \quad 3c_i > \sum_{j=1}^{n} (|a_{ij}| \mu_j^{3\alpha_j^*} + |a_{ij}| \mu_j^{3\beta_j^*} + |b_{ij}| |\sigma_j| \mu_j^{3\alpha_j} + |b_{ij}| |\sigma_j| \mu_j^{3\beta_j}) + \sum_{j=1}^{n} (|a_{ji}| \mu_i^{3\gamma_i^*} + |b_{ji}| |\sigma_i| \mu_i^{3\gamma_i}),$$

where $\alpha_i, \beta_i, \gamma_i, \alpha_i^*, \beta_i^*, \gamma_i^*$ are constants, and $\alpha_i + \beta_i + \gamma_i = 1$, $\alpha_i^* + \beta_i^* + \gamma_i^* = 1$, $i=1,2,\ldots,n$.

Then the equilibrium x^* of DCNN model (3) is globally exponentially stable.

Proof. Assume condition (I) holds; then there exists a number $\alpha > 1$ such that

$$c_i > \frac{\alpha - 1}{\alpha} \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}| |\sigma_j|) + \frac{1}{\alpha d_i} \sum_{j=1}^{n} (|a_{ji}| d_j \mu_i^{\alpha} + |b_{ji}| |\sigma_i| d_j \mu_i^{\alpha}), \quad i=1,2,\ldots,n;$$

which is the special case of hypothesis (H3) as $\alpha_{ij} = \beta_{ij} = 0$ $(i=1,2,\ldots,m)$, $\alpha_{m+1,j} = \beta_{m+1,j} = 1$, $j=1,2,\ldots,n$.

Assume condition (II) holds; then there exists a number $\alpha > 1$ such that

$$c_i > \frac{\alpha - 1}{\alpha} \sum_{j=1}^{n} (|a_{ij}| \mu_j^{\alpha/(\alpha-1)} + |b_{ij}| |\sigma_j| \mu_j^{\alpha/(\alpha-1)}) + \frac{1}{\alpha d_i} \sum_{j=1}^{n} (|a_{ji}| d_j + |b_{ji}| |\sigma_i| d_j), \quad i=1,2,\ldots,n;$$

which is the special case of hypothesis (H3) as $m=1$, $\alpha_{1j} = \beta_{1j} = 1$, $\alpha_{2j} = \beta_{2j} = 0$, $j=1,2,\ldots,n$.

Condition (III) is the special case of hypothesis (H3) as $m=1$, $\alpha=2$, $d_i=1$ $(i=1,2,\ldots,n)$.

Condition (IV) is the special case of hypothesis (H3) as $m=2$, $\alpha=3$, $p_1=p_2=1$, $d_i=1$ $(i=1,2,\ldots,n)$.

Remark 2.

(1) Corollary (I) extends and improves Proposition 1 in Ref. [1], Conclusion (i) of Theorem in Ref. [2], Conclusion (I) of Theorem 1 in Ref. [3]; and extends Theorem 1 in Ref. [8].

(2) Corollary (II) extends and improves Theorem 1 in Ref. [5].

(3) Corollary (II) extends Theorem 1 in Ref. [4], Theorem 1 in Ref. [6].

(4) Corollary (III) extends and improves Conclusions (ii)–(v) of Theorem in Ref. [2], Theorem 2 in Ref. [6], Conclusions (II)–(IV) of Theorem 1 in Ref. [3], Theorem 1 in Ref. [7].

(5) Corollary (IV) extends and improves Theorem 2 in Ref. [7].

III. EXISTENCE OF PERIODIC SOLUTIONS

In this section, we study the periodic solutions of the DCNN of the type

$$x_i'(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(\sigma_j x_j(t - \tau_{ij})) + I_i(t), \quad c_i > 0, \quad i=1,2,\ldots,n; \tag{7}$$

in which $I_i: R^+ \to R$, $i=1,2,\ldots,n$ are continuously periodic functions with period ω, i.e., $I_i(t+\omega) = I_i(t)$. Other symbols possess the same meaning as that Eq. (1).

Theorem 2. Suppose that the output of the cell function f_i $(i=1,2,\ldots,n)$ satisfies the hypotheses (H1) and (H2) above. Assume, furthermore, that the system parameters satisfy the condition (H3), as given earlier. Then there exists exactly one $\omega-$ periodic solution of Eq. (7) and all other solutions of Eq. (7) converge exponentially to it as $t \to +\infty$.

Proof. Let $C = C([-\tau, 0], R^n)$ are the Banach space of continuous functions which map $[-\tau, 0]$ into R^n with the topology of uniform convergence. For any $\phi \in C$, we define $\|\phi\| = \sup_{-\tau \leqslant \theta \leqslant 0} |\phi(\theta)|$, in which $|\phi(\theta)| = \max_{1 \leqslant i \leqslant n} |\phi_i(\theta)|$.

For $\forall$ $\phi, \psi \in C$, we denote the solutions of Eq. (7) through $(0, \phi)$ and $(0, \psi)$ as

$$x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \ldots, x_n(t, \phi))^T,$$

$$x(t, \psi) = (x_1(t, \psi), x_2(t, \psi), \ldots, x_n(t, \psi))^T,$$

respectively.

Define

$$x_t(\phi) = x(t + \theta, \phi), \quad \theta \in [-\tau, 0], \quad t \geqslant 0,$$

Then $x_t(\phi) \in C$ for $\forall \; t \geqslant 0$.

Thus, we follow from system (7) that

$$[x_i(t,\phi)-x_i(t,\psi)]' = -c_i[x_i(t,\phi)-x_i(t,\psi)] + \sum_{j=1}^{n} a_{ij}[f_j(x_j(t,\phi))-f_j(x_j(t,\psi))] + \sum_{j=1}^{n} b_{ij}[f_j(\sigma_j x_j(t-\tau_{ij},\phi)) - f_j(\sigma_j x_j(t-\tau_{ij},\psi))] \quad (8)$$

for $t \geqslant 0, \quad i = 1,2,\ldots,n$.

Let

$$z_i(t) = [x_i(t,\phi)-x_i(t,\psi)]e^{\lambda t} \quad (i=1,2,\ldots,n),$$

where $\lambda < C_i$, as given earlier.

Substituting them into Eq. (8) and simplifying, we get

$$z_i'(t) = (\lambda - c_i)z_i(t) + e^{\lambda t}\left\{ \sum_{j=1}^{n} a_{ij}[f_j(x_j(t,\psi)+e^{-\lambda t}z_j(t)) - f_j(x_j(t,\psi))] + \sum_{j=1}^{n} b_{ij}\{f_j(\sigma_j[x_j(t-\tau_{ij},\psi) + e^{-\lambda(t-\tau_{ij})}z_j(t-\tau_{ij})]) - f_j(\sigma_j x_j(t-\tau_{ij},\psi))\}\right\}.$$

Consider the Lyapunov functional $V(t)$, as given earlier.

By a minor modification of the proof of Theorem 1, one can easily get

$$|x_i(t,\phi)-x_i(t,\psi)| \leqslant K e^{-\lambda t}\|\phi - \psi\|,$$

where $K > 1$ is a constant.

The rest of the proof is similar to that of Theorem 4 in [8]. We omit it here to avoid repetition.

Remark 3. Theorem 4 in Ref. [8] is an immediate consequence of Theorem 2 above.

IV. EXAMPLES

Example 1. Consider the following neural networks with delays

$$\begin{aligned} x_1'(t) &= -c_1x_1(t) + a_{11}f(x_1(t)) + a_{12}f(x_2(t)) \\ &\quad + b_{11}f(\sigma_1 x_1(t-\tau_1)) + b_{12}f(\sigma_2 x_2(t-\tau_2)) + I_1, \\ x_2'(t) &= -c_2x_2(t) + a_{21}f(x_1(t)) + a_{22}f(x_2(t)) \\ &\quad + b_{21}f(\sigma_1 x_1(t-\tau_1)) + b_{22}f(\sigma_2 x_2(t-\tau_2)) + I_2, \end{aligned} \quad (9)$$

where the relation between the output of the cell and the state of the cell is described by a piecewise-linear function $f_i(x) \equiv f(x) = \frac{1}{2}(|x+1|-|x-1|)$, $\tau_i \geqslant 0$, $i=1,2$. It is easy to see that the function f_i clearly satisfies the hypotheses (H1) and (H2) above, and $\mu_1 = \mu_2 = 1$. Hence, the condition (H3) in Theorem 1 can be reduced to

$$\begin{aligned} c_1 &> \frac{\alpha-1}{\alpha}(|a_{11}|+|a_{12}|+|b_{11}||\sigma_1|+|b_{12}||\sigma_2|) \\ &\quad + \frac{1}{\alpha d_1}(|a_{11}|d_1+|a_{21}|d_2+|b_{11}||\sigma_1|d_1+|b_{21}||\sigma_1|d_2), \\ c_2 &> \frac{\alpha-1}{\alpha}(|a_{21}|+|a_{22}|+|b_{21}||\sigma_1|+|b_{22}||\sigma_2|) \\ &\quad + \frac{1}{\alpha d_2}(|a_{12}|d_1+|a_{22}|d_2+|b_{12}||\sigma_2|d_1+|b_{22}||\sigma_2|d_2). \end{aligned} \quad (10)$$

Consider the special case of the model (9) as

$$c_1 = 0.6, \quad c_2 = 0.7, \quad a_{11} = 0.1, \quad a_{12} = 0.1,$$

$$a_{21} = 0.2, \quad a_{22} = 0.1,$$

$$b_{11} = 0.2, \quad b_{12} = 0.5, \quad b_{21} = 0.4, \quad b_{22} = 0.5,$$

$$\sigma_1 = 0.5, \quad \sigma_2 = 0.4,$$

$$I_1 = 0.1, \quad I_2 = 0, \quad \tau_1 = 0.2, \quad \tau_2 = 0.3.$$

In this case, Eq. (10) can be reduced to

$$(0.1\alpha + 0.3)d_1 > 0.4d_2, \quad 0.4d_2 > 0.3d_1.$$

Therefore, if we take $\alpha = 2$, $d_1 = d_2 = 1$, then Eq. (10) holds. Thus by Theorem 1, the equilibrium of the model (9) is globally exponentially stable. It is easy to verify that (1,1) is an equilibrium of the model (9).

Example 2. Consider the following neural networks with delays

$$\begin{aligned} x_1'(t) &= -c_1x_1(t) + a_{11}f(x_1(t)) + a_{12}f(x_2(t)) \\ &\quad + b_{11}f(\sigma_1 x_1(t-\tau_1)) + b_{12}f(\sigma_2 x_2(t-\tau_2)) + I_1(t), \\ x_2'(t) &= -c_2x_2(t) + a_{21}f(x_1(t)) + a_{22}f(x_2(t)) \\ &\quad + b_{21}f(\sigma_1 x_1(t-\tau_1)) + b_{22}f(\sigma_2 x_2(t-\tau_2)) + I_2(t), \end{aligned} \quad (11)$$

where the relation between the output of the cell and the state of the cell is described by $f_i(x) \equiv f(x) = \tanh(x)$, $\tau_i \geqslant 0$, $i = 1,2$. It is easy to see that the function f_i clearly satisfies the hypotheses (H1) and (H2) above, and $\mu_1 = \mu_2 = 1$. Hence, the condition (H3) in Theorem 2 can be reduced to Eq. (10).

Consider the special case of the model (11) as

$$c_1 = 1, \quad c_2 = 0.9, \quad a_{11} = 0.1, \quad a_{12} = 0.2,$$

$$a_{21} = 0.1, \quad a_{22} = 0.1,$$

$$b_{11} = 0.4, \quad b_{12} = 0.5, \quad b_{21} = 0.6, \quad b_{22} = 0.5,$$

$$\sigma_1 = 0.5, \quad \sigma_2 = 0.8, \quad \tau_1 = 0.3, \quad \tau_2 = 0.4,$$

$$I_1 = \cos t + \sin t - 0.1\tanh(\sin t) - 0.2\tanh(\cos t) - 0.4\tanh[0.5\sin(t-\tau_1)] - 0.5\tanh[0.8\cos(t-\tau_2)],$$

$$I_2 = 0.9\cos t - \sin t - 0.1\tanh(\sin t) - 0.1\tanh(\cos t) - 0.6\tanh[0.5\sin(t-\tau_1)] - 0.5\tanh[0.8\cos(t-\tau_2)].$$

In this case, Eq. (10) can be reduced to

$$(0.1\alpha + 0.6)d_1 > 0.4d_2, \quad 0.4d_2 > 0.6d_1.$$

Therefore, if we take $\alpha = 3$, $d_1 = 1$, $d_2 = 2$, then Eq. (10) holds. Thus by Theorem 2, the model (11) has a unique 2π-periodic solution, and all other solutions of the model (11) converge exponentially to it as $t \to +\infty$. It is easy to verify that $(\sin t, \cos t)$ is the 2π-periodic solution of the model (11).

V. CONCLUSION

All of the obtained criteria are independent of delays. Since the conditions of Theorems 1 and 2 include some adjustable parameters, the results have a wider adaptive range. Specially, the conditions of Corollary 1 are easily verified. These factors play an important role in the design of unconditioned globally exponentially stable DCNN's and periodic oscillatory DCNN's.

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