Velocity-fluctuation-induced anomalous kinetics of the $A+A\rightarrow\varnothing$ reaction

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The effect of a random velocity field on the kinetics of the single-species annihilation reaction $A+A\rightarrow\varnothing$ is analyzed near two dimensions with the aid of the perturbative renormalization group. The previously found asymptotic behavior induced by density fluctuations only in the diffusion-limited reaction is shown to be unstable to any velocity fluctuations (including thermal fluctuations near equilibrium) in spatial dimensions $d\leqslant d_c=2$. Four different stable long-time asymptotic regimes induced by the combined effect of velocity and density fluctuations are identified and the corresponding decay rates calculated in the leading order.

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I. INTRODUCTION

Investigation of the effect of density fluctuations on reaction kinetics has gained considerable attention recently [1–3]. This analysis has shown that in low space dimensions the usual rate equation is not sufficient for a description of the reaction kinetics, and the fluctuations of the density of the reactants must be taken into account. Corresponding asymptotic expressions for densities have been calculated for processes like $A+A\rightarrow\varnothing$ [1,4] and $A+B\rightarrow\varnothing$ [5]. The behavior of the density and effective reaction rate may also be affected by fluctuations of an advective velocity field. So far the analysis of this effect has been concentrated on the case of time-independent random drift [6–10]. However, for a more realistic analysis of the effects of the velocity fluctuations on the reaction kinetics a time-dependent random drift field would be preferable.

In this paper we show that the previously found anomalous behavior at and below the critical dimension $d_c=2$ in the reaction $A+A\rightarrow\varnothing$ [1,4] is unstable to any drift field fluctuations, including those generated by the ubiquitous thermal noise. It is shown that anomalous behavior with faster decay rate of the reactant density is induced at $d\leqslant 2$ by the combined effect of the density and drift fluctuations.

We study the problem of advection of a reactive scalar using a random velocity field generated by the stochastically forced Navier-Stokes equation, which has been widely used to produce a stochastic velocity field corresponding to both thermal fluctuations near equilibrium [11–13] and a turbulent velocity field with the Kolmogorov scaling behavior [14,15]. Attempts have been made to investigate the problem of a reactive scalar advected by a turbulent velocity field [16,17], in which a Langevin-equation approach was used to generate the concentration fluctuations. The Langevin equation works well in linear problems, but it may be problematic in nonlinear ones (see, e.g., Ref. [18]). Therefore, we use the master-equation approach [18] to take into account the reactant density fluctuations.

We use the second-quantization formalism of Doi [19] to cast the stochastic problem into a field-theoretic form. The critical dimension of the pure annihilation problem $A+A\rightarrow\varnothing$ is 2 [4], therefore we have used the recently proposed two-parameter expansion [20,21] for the calculation of the random velocity field generated by the stochastic Navier-Stokes equation. This gives rise to a two-parameter expansion of all physical quantities in the problem of advection of a reactive scalar.

This paper is organized as follows. In Sec. II a field-theoretic model for the general reaction $sA \underset{K_{-0}}{\overset{K_{+0}}{\rightleftharpoons}} rA$ is constructed on the basis of the second-quantization approach to the reaction, whereas the stochastic Navier-Stokes equation is used to generate the random velocity field. Renormalization of the field-theoretic model corresponding to the annihilation reaction $A+A\rightarrow\varnothing$ is carried out in Sec. III, where also the fixed points of the renormalization group (RG) differential equations are classified, and it is shown that in and below two dimensions the velocity fluctuations always drive the system in an asymptotic regime different from that induced by density fluctuations only [1,4]. Section IV is devoted to the explicit calculation of the average number density at the tree level. Anomalous decay rates corresponding to the four different fixed points of the RG are calculated at the leading approximation, as well as the logarithmic corrections on the borderlines between the basins of attraction of the fixed points. Conclusions are presented in Sec. V.

II. FIELD-THEORETIC MODEL FOR THE NONLINEAR ADVECTION-DIFFUSION PROBLEM

The second-quantized formulation of classical reaction processes is based on the use of the creation and annihilation operators $\psi^\dagger$ and ψ and the vacuum state $|0\rangle$ [19] (see also [3,22]),

$$[\psi(\mathbf{x}),\psi^\dagger(\mathbf{x}')]=\delta(\mathbf{x}-\mathbf{x}'),$$

$$[\psi(\mathbf{x}),\psi(\mathbf{x}')]=[\psi^\dagger(\mathbf{x}),\psi^\dagger(\mathbf{x}')]=0,$$

$$\psi(\mathbf{x})|0\rangle=0,\ \langle 0|\psi^\dagger(\mathbf{x})=0,\ \langle 0|0\rangle=1. \qquad (2.1)$$

Let $P(\{n_i\},t)$ be the joint probability distribution function (PDF) to observe n_i particles at positions $\mathbf{x}_i$. The state vector of the classical many-particle system is defined as the sum over all occupation numbers,

$$|\Phi(t)\rangle=\sum_{\{n_i\}} P(\{n_i\},t)|\{n_i\}\rangle,$$

where the basis vectors are defined as

$$|\{n_i\}\rangle=\prod_i [\psi^\dagger(\mathbf{x}_i)]^{n_i}|0\rangle.$$

Averages of observables may be calculated with the use of a coherent-state vector,

$$\langle A(t)\rangle=\sum_{\{n_i\}} A[\{n_i\}]P(\{n_i\},t)=\langle 0|e^{\int d\mathbf{x}\,\psi}A(\psi^\dagger\psi)|\Phi(t)\rangle, \tag{2.2}$$

where $\hat{n}(\mathbf{x})=\psi^\dagger(\mathbf{x})\psi(\mathbf{x})$ is the number-density operator [19].

The set of coupled master equations for the PDFs may be cast in the form of a ''master equation'' for the state vector [1,19,22]:

$$\frac{\partial}{\partial t}|\Phi(t)\rangle=-\hat{H}|\Phi(t)\rangle, \tag{2.3}$$

with the initial condition $|\Phi(0)\rangle=\Sigma_{\{n_i\}}P(\{n_i\},0)|\{n_i\}\rangle$. In Eq. (2.3) the kinetic operator $\hat{H}$ is the sum of terms

$$\hat{H}=\hat{H}_A+\hat{H}_D+\hat{H}_R \tag{2.4}$$

corresponding to advection, diffusion, and chemical reaction, respectively. For the generic reaction involving only one type of particle,

$$sA \underset{K_{-0}}{\overset{K_{+0}}{\rightleftharpoons}} rA, \tag{2.5}$$

with the unrenormalized (mean field) rate constants K_{+0} and K_{-0}, the operators in Eq. (2.4) in terms of the field operators are [19,22]

$$\hat{H}_A=\int d\mathbf{x}\,\psi^\dagger(\mathbf{x})\nabla[\mathbf{v}(\mathbf{x},t)\psi(\mathbf{x})],$$

$$\hat{H}_D=-D_0\int d\mathbf{x}\,\psi^\dagger(\mathbf{x})\nabla^2\psi(\mathbf{x}),$$

$$\hat{H}_R=K_{+0}\int d\mathbf{x}[(\psi^\dagger)^s-(\psi^\dagger)^r]\psi^s+K_{-0}\times\int d\mathbf{x}[(\psi^\dagger)^r-(\psi^\dagger)^s]\psi^r. \tag{2.6}$$

In Eq. (2.6) the subscript 0 refers to unrenormalized parameters of the model, which are the physical ones in the statistical applications of the renormalization group. In the following, we denote the corresponding renormalized parameters by the same symbols without the subscript.

With the use of the formal solution of the master equation (2.3) the average (2.2) may be written as

$$\langle A(t)\rangle=\langle 0|e^{\int d\mathbf{x}\,\psi}A(\psi^\dagger\psi)e^{-\hat{H}t}|\Phi(0)\rangle. \tag{2.7}$$

However, we find it convenient to use the form in which the coherent-state exponential is commuted to the right, which leads to the expression

$$\langle A(t)\rangle=\langle 0|A[(\psi^\dagger+1)\psi]e^{-\hat{H}'t}e^{\int d\mathbf{x},\psi}|\Phi(0)\rangle, \tag{2.8}$$

corresponding to the use of the ''inclusive'' scalar product of Ref. [22]. The connection between the kinetic operators in Eqs. (2.7) and (2.8) is [3,22]

$$\hat{H}'(\psi^\dagger,\psi)=\hat{H}(\psi^\dagger+1,\psi),$$

i.e., the field argument $\psi^\dagger$ is replaced by $\psi^\dagger+1$ due to the transfer of the coherent-state exponential from the left to the right. The average number density, for instance, in the form (2.8) is

$$n(\mathbf{x},t)=\langle 0|e^{\int d\mathbf{x}\,\psi}\psi^\dagger(\mathbf{x})\psi(\mathbf{x})|\Phi(t)\rangle=\langle 0|\psi(\mathbf{x})e^{-\hat{H}'t}e^{\int d\mathbf{x}\,\psi}|\Phi(0)\rangle. \tag{2.9}$$

It is convenient to use the Poisson distribution as the initial condition [3,19,22], which in terms of the field operators and vacuum state yields for the initial state the following expression:

$$|\Phi(0)\rangle=e^{-n_0V+n_0\int d\mathbf{x}\,\psi^\dagger}|0\rangle,$$

where n_0 is the initial number density and V the volume of the system. With this choice of the initial state it is readily seen that

$$e^{\int d\mathbf{x}\,\psi}|\Phi(0)\rangle=e^{n_0\int d\mathbf{x}\,\psi^\dagger}|0\rangle.$$

To construct a perturbative expansion of the expectation value (2.8) we introduce the time-ordered exponential form of the evolution operator,

$$U(t,t_0)=e^{\hat{H}_0't}e^{-\hat{H}'(t-t_0)}e^{-\hat{H}_0't_0}=Te^{-\int_{t_0}^t\hat{H}_I'dt},$$

where the kinetic operator has been decomposed to a free-field operator $\hat{H}_0'$ and an interaction operator $\hat{H}_I'=\hat{H}'-\hat{H}_0'$. In the interaction operator the field operators (2.1) are replaced by the time-dependent operators

$$\psi^\dagger(\mathbf{x},t)=e^{\hat{H}_0't}\psi^\dagger(\mathbf{x})e^{-\hat{H}_0't},\ \psi(\mathbf{x},t)=e^{\hat{H}_0't}\psi(\mathbf{x})e^{-\hat{H}_0't},$$

in which we have chosen $t_0=0$. With the same replacement in the operator $A[(\psi^\dagger+1)\psi]$ we cast the expectation value (2.8) in the form

$$\langle A(t)\rangle=\langle 0|T(A\{[(\psi^\dagger(t)+1]\psi(t)\}\times e^{-\int_0^t\hat{H}_I'dt+n_0\int d\mathbf{x}\,\psi^\dagger(\mathbf{x},0)})|0\rangle, \tag{2.10}$$

where the correct ordering of time arguments in the operators $A\{[\psi^\dagger(t)+1]\psi(t)\}$ and $e^{n_0\int d\mathbf{x}\,\psi^\dagger}=e^{n_0\int d\mathbf{x}\,\psi^\dagger(\mathbf{x},0)}$ allows us to introduce them into the time-ordered product together with the evolution operator.

In order to use the standard Laplace (or Fourier, with the choice $t_0\to-\infty$) transform with respect to time we use the following trick to send the upper limit of the time integral in the interaction operator to infinity. Using the identities

$$\langle 0|\hat{H}_0'=\langle 0|\hat{H}_I'=0$$

valid for the operators $\hat{H}_0'$ and $\hat{H}_I'$ generated from the operators $\hat{H}_A$, $\hat{H}_D$, and $\hat{H}_R$ [Eq. (2.6)] by the shift $\psi^\dagger\to\psi^\dagger+1$ (note that, in general, $\langle 0|\hat{H}_R\neq 0$!; this is one reason why we prefer to commute the coherent-state exponential to the right), we obtain the identity

$$\langle 0|=\langle 0|e^{\hat{H}_0't}e^{-\hat{H}'(t-\tau)}e^{-\hat{H}_0'\tau}Te^{-\int_t^\tau\hat{H}_I'dt}=\langle 0|Te^{-\int_t^\tau\hat{H}_I'dt}. \tag{2.11}$$

Choosing $\tau>t$ in the identity (2.11), introducing it into the expectation value (2.10) and taking the limit $\tau\to\infty$, we arrive at the expression

$$\langle A(t)\rangle=\langle 0|T(A\{[\psi^\dagger(t)+1]\psi(t)\}\times e^{-\int_0^\infty\hat{H}_I'dt+n_0\int d\mathbf{x}\,\psi^\dagger(\mathbf{x},0)})|0\rangle, \tag{2.12}$$

where the integration limits in the interaction operator are now independent of the current time argument t. With the choice of the lower limit in Eq. (2.12), the use of the Laplace transform with respect to time is implicitly assumed.

Using standard procedures (see, e.g., Ref. [23]) the expectation value (2.12) may be cast into the form of a functional integral over complex-conjugate scalar fields $\psi^\dagger(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$:

$$\langle A(t)\rangle=\int\mathcal{D}\psi^\dagger\mathcal{D}\psi\,A\{[\psi^\dagger(t)+1]\psi(t)\}e^{S_1}, \tag{2.13}$$

where the unrenormalized dynamic action S_1 for the generic single-species reaction (2.5) with an initial Poisson distribution is

$$\begin{aligned}S_1(\psi^\dagger,\psi,\mathbf{v})=-\int d\mathbf{x}\int_0^\infty dt\Big\{&\psi^\dagger(\mathbf{x},t)\partial_t\psi(\mathbf{x},t)\\&+\psi^\dagger(\mathbf{x},t)\boldsymbol{\nabla}[\mathbf{v}\psi(\mathbf{x},t)]-D_0\psi^\dagger(\mathbf{x},t)\nabla^2\psi(\mathbf{x},t)\\&+K_{+0}\Bigg[\sum_{j=1}^{s}\binom{s}{j}[\psi^\dagger(\mathbf{x},t)]^j-\sum_{j=1}^{r}\binom{r}{j}\\&\times[\psi^\dagger(\mathbf{x},t)]^j\Bigg]\psi(\mathbf{x},t)^s\\&+K_{-0}\Bigg[\sum_{j=1}^{r}\binom{r}{j}[\psi^\dagger(\mathbf{x},t)]^j-\sum_{j=1}^{s}\binom{s}{j}\\&\times[\psi^\dagger(\mathbf{x},t)]^j\Bigg]\psi(\mathbf{x},t)^r\Big\}+n_0\int d\mathbf{x}\,\psi^\dagger(\mathbf{x},0).\end{aligned} \tag{2.14}$$

This form of the action implies that the time-ordered product at coinciding time arguments is defined as the normal-ordered product. Therefore, in the perturbative expansion of the expectation value (2.13) no contractions between field arguments at the same time instant occur in the interaction vertices.

The stationarity conditions for the action (2.14) with respect to variations of the fields $\psi^\dagger$ and ψ are

$$\begin{aligned}0=\frac{\delta S_1}{\delta\psi^\dagger}=&-\frac{\partial\psi}{\partial t}-\boldsymbol{\nabla}(\mathbf{v}\psi)+D_0\nabla^2\psi\\&-K_{+0}\Bigg[\sum_{j=1}^{s}\binom{s}{j}j(\psi^\dagger)^{j-1}-\sum_{j=1}^{r}\binom{r}{j}j(\psi^\dagger)^{j-1}\Bigg]\psi^s\\&-K_{-0}\Bigg[\sum_{j=1}^{r}\binom{r}{j}j(\psi^\dagger)^{j-1}\\&-\sum_{j=1}^{s}\binom{s}{j}j(\psi^\dagger)^{j-1}\Bigg]\psi^r+n_0\delta(t),\end{aligned} \tag{2.15}$$

$$\begin{aligned}0=\frac{\delta S_1}{\delta\psi}=&\frac{\partial\psi^\dagger}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}\psi^\dagger+D_0\nabla^2\psi^\dagger-sK_{+0}\Bigg[\sum_{j=1}^{s}\binom{s}{j}(\psi^\dagger)^j\\&-\sum_{j=1}^{r}\binom{r}{j}(\psi^\dagger)^j\Bigg]\psi^{s-1}-rK_{-0}\Bigg[\sum_{j=1}^{r}\binom{r}{j}(\psi^\dagger)^j\\&-\sum_{j=1}^{s}\binom{s}{j}(\psi^\dagger)^j\Bigg]\psi^{r-1}.\end{aligned} \tag{2.16}$$

Substituting the homogeneous solution $\psi^\dagger=0$ of the latter equation (2.16) in the former equation (2.15) we arrive at the usual mean-field rate equation [18],

$$\frac{\partial\psi}{\partial t}+\boldsymbol{\nabla}(\mathbf{v}\psi)=D_0\nabla^2\psi+K_{+0}(r-s)\psi^s+K_{-0}(s-r)\psi^r+n_0\delta(t)$$

for ψ, which, according to Eq. (2.9), at this level may be identified with the average number density $n(\mathbf{x},t)$. Further, we will study the annihilation process $A+A\to\varnothing$, and therefore put $r=0$, $s=2$, and $K_{-0}=0$ in Eq. (2.14).

In order to analyze the effect of velocity fluctuations on the reaction we average the expectation value (2.13) over the random velocity field $\mathbf{v}$. Due to the incompressibility conditions $\boldsymbol{\nabla}\cdot\mathbf{v}=0$, $\boldsymbol{\nabla}\cdot\mathbf{f}^v=0$ imposed on the velocity field $\mathbf{v}$ and the random-force field $\mathbf{f}^v$ it is sufficient to consider the transverse components

$$\partial_t\mathbf{v}+P(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}-\nu_0\nabla^2\mathbf{v}=\mathbf{f}^v \tag{2.17}$$

of the stochastic Navier-Stokes equation. In Eq. (2.17) ν_0 is the unrenormalized kinematic viscosity and P is the transverse projection operator.

In the approach based on the assumption of maximal randomness [24] the random force is assumed to have Gaussian distribution with zero mean and the correlation function

$$\langle f_m^v(\mathbf{x}_1,t_1) f_n^v(\mathbf{x}_2,t_2)\rangle = \delta(t_1-t_2) \times \int \frac{d\mathbf{k}}{(2\pi)^d} P_{mn}(\mathbf{k}) d_f(k) e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}, \tag{2.18}$$

where $P_{mn}(\mathbf{k})=\delta_{mn}-k_m k_n/k^2$ is the transverse projection operator in the wave-vector space, and $d_f(k)$ is a function of the wave number k and the parameters of energy pumping.

The critical dimension of the pure annihilation problem $A+A\to\varnothing$ is 2 [4]; therefore for the calculation of the random velocity field we have used the recently proposed two-parameter expansion, in which the kernel function is [20]

$$d_f(k)=g_{10}\nu_0^3 k^{4-d-2\epsilon}+g_{20}\nu_0^3 k^2. \tag{2.19}$$

The nonlocal term is often used to generate the turbulent velocity field with Kolmogorov's scaling [14,15] (which is achieved by choosing $\epsilon=2$). The local term is produced in the course of renormalization near two dimensions [20], but it also has an important physical meaning: such a term in the force correlation function, with a suitable choice of the parameter g_{20}, is believed to generate thermal fluctuations of the velocity field near equilibrium [11–13]. The choice (2.19) of the correlation kernel fixes the critical dimension of the model at the value $d_c=2$, which is also the critical dimension of the reaction $A+A\to\varnothing$ in the master-equation approach. This is drastically different from the Langevin-equation approach, in which the critical dimension for this reaction $d_c=6$ [16,17].

Expectation values of functionals of the velocity field generated by the stochastic differential equations (2.17) and (2.18) may be calculated with the use of the ''weight'' functional [14,15] $\mathcal{W}_2=e^{S_2}$, where the action S_2 is a functional of divergenceless vector fields $\mathbf{v}(\mathbf{x},t)$ and $\tilde{\mathbf{v}}(\mathbf{x},t)$ of the form

$$S_2=\frac{1}{2}\int dt\, d\mathbf{x}\, d\mathbf{x}'\, \tilde{\mathbf{v}}(\mathbf{x},t)\cdot\tilde{\mathbf{v}}(\mathbf{x}',t) d_f(|\mathbf{x}-\mathbf{x}'|) + \int dt\, d\mathbf{x}\, \tilde{\mathbf{v}}\cdot[-\partial_t\mathbf{v}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\nu_0\nabla^2\mathbf{v}]. \tag{2.20}$$

Here, $d_f(|\mathbf{x}-\mathbf{x}'|)$ is the coordinate-space representation of the correlation kernel (2.19).

Thus we see that the expectation value of any observable with respect to the concentration and velocity fluctuations may be calculated using the weight functional $\mathcal{W}=e^{S_1+S_2}$, where the unrenormalized dynamic action S_1 is

$$S_1=-\int_0^\infty dt\int d\mathbf{x}\{\psi^\dagger\partial_t\psi+\psi^\dagger\boldsymbol{\nabla}(\mathbf{v}\psi)-D_0\psi^\dagger\nabla^2\psi + \lambda_0 D_0[2\psi^\dagger+(\psi^\dagger)^2]\psi^2\}+n_0\int d\mathbf{x}\,\psi^\dagger(\mathbf{x},0). \tag{2.21}$$

This action corresponds to the reaction $A+A\to\varnothing$, and we have extracted the diffusion coefficient D_0 in the rate constant $K_{+0}=\lambda_0 D_0$ for dimensional reasons.

III. FIXED POINTS OF THE RENORMALIZATION GROUP

For power counting in the action (2.20) and (2.21) we use the scheme [15], in which to any quantity Q separate canonical dimensions with respect to the wave number d_Q^k and frequency d_Q^ω are assigned with the normalization $d_\omega^\omega=-d_t^\omega=1$, $d_k^k=-d_x^k=1$, and $d_k^\omega=d_\omega^k=0$. These canonical dimensions for fields and parameters are determined from the requirement that the action (2.20) and (2.21) is dimensionless with respect to temporal and spatial variables separately. The total canonical dimension $d_Q=2d_Q^\omega+d_Q^k$ is determined from the condition that the parabolic differential operators of the diffusion and Navier-Stokes equations scale uniformly under the transformation $k\to\mu k$, $\omega\to\mu^2\omega$.

The quadratic part of the action (2.21) determines only the canonical dimensions of the product of fields $\psi^\dagger\psi$. However, in order to keep both terms in the nonlinear part $\lambda_0 D_0\int dt\, d\mathbf{x}[2\psi^\dagger+(\psi^\dagger)^2]\psi^2$ of the action, the field $\psi^\dagger$ must be dimensionless. If the field $\psi^\dagger$ has positive canonical dimension, then the quartic term should be discarded as irrelevant by power counting. The action with the cubic term only, however, does not generate any loop integrals corresponding to density fluctuations and thus is uninteresting for the analysis of fluctuation effects.

Using this choice we arrive at the following canonical dimensions for fields and parameters in d-dimensional space:

$$d_\psi^k=d,\quad d_\psi^\omega=0,\quad d_\psi=d;$$

$$d_{\psi^\dagger}^k=0,\quad d_{\psi^\dagger}^\omega=0,\quad d_{\psi^\dagger}=0;$$

$$d_v^k=-1,\quad d_v^\omega=1,\quad d_v=1;$$

$$d_{\tilde{v}}^k=d+1,\quad d_{\tilde{v}}^\omega=-1,\quad d_{\tilde{v}}=d-1;$$

$$d_{\nu_0}^k=-2,\quad d_{\nu_0}^\omega=1,\quad d_{\nu_0}=0;$$

$$d_{D_0}^k=-2,\quad d_{D_0}^\omega=1,\quad d_{D_0}=0;$$

$$d_{\lambda_0}^k=2-d,\quad d_{\lambda_0}^\omega=0,\quad d_{\lambda_0}=2-d;$$

$$d_{g_{01}}^k=2\epsilon,\quad d_{g_{01}}^\omega=0,\quad d_{g_{01}}=2\epsilon;$$

$$d_{g_{02}}^k=2-d,\quad d_{g_{02}}^\omega=0,\quad d_{g_{02}}=2-d.$$

For one-particle irreducible (1PI) graphs with n_v, $n_{\tilde{v}}$, n_ψ, and $n_{\psi^\dagger}$ external arguments of the fields v, $\tilde{v}$, ψ, and $\psi^\dagger$, respectively, in the logarithmic theory (i.e., when the canonical dimensions of the coupling constants vanish: $d_{g_{01}}=d_{g_{02}}=d_{\lambda_0}=0$) the superficial degree of divergence is $\delta=4-n_v-n_{\tilde{v}}-2n_\psi$. Due to the vanishing of the canonical dimension of the field $\psi^\dagger$, it might seem that there is an infinite number of superficially divergent graphs with an arbitrary number of external $\psi^\dagger$ legs. However, the number of $\psi^\dagger$ fields at the

interaction vertices is less than or equal to the number of ψ fields, and only lines connecting a $\psi^\dagger$ argument with a ψ argument are present in the graphs. By inspection of graphs it is readily seen that because of this $n_{\psi^\dagger} \leqslant n_\psi$ and thus no proliferation of superficially divergent graphs occurs [1].

Contrary to the pure reaction case [1,4], the 1PI Green function $\Gamma_{\psi^\dagger\psi}$ is renormalized here, which gives rise to the renormalization of the diffusion coefficient D. Because of the Galilei invariance of the action the superficially divergent 1PI Green functions $\Gamma_{\psi^\dagger\psi v}$ and $\Gamma_{vv\tilde{v}}$ are actually convergent [12,14,15]. Graphs corresponding to $\Gamma_{\psi^\dagger\psi\psi}$ and $\Gamma_{\psi^\dagger\psi^\dagger\psi\psi}$ differ only by one external vertex and thus give rise to equal renormalization of the rate constant $\lambda_0 D_0$, preserving the symmetry of coupling constants in the reactive part of the action.

Thus, the renormalized action for the $A+A\to\varnothing$ reaction in a turbulent velocity field is

$$\begin{aligned} S = -\int d\mathbf{x}\, dt \Big\{ & \psi^\dagger \partial_t \psi + \psi^\dagger \boldsymbol{\nabla}(\mathbf{v}\psi) - u\nu Z_2 \psi^\dagger \nabla^2 \psi \\ & + \lambda u \nu \mu^{-2\delta} Z_4 [2\psi^\dagger + (\psi^\dagger)^2]\psi^2 \\ & - \frac{1}{2}\tilde{\mathbf{v}}[g_1 \nu^3 \mu^{2\epsilon}(-\nabla^2)^{1-\delta-\epsilon} - g_2 \nu^3 \mu^{-2\delta} Z_3 \nabla^2]\tilde{\mathbf{v}} \\ & + \tilde{\mathbf{v}}\cdot[\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu Z_1 \nabla^2 \mathbf{v}] \Big\} + n_0 \int d\mathbf{x}\, \psi^\dagger(\mathbf{x},0), \end{aligned} \tag{3.1}$$

where we have introduced the inverse Prandtl number $u = D/\nu$ and the deviation of the space dimension from 2, $2\delta = d-2$, as dimensionless parameters. The parameter μ sets the wave-number scale of the renormalized model.

As in the case of the diffusion-limited reaction [1,4], it is readily seen that essentially the same renormalized action describes also the coagulation reaction $A+A\to A$. Indeed, starting from the corresponding dynamic action we arrive at the renormalized action (3.1) with the only difference that in the coefficient of the term $\psi^\dagger\psi^2$ the number 2 is replaced by unity. This changes the scaling functions slightly (see Ref. [25] for details) but not the scaling dimensions, and thus both reactions are in the same universality class.

We use a combination of dimensional and analytic regularization with the minimal subtraction scheme [26], in which a one-loop calculation yields the following renormalization constants:

$$\begin{aligned} Z_1 &= 1 - \frac{1}{64\pi}\left(\frac{g_1}{\epsilon} - \frac{g_2}{\delta}\right), \\ Z_2 &= 1 - \frac{1}{16\pi u(1+u)}\left(\frac{g_1}{\epsilon} - \frac{g_2}{\delta}\right), \\ Z_3 &= 1 - \frac{1}{64\pi}\left(\frac{g_1^2}{g_2}\frac{1}{2\epsilon+\delta} + \frac{2g_1}{\epsilon} - \frac{g_2}{\delta}\right), \\ Z_4 &= 1 - \frac{\lambda}{4\pi\delta}. \end{aligned} \tag{3.2}$$

Note that in contrast to the pure reaction case [1,4] the renormalization constant Z_4 cannot be calculated exactly due to the presence of the velocity field.

The bare (unrenormalized) and renormalized parameters are related as

$$\begin{aligned} g_1 &= g_{10}\mu^{-2\epsilon} Z_1^3, \quad g_2 = g_{20}\mu^{2\delta} Z_1^3 Z_3^{-1}, \\ \lambda &= \lambda_0 \mu^{2\delta} Z_2 Z_4^{-1}, \\ \nu &= \nu_0 Z_1^{-1}, \quad u = u_0 Z_1 Z_2^{-1}. \end{aligned} \tag{3.3}$$

The coefficient functions of the RG differential operator

$$D_{\mathrm{RG}} = \mu \left.\frac{\partial}{\partial\mu}\right|_0 = \mu\frac{\partial}{\partial\mu} + \sum_g \beta_g \frac{\partial}{\partial g} - \gamma_1 \nu \frac{\partial}{\partial\nu},$$

where the subscript ''0'' refers to partial derivatives taken at fixed values of the bare (unrenormalized) parameters, are defined as

$$\gamma_1 = \mu \left.\frac{\partial \ln Z_1}{\partial\mu}\right|_0, \quad \beta_g = \mu \left.\frac{\partial g}{\partial\mu}\right|_0, \tag{3.4}$$

with $g = \{g_1, g_2, u, \lambda\}$.

From the definitions (3.3) and (3.4) it follows that

$$\begin{aligned} \beta_{g1} &= g_1(-2\epsilon + 3\gamma_1), \quad \beta_{g2} = g_2(2\delta + 3\gamma_1 - \gamma_3), \\ \beta_\lambda &= \lambda(2\delta - \gamma_4 + \gamma_2), \quad \beta_u = u(\gamma_1 - \gamma_2), \end{aligned} \tag{3.5}$$

where all γ functions are defined according to Eq. (3.4). Explicit expressions calculated from Eq. (3.2) are

$$\begin{aligned} \gamma_1 &= \frac{g_1+g_2}{32\pi}, \quad \gamma_2 = \frac{g_1+g_2}{8\pi u(u+1)}, \\ \gamma_3 &= \frac{(g_1+g_2)^2}{32\pi g_2}, \quad \gamma_4 = -\frac{\lambda}{2\pi}. \end{aligned}$$

There are one infrared-unstable and four stable fixed points of the RG [zeros of the β functions (3.5)] in the physical region of the parameter space (i.e., with non-negative values of the coupling constants), which we have listed in Table I. It is important to note that the unstable fixed point corresponds to anomalous decay brought about by density fluctuations only [1,4]. Since it is not stable, the system does not exhibit long-time asymptotic behavior corresponding to it, whenever the reaction takes place in an environment with a physical drift field (i.e., at least in real liquid and gaseous media), but rather shows asymptotic behavior governed by one of the stable fixed points, at which the anomalous behavior is produced by the combined effect of density and drift fluctuations and leads to faster decay of the reactant density than the density fluctuations only.

IV. LONG-TIME ASYMPTOTICS OF NUMBER DENSITY

The simplest way to find the average number density is to calculate it from the stationarity condition of the functional Legendre transform [23] (which is often called the effective action) of the generating functional of renormalized Green

TABLE I. Physical fixed points of the RG, their regions of stability in the $(\varepsilon, \delta = d/2 - 1)$ plane, and the scaling dimension α of the density decay rate $[n(t) \sim t^{-\alpha}]$.

Fixed point	g_1^*	g_2^*	u^*	λ^*	α	Region of stability
Gaussian	0	0	not fixed	0	1	$\epsilon<0,\ \delta>0$
Thermal	0	$-32\pi\delta$	$(\sqrt{17}-1)/2$	$-2\pi\delta$	$1+\frac{\delta}{2}$	$\delta<0,\ 2\epsilon+3\delta<0$
Anomalous kinetic	$\frac{64\pi}{9}\frac{\epsilon(2\epsilon+3\delta)}{\epsilon+\delta}$	$\frac{64\pi}{9}\frac{\epsilon^2}{\delta+\epsilon}$	$(\sqrt{17}-1)/2$	$-\frac{4\pi}{3}(\epsilon+3\delta)$	$\frac{1+\delta}{1-\epsilon/3}$	$\epsilon>0,\ -\frac{2\epsilon}{3}<\delta<-\frac{\epsilon}{3}$
Normal kinetic	$\frac{64\pi}{9}\frac{\epsilon(2\epsilon+3\delta)}{\epsilon+\delta}$	$\frac{64\pi}{9}\frac{\epsilon^2}{\delta+\epsilon}$	$(\sqrt{17}-1)/2$	0	1	$\epsilon>0,\ \delta>-\frac{\epsilon}{3}$
Driftless	0	0	not fixed	$-4\pi\delta$	$1+\delta$	unstable

functions obtained by replacing the unrenormalized action by the renormalized one in the weight functional. This is a convenient way to avoid any summing procedures used [1,27] to take into account the higher-order terms in the initial number density n_0. Since the renormalization and calculation of the fixed points of the RG are carried out at one-loop level, we can find only the leading term of the ϵ, δ expansion of the average number density, which corresponds to solving the stationarity equations at the tree level. For a spatially homogenous solution this leads to the rate equation

$$\frac{dn(t)}{dt} = -2\lambda u \nu \mu^{-2\delta} n^2(t) \tag{4.1}$$

with the initial condition $n(0) = n_0$ for the average number density $n(t) = \langle \psi(t) \rangle$.

Solving the Callan-Symanzik equation for the number density,

$$\left((2-\gamma_1) t \frac{\partial}{\partial t} + \sum_g \beta_g \frac{\partial}{\partial g} - d n_0 \frac{\partial}{\partial n_0} + d \right) n(t) = 0,$$

by the method of characteristics, we arrive at the solution of the tree-level rate equation (4.1) in the form

$$n(t) = \frac{\overline{n} e^{-d \int_\tau^t [(2-\gamma_1)s]^{-1} ds}}{1 + 2\overline{\lambda u} \nu \mu^{-2\delta} \overline{n} \tau}, \tag{4.2}$$

where τ is the time scale, which here has been left independent of the wave-number scale μ. In Eq. (4.2), $\overline{\lambda}$, $\overline{u}$, and $\overline{n}$ are the first integrals of the system of differential equations

$$t \frac{d}{dt} \overline{g} = -\frac{\beta_g(\overline{g})}{2 - \gamma_1(\overline{g})}, \quad t \frac{d}{dt} \overline{n} = d \frac{\overline{n}}{2 - \gamma_1(\overline{g})},$$

where $\overline{g} = \{\overline{g_1}, \overline{g_2}, \overline{u}, \overline{\lambda}\}$, with the initial conditions $\overline{g}|_{t=\tau} = g$, $\overline{n}|_{t=\tau} = n_0$.

In the basin of attraction of the Gaussian (trivial) fixed point the tree-level expression for the number density is that of the unrenormalized mean-field theory,

$$n(t) = \frac{n_0}{1 + 2\lambda_0 D_0 n_0 t} \underset{t \to \infty}{\sim} \frac{1}{2K_{+0} t}, \tag{4.3}$$

which physically corresponds to the case when fluctuation effects are negligible.

At the thermal fixed point local correlations of the random force dominate over the long-range correlations. Physically, velocity fluctuations generated by a locally correlated random force correspond to thermal fluctuations near equilibrium [11–13]. The tree-level expression for the number density is

$$n(t) = \frac{n_0}{1 - 2\pi\delta(\sqrt{17}-1)\nu_0 n_0 \mu^{-2\delta} \tau^{-\delta/2} t^{1+\delta/2}}$$

$$\underset{t \to \infty}{\sim} -\frac{\mu^{2\delta} \tau^{\delta/2}}{2\pi\delta(\sqrt{17}-1)\nu_0 t^{1+\delta/2}}, \tag{4.4}$$

where the exponents of temporal variables t and τ have been written to the order $O(\delta)$. Since the thermal fixed point is stable below two dimensions ($\delta < 0$), the asymptotic decay rate $n \sim t^{-1-\delta/2}$ predicted by Eq. (4.4) is faster than that $n \sim t^{-d/2}$ brought about by the density fluctuations only [1,4]. This reflects the enhancement of the effective reaction rate due to the local random stirring, which increases the opportunities for a particle to come into contact with another. We emphasize that this change in the asymptotic reaction rate is brought about not only by a suitable stirring, but also by thermal fluctuations of the velocity field. Therefore, even when there is no external stirring, the actual asymptotic decay rate of the number density is given by Eq. (4.4) at least in a liquid or gaseous environment, where thermal fluctuations of the velocity are inevitable.

In the regime governed by the kinetic fixed point [21] with an anomalous reaction rate the fixed-point value of $\gamma_1^* = 2\epsilon/3$ exactly [14,15]. Therefore, in the long-time asymptotic expression

$$n(t) \underset{t \to \infty}{\sim} -\frac{3\mu^{2\delta} \tau^{(3\delta+\epsilon)/(3-\epsilon)}}{4\pi(\sqrt{17}-1)(\epsilon+3\delta)\nu_0 t^{(1+\delta)/(1-\epsilon/3)}} \tag{4.5}$$

for the tree-level average number density, the exponents of the temporal variables are also exact. The power-law behavior $n \sim t^{-(1+\delta)/(1-\epsilon/3)}$ corresponds to faster decay than in the absence of stirring, because $\epsilon > 0$. The decay is also faster than that induced by locally correlated components of the random force, but still slower than the mean-field decay rate.

Note that this is different from the case of a time-independent divergenceless random velocity field, in which there is no fixed point with $\lambda^* \neq 0$ [9].

The kinetic fixed point with normal reaction rate is stable when the long-range correlations of the random force are dominant and corresponds to reaction kinetics with the normal (mean-field-like) decay rate:

$$n(t) \underset{t\to\infty}{\sim} \frac{1}{(\sqrt{17}-1)\,\nu_0 \lambda_0 t}, \tag{4.6}$$

but with an amplitude different from that in the Gaussian case (4.3). The effect of random stirring with long-range correlations is drastic in the sense that it completely wipes off the effect of density fluctuations below two dimensions, restoring the mean-field decay of the number density with renormalized amplitude. Above two dimensions the effect of the random stirring reduces to the renormalization of the decay amplitude.

The exponent of time in Eqs. (4.5) and (4.6) is a continuous function of ϵ and δ on the borderline of their respective basins of attraction. The continuity of the exponents on the borderlines between the basins of attraction of the fixed points is, however, present up to logarithmic corrections only. We have calculated these logarithmic corrections also, with the following results.

On the borderline between the basins of attraction of the Gaussian and thermal fixed points ($\delta=0,\ \epsilon<0$),

$$n(t)\sim \frac{\ln^{1/2}(t/\tau)}{2\sqrt{2}\,\pi(\sqrt{17}-1)\,\nu_0 t}.$$

This is slower than the mean-field decay rate ($n\sim t^{-1}$), but faster than the density-fluctuation-induced decay rate ($n \sim t^{-1}\ln t$).

The following anomalous decay rate:

$$n(t)\sim -\frac{\mu^{2\delta}\tau^{\delta/2}}{2\pi 3^{(1+\delta/2)/3}(\sqrt{17}-1)\,\delta\nu_0 t^{1+\delta/2}\ln^{(1+\delta/2)/3}(t/\tau)}$$

is found on the borderline ($2\epsilon+3\delta=0,\ \delta<0$) between the regions of stability of the thermal fixed point and the anomalous kinetic fixed point.

The transition from the regime governed by the anomalous kinetic fixed point to that governed by the normal kinetic fixed point on the ray $\epsilon+3\delta=0,\ \delta<0$, is characterized by the logarithmic correction

$$n(t)\sim \frac{\mu^{2\delta}\ln(t/\tau)}{4\pi(\sqrt{17}-1)(1+\delta)\,\nu_0 t}.$$

On the borderline ($\epsilon=0,\ \delta>0$) between the basins of the normal kinetic fixed point and the Gaussian fixed point there are no logarithmic corrections to the mean-field-like decay rate.

V. CONCLUSION

In conclusion, we have analyzed the effects of density and velocity fluctuations on the reaction kinetics of the single-species decay $A+A\to\varnothing$ universality class and found that the anomalous behavior predicted by the analysis based on the density fluctuations only [1,4] is unstable to any drift field fluctuations and thus is not realized in the long-time limit. Instead, the asymptotic behavior of the system is governed by four stable fixed points, at which the anomalous behavior is due to both drift and density fluctuations.

We have carried out the renormalization of the model at one-loop level and calculated the long-time asymptotic behavior of the number density of the reacting particles in these regimes. The results of our analysis show that the velocity fluctuations affect the asymptotic decay of the number density significantly at and below two dimensions, enhancing the effective decay rate in comparison with the anomalous decay rate due to density fluctuations only. Since this effect is brought about already by the ubiquitous thermal fluctuations, we conjecture that the velocity-fluctuation-driven asymptotic behavior is the true asymptotic behavior whenever a physical velocity field may be present, i.e., at least in gaseous or liquid media. This effect of increasing the asymptotic reaction rate is more pronounced for the long-range correlated force components, so that when the correlations of random forcing fall off slowly enough, the random stirring destroys the effect of density fluctuations completely and restores a mean-field-like decay of the number density.

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