Fluid Lagrangian approach to the classical-quantum transition

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With a two-field Lagrangian of the Hamilton-Jacobi equation, a coupling between the two fields can be introduced to yield an appropriate quantum Lagrangian. The quantum effect is nothing more than a quantum pressure force acting on the otherwise classical pressureless fluid when one of the fields is properly renormalized. Inclusion of the electromagnetic coupling under the present formulation shows that the collisionless quantum fluid picture remains valid. Similar, but not identical, to the classical collisionless fluids, the flow circulation and magnetic flux can remain locked together even for a spatially and temporally varying magnetic field. Extensions beyond the present scope of quantum formulation are also discussed.

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I. INTRODUCTION

In the early development of quantum mechanics, the action-angle-variable approach has been the key procedure for identifying the appropriate conserved quantities [1], which may then be quantized by the semiclassical quantization rules. The rationale behind such an approach is rooted to the recognition that the WKB approximation of the Schrödinger equation is exactly the Hamilton-Jacobi equation of classical mechanics, which provides a good scheme to identify the invariant action [2]. However, the WKB approximation is after all an approximation, and it becomes inaccurate when the wavelength is long. Even with the Airy-function type, 1/4-wavelength phase shift taken into account, the WKB approximation is still inaccurate for the low-energy particles in the reflection problems. Moreover, in the WKB approximation of the Schroedinger equation, it is always confined to the regime where the external potential is stationary and the energy conserved. In fact, a serious problem of the WKB approximation arises in the nonstationary situation. The time evolution of the Schrödinger equation and the Hamilton-Jacobi equation are very different. The Hamilton-Jacobi equation has a natural tendency, as will be elaborated below, for its solution to get focused and develops singular caustics, but the Schrödinger equation tends to be dispersive so that the classical caustics can be made disappeared [3–6]. After all, the WKB approximation is simply a working rule and it fundamentally does not address why the quantum dynamics should be wavelike. Therefore, to understand the origin of the transition from classical mechanics to quantum mechanics, one needs to seek an insight beyond the WKB approximation.

In fact, the Hamilton-Jacobi equation is identical to the time-dependent Bernoulli equation for a pressureless classical potential flow. Regardless of whether a finite pressure may ever exist, it is well known that the classical potential flow has a tendency to get steepened and forms shock waves [7]. The steepening is caused by the nonlinear convection of flow, but in the WKB approximation, it is exactly this nonlinear convection term that approximates the dispersive effect in the Schrödinger equation. Moreover, the pressureless classical potential flow field is nonintegrable even without any forcing, but the quantum mechanics is integrable with no forcing. In this regard, the WKB approximation can be fundamentally incorrect in making the semiclassical connection between quantum and classical behaviors when one considers situations beyond the stationary cases.

Apart from the above considerations from the viewpoint of fundamental concept in physics, the fluid description can be a rather useful picture to depict the many-body behaviors of bosonic fields. Recent experimental progresses in the Bose-Einstein condensation has called for a theoretical understanding of the collective quantum behaviors. The quantum fluid picture to be explored below can serve as a good methodology, by which the potential rich phenomena in the Bose-Einstein condensates may be understood.

The aim of this paper is to explore the possibility of making a minimal extension of classical mechanics to arrive at quantum mechanics. An understanding of such an extension may help us gain an insight to the transition from classical dynamics to quantum mechanics. In Sec. II, we shall start with the Hamilton-Jacobi equation and attempt to obtain the Schrödinger equation under the minimum coupling framework. This procedure allows us to pin down what causes the transition from the classical to the quantum. In Sec. III, we extend the formulation to situations that couple to electromagnetic fields. In particular, we confirm that the fluid picture of quantum mechanics remains valid. In Sec. IV, we shall discuss the implications of such a classical-quantum transition and explore the possible extension beyond the standard quantum mechanics.

II. QUANTUM COUPLING TO CLASSICAL FLUIDS

As mentioned above, the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2}=-U(\mathbf{r},\mathbf{t}), \tag{1}$$

is nothing more than the Bernoulli equation of a pressureless potential flow for a compressible fluid, where S corresponds to the velocity potential for the fluid (or the generation function for the Hamilton-Jacobi equation). That is, the velocity field $\mathbf{V}\equiv\nabla S$. Here, U is the force potential. In the fluid description, one must define what the velocity field ∇S means. We hence impose the continuity equation

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\nabla S)=0, \tag{2}$$

where ρ is the mass density, or number density if all particles in the fluid have the same masses. It defines this velocity field to be one which advects the fluid elements in such a way that the mass within each element is always conserved.

In this classical description of a pressureless fluid, there exist two independent real scalar fields S and ρ. Equations (1) and (2) can be derived from the following Lagrangian:

$$L=\rho\left[\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2}+U(\mathbf{r},t)\right], \tag{3}$$

by a straightforward algebra. Comparing Eqs. (1) and (2), one notes that S can be solved without the knowledge of ρ, but ρ can only be solved after S is obtained, a characteristic of the single-particle description. However, if U is not an external potential but a function of only ρ, we may replace U by $(1/\rho)\int U(\rho)d\rho$, which can be regarded as the internal energy density and U the enthalpy density. In this situation, both S and ρ are well coupled and it reflects the characteristic of collective behaviors.

This classical Lagrangian can be modified to become one that yields the Schrödinger equation. The modification is motivated by the recognition that the Hamilton-Jacobi equation can describe the dynamics of a large number of noninteracting identical particles which manifest themselves as a pressureless fluid, and that the single-particle Schrödinger equation can also describe the dynamics of a large number of noninteracting identical particles, such as the Bose gas, when the probability density is interpreted as the number density. The extension from classical dynamics to quantum dynamics can be conducted by introducing a coupling Lagrangian between ∇S and $\nabla\rho$, with a coupling constant equal to $i\hbar/2$. That is, the interacting Lagrangian reads

$$L_{\rm int}=\pm\frac{i\hbar}{2}\nabla S\cdot\nabla\rho, \tag{4}$$

and either sign is acceptable. The particle mass has been taken to be unity and it will remain so hereafter. The resulting equations of motion are

$$\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2}\mp\frac{i\hbar}{2}\nabla^2 S=-U, \tag{5}$$

and

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\nabla S)+\frac{\pm i\hbar}{2}\nabla^2\rho=0. \tag{6}$$

They do not yet bear any resemblance to the Schrödinger equation. To transform into a recognizable form, we renormalize the field variable $\hat{S}=S\pm(i\hbar/2)\ln\rho$ and identify $\hat{S}$ to be the physical velocity potential field. Such a transformation keeps the density ρ satisfying the continuity equation:

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\nabla\hat{S})=0, \tag{7}$$

thereby giving $\nabla\hat{S}$ a well-defined physical meaning of being a velocity field $\mathbf{V}$. On the other hand, the dynamical equation for $\hat{S}$ becomes

$$\frac{\partial \hat{S}}{\partial t}+\frac{(\nabla\hat{S})^2}{2}+U-\frac{\hbar^2}{4}\left[\nabla^2\ln\rho+\frac{(\nabla\ln\rho)^2}{2}\right]=0. \tag{8}$$

Therefore, both $\hat{S}$ and ρ can be real fields. Let the wave function be $\psi\equiv\sqrt{\rho}\exp(i\hat{S}/\hbar)$, and Eq. (5) can be derived from the imaginary part of the time-dependent Schrödinger equation, and Eq. (6) from the real part [8–10]. Comparing Eqs. (1) and (8), one finds that the quantum effect results in a pressurelike force that contributes to evolve the velocity field, though Eq. (8) means to decribe a collisionless quantum fluid. The familiar Euler equation for a fluid can be derived by taking a gradient on both sides of Eq. (8) [see Eq. (10) below].

That is, the introduction of an interaction Lagrangian into the classical Lagrangian gives rise to a quantum Lagrangian, which is reduced to the usual Schrödinger description when the S is properly renormalized to $\hat{S}$. Furthermore, if $U=a+b\rho$ where a and b are constant, Eqs. (5) and (6) represent the Landau-Ginzberg equation for the bosonic particles near phase transitions. The two-field Lagrangian of quantum mechanics can finally be expressed as

$$L=\rho\left[\frac{\partial \hat{S}}{\partial t}+\frac{(\nabla\hat{S})^2}{2}+\frac{\hbar^2}{2}(\nabla\ln\sqrt{\rho})^2+W\right], \tag{9}$$

where $W=U$ if U is an external potential, and $W=(1/\rho)\int U(\rho)d\rho$ if $U=U(\rho)$. We note that the last three terms of Eq. (9) constitute the energy density of the quantum fluid [8], which can also be taken as the Hamiltonian density $H(\nabla\hat{S},\rho,\nabla\rho)$. Given such a fluid Hamiltonian density, we note that Eqs. (7) and (8) can alternatively be derived from the Hamiltonian formulation. In other words, $\partial\hat{S}/\partial t=-\delta H/\delta\rho$ and $\partial\rho/\partial t=\delta H/\delta\hat{S}$, with δ being the variational derivative [11].

III. MINIMAL COUPLING TO ELECTROMAGNETIC FIELDS

Given the above framework that describes the classical-quantum transition, it is now a straightforward matter to extend it to situations that contain an electromagnetic coupling. We will not repeat all the algebra but to mention the key steps. The pressureless classical fluids, described by the Hamilton-Jacobi equation, have already had a well-known procedure to get coupled to the electromagnetic fields. That is simply to insert to ∇S a diamagnetic current $-(e/c)\mathbf{A}$ and add $e\Phi$ to $\partial S/\partial t$, where e and c are the electric charge and light speed, respectively, $\mathbf{A}$ and Φ the vector potential and electric potential. Such a prescription warrants the gauge invariance of the Hamilton-Jacobi equation, i.e., invariant to the simultaneous transformations $S\rightarrow S+(e/c)\chi$, $\mathbf{A}\rightarrow\mathbf{A}+\nabla\chi$, and $\Phi\rightarrow\Phi-(1/c)(\partial\chi/\partial t)$, and hence the classical Lagrangian density, Eq. (3), must be modified accordingly. For the quantum mechanical coupling, it also needs to satisfy the gauge invariance, and therefore the coupling should be in

between $\nabla\rho$ and $\nabla S-(e/c)\mathbf{A}$, instead of ∇S alone. That is, in the presence of electromagnetic fields, the velocity field $\mathbf{V}$ appearing in Eqs. (8) and (9) must be replaced by $\nabla\hat{S}-(e/c)\mathbf{A}$, and $\partial\hat{S}/\partial t$ replaced by $\partial\hat{S}/\partial t+e\Phi$, where $\hat{S}$ is defined in the same way as in the neutral particle case.

Inclusion of electromagnetic coupling in fact does yield the same important conservation equations as in the classical magnetized fluid. The counterpart of the classical Euler equation, which is equivalent to the momentum conservation, in quantum mechanics now reads

$$\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}+\left[\left(\frac{e}{c}\right)\frac{\partial\mathbf{A}}{\partial t}+e\nabla\Phi-\left(\frac{e}{c}\right)\mathbf{V}\times\mathbf{B}\right]+\nabla\left\{U-\left[\frac{\hbar^2}{4}\left(\nabla^2\ln\rho+\frac{(\nabla\ln\rho)^2}{2}\right)\right]\right\}=0. \tag{10}$$

The third term accounts for the additional electromagnetic forces acting on the fluid. Aside from the quantum pressure force, this equation is what should be expected for classical magnetized fluids. This equation of motion has a rather delicate control of dynamics that warrants the *smooth* rotational component of velocity field to be always locked with the diamagnetic current, $-(e/c)\mathbf{A}$. To see this point clearly, we may for the time being pretend $\mathbf{V}$ to be unrelated to $\mathbf{B}$ and take a *curl* operation on both sides of Eq. (10). It gives

$$\frac{\partial}{\partial t}[\mathbf{\Omega}+(e/c)\mathbf{B}]+\nabla\times\{\mathbf{V}\times[\mathbf{\Omega}+(e/c)\mathbf{B}]\}=0, \tag{11}$$

where the vorticity $\mathbf{\Omega}\equiv\nabla\times\mathbf{V}$. Equation (11) represents the local conservation law of the vector field $\mathbf{\Omega}+(e/c)\mathbf{B}$, much as Eq. (2) represents the local conservation law of the scalar field ρ. Equation (11) also means that the angular momentum of a quantum particle has to be tied to the magnetic flux not only in stationary states but also in dynamical states. If initially $\mathbf{\Omega}+(e/c)\mathbf{B}=0$, this condition will remain true in the whole evolution. This shows the delicacy of Eq. (10), in that no other *smooth* rotational component of velocity field can ever exist in quantum fluids, and the only extra ones are singular vortex lines or sheets [9]. Smooth rotational velocity component is typically absent in the classical magnetized fluid of no pressure, but generally not in the classical magnetized fluid with finite pressure. In this regard, the quantum fluid, having a finite quantum pressure, has a rather restricted built-in kinematics from that of a classical fluid.

Having obtained Eq. (10) and discussed the conservation of linear and angular momenta, we now turn to a close examination of Eq. (10) and point out a possible paradox of the fluid description. We shall examine the simplest case, with a static uniform magnetic field $B\hat{z}$, to illustrate the issue. Given that the velocity field $\mathbf{V}=\nabla\hat{S}-(e/c)\mathbf{A}$, we find that the velocity field $\mathbf{V}$ can become unbound at a large distance since its rotational component is so. For example, the gauge $\mathbf{A}=(1/2)\mathrm{Br}\hat{\phi}$ gives a uniform rotation to the entire fluid. This situation seems very different from what one would have anticipated for a collection of noninteracting magnetized particles that are distributed more or less uniformly in space and each is circulating around its own guiding center without any systematic flow. In fact, this paradox derives its origin from the Hamilton-Jacobi equation.

To understand what the uniform rotation does to the fluid, we consider the radial force balance of Eq. (10). Split the velocity field into $\mathbf{V}=\mathbf{V}_r+\mathbf{V}_p$, with the former being the *static* rotational component and the latter the dynamical potential component. Upon undergoing a uniform rotation, the fluid must experience an outward centrifugal force, which arises from the inertia force term $\mathbf{V}_r\cdot\nabla\mathbf{V}_r$ of Eq. (10), but this force is over-balanced by an inward Lorentz force $(e/c)(\mathbf{V}_r\times\mathbf{B})$, resulting in a net inward force. This inward force has nothing to balance for the classical pressureless fluid described by the Hamilton-Jacobi equation, and it hence yields radial-orbit oscillations, rushing into the origin $r=0$ at a finite speed. For axisymmetric boundary conditions, such a fluid will certainly develop a density singularity at $r=0$. However, equipped with additional quantum pressure opposing the converging flow, the quantum fluid can attain a regular state, for which the density peaks at $r=0$ with a finite value.

The paradox given above can be removed when the classical particles possess additional angular momenta other than that arising from the diamagnetic current. The guiding centers of particles can be located at a finite radius r. The additional angular momenta makes the particles to acquire a counter-clockwise rotating velocity with the same speed as the local magnetization current, i.e., $(\partial S/\partial\phi)/r=(e/c)A_\phi$ at the guiding centers, thereby always yielding static guiding centers. By increasing the angular momentum, the particle guiding centers can be situated at any large radius. Such a picture of static guiding centers can be extended to quantum mechanics, where a particular angular momenta yield static density peaks at some particular radii; furthermore, the probability density can be made more or less uniform on a coarse scale when many modes of different angular momenta and energies are superposed. Consequently, the fluid description of quantum mechanics has no contradiction to the conventional picture of non-interacting particles that are subject to magnetic forces.

As a result of superposition of many counter-rotating modes of different angular momenta in this system, there exist many topological defects, i.e., singular vortex lines, located where $\rho=0$ [8], and the quantum system is generally nonstationary. This leads us to the final issue about how the vortex lines should evolve in the general time-dependent situations. If initially S is a multivalued function, e.g., containing a term such as $b\phi$ where b is a constant and ϕ some azimuthal angle around a three-dimensional line, the vector field $\mathbf{\Omega}+(e/c)\mathbf{B}$ will not vanish at the vortex line in Eq. (11). This equation demands that the vortex line must be frozen to the quantum fluid and carried about by the fluid motion. In general the constant b is arbitrary, but in quantum mechanics the constant b is required to be $2m\pi$, with m being an integer, in order to make the wave function single-valued. Once the vortex line is initially quantized, it should remain so throughout the evolution. Thus, from the dynamical viewpoint, the quantization of angular momentum is therefore not a consequence of dynamics but a consequence of the initial condition.

IV. SUMMARY AND DISCUSSION

In this paper, we report a field-theoretical approach that yields the transition from the classical equation of motion to the quantum equation of motion by introducing a coupling between the classical velocity potential (or the generation function) S and the classical density field ρ. The coupling constant is found to be imaginary and equal to $i\hbar/2$. Such a imaginary coupling constant makes the originally real field S to become complex. However, if one appropriately subtracts the imaginary part from S and redefines a real field $\hat{S}$, one can arrive at the two-field description of the Schrödinger equation.

We find, from Eq. (5), that the quantum mechanical effect is nothing more than an imaginary viscous force acting on an otherwise classical pressureless potential flow. With these results, two questions naturally arise. What can be the microscopic origin of the imaginary viscosity? What does the imaginary viscous force do to the dynamics of the flow? Equation (8) reveals an answer to the second question. The imaginary viscous force yields an effective pressure force [8], which turns out to be dispersive; it prevents the formation of caustics produced by the nonlinear steepening of convection. As to the first question, a deep insight is needed. More than three decades ago, Nelson proposed the existence of some intrinsic fluctuations in the space that force the particle trajectory to fluctuate about the classical orbit [12]. The intrinsic fluctuations must instantaneously adjust themselves according to a certain rule that depends on the instantaneous probability density of the particle. Such an interpretation of quantum mechanics, the so called stochastic mechanics, has acquired continual attentions in the past years [13–16]. In Refs. [15,16], the quantity ∇S defined above has been used to calculate the particle trajectory in the complex space dimension for making a contact with the weak measurement theory. Recently, stochastic mechanics been also refined to relax the instantaneity condition [17]. A similar but somewhat different idea, the so-called de Broglie-Bohm theory [10,18], had also been put forth prior to that of Nelson. In that theory, the Schrödinger equation evolves the pilot wave, whose phase is the velocity potential $\hat{S}$ given above. The velocity $\nabla\hat{S}$ then evolves the particle position and ρ describes the self-consistent probability density of the particle appearing at a certain location. In spite of these efforts, it remains un-interpreted what the nature of such intrinsic fluctuations should be. The present work is no exception; we do not understand what the underlying physical mechanism is for the imaginary diffusion that appears in Eq. (5).

In the presence of electromagnetic fields, we find that the dynamical picture of a fluid still holds for describing the quantum dynamics. An important result derived from our analysis is that the *smooth* rotational component of the velocity fields must always remain the same as the diamagnetic current, no matter how rapidly the electromagnetic fields may vary. Though this result is the same as that of the dynamics of a classical pressureless magnetized fluid, the result is however different from that of a fluid with finite pressure, which can have a finite *smooth* rotational component of velocity other than the diamagnetic current if this rotational component is initially given. The fact that quantum mechanics shares the same property of angular momentum with the pressureless fluid is not surprising since the Schrödinger equation is based on an extension of the Hamilton-Jacobi equation. However, the fact that it does not possess the general characteristics of finite-pressure classical fluids casts a question mark on the foundation of quantum mechanics based on the Hamilton-Jacobi equation, since the ''quantum'' fluid also contains a finite quantum pressure.

The Hamilton-Jacobi equation was originally devised to solve the particle dynamics under the assumption that the particle trajectory is solvable. But this is true only for integrable systems. As the Schrödinger equation is built upon a straightforward extension of the Hamilton-Jacobi equation. It is therefore of interest to ponder whether there may exist a more general formulation of particle dynamics than the Hamilton-Jacobi equation that can be extended to the quantum regime. A hint to this issue is that in a steady state, the Hamilton-Jacobi equation yields highly foliated solutions that are multivalued almost everywhere when the particle dynamics is highly chaotic, indicative of that the simple-minded expression of classical momentum in terms of the gradient of a scalar field ∇S is far from adequate. On this note, we shall stress that the highly foliated S of the Hamilton-Jacobi equation means that S contains densely distributed singularities such as the branch cuts, and with proper regularizations the classical momentum may acquire a smooth rotational component. Thus, the conventional quantum mechanics can be inappropriate in the highly chaotic regime. Actually, quantum mechanics of present formulation has only been tested at high precision for the integrable systems, or weakly chaotic systems where perturbation theories still hold and the classical action space is densely filled with invariant KAM surfaces. In this regime, the Hamilton-Jacobi equation is a good starting point for its extension to quantum mechanics. However, in the highly chaotic regime where the Hamilton-Jacobi equation becomes a bad description of particle dynamics, will the Schrödinger equation remain a reliable description of quantum mechanics? Can Eq. (10), with a general vector field $\mathbf{V}$, together with the density continuity equation be a better description of the chaotic quantum world? An obvious result the latter formulation produces is that the angular momentum no longer needs to be quantized, unless the initial condition specifies. This is because now $\mathbf{V}$ and ρ, rather than the wave function ψ, are the physical fields.

We now turn to a different discussion that also extends quantum mechanics beyond the present scope. From the present formulation based on the Hamilton-Jacobi equation, it is also natural to contemplate why the coupling constant must be imaginary. What if the coupling constant is a real quantity? Consider the situation where $\kappa \equiv \pm i\hbar$ which is a positive real quantity. The last term in Eq. (5) is nothing more than a viscous force, with κ playing a similar role as the viscosity, and the last term in Eq. (6) has an effect of negative diffusion. Alternatively, one may choose κ to be a negative real quantity, yielding a negative viscosity and positive density diffusion. While the physics of viscous force is well understood, the negative diffusion is a much less commonly known phenomenon. The negative diffusion may occur in turbulent media. Under the action of small-scale turbulent motion, the large-scale flow may self-organize to form coherent patterns. The self-organization process can be real-

ized as a negative diffusion process [19,20]. In fact, with a real κ and if U is indeed an external potential, Eq. (5) can be reduced to a Schrödinger equation that has an imaginary time, for which the wave function can be expressed as $\psi = e^{-S/\kappa}$. The quantity S can be identified as the partition function in statistical mechanics and the imaginary time identified as β, the inverse temperature. On the other hand, Eq. (6) has no immediate physical significance in statistical mechanics when U is an external potential. We will return to this issue again.

Next, we may further inquire what if $\hbar$ takes a complex value. This case may have a correspondence to certain physical situations. To see how it may happen, let $\hbar = \kappa e^{i\alpha}$, where both α and κ are real constants. We may transform away the phase factor $e^{i\alpha}$ with the following new variables:

$$S_1 = Se^{i\alpha}, \quad \tau = te^{-i\alpha}, \quad U_1 = Ue^{2i\alpha}. \tag{12}$$

To be useful, we technically force τ to be a real quantity. That is, the Schrödinger equation with a complex $\hbar$ can be transformed into a quantum mechanical problem with a complex potential U_1 and real ''time'' τ, for which the wave function is taken to be $\psi = \sqrt{\rho} e^{iS_1/\kappa}$. A complex potential in quantum mechanics is encountered usually in an effective theory where inelastic processes take place.

Aside from the fundamental issues concerning a better insight of quantum mechanics, we now address a technical application of a complex $\hbar$. It is sometimes useful to find a quantum partition function from the corresponding quantum transition probability by analytically continuing an imaginary β to a real β and then sum over all initial states [21]. The analytical continuation may be conducted in the complex β domain. But from the present formulation, the analytical continuation may also be carried out in the complex $\hbar$ domain. The path of analytical continuation for a complex $\hbar$ is different from the path for a complex time taken with a real potential. The former can be useful when the latter encounters extended singularities, such as branch cuts, across which the analytical continuation cannot pass. Taking a complex $\hbar$ is equivalent to taking both the potential U_1 and t to be complex by forcing τ to be real, and it may provide a physical avenue to get around the extended singularities. This is so because the situations with a complex $\hbar$ can be mapped to real quantum mechanical problems with complex potentials, and they have a physical correspondence to the inelastic scattering problems, for which singularities are less likely to occur.

In fact, an interesting result a complex $\hbar$ can produce is that one may evolve the system as an initial-valued problem. By choosing a fixed angle α and a real time t, one may observe, at least numerically, how the wave function evolves in time. Unlike the situation with a complex time but real $\hbar$, at any instant in time, one now has a well-defined manner (explained below) to analytically continue the wave function with respect to $\hbar$ to its actual value, without the need of the past and future wave functions. The idea is to make use of $\delta S/\delta\alpha$, $\delta\rho/\delta\alpha$, and their higher-order derivatives. One needs to numerically compute $S(r,t)$ and $\rho(r,t)$ for several α's of small differences, so that these derivatives can be evaluated. The accuracy of the analytically continued solution depends on how many higher derivatives to be retained in the solution construction. As the parallel computation has become the state of art, such a strategy should be easily realizable.

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