Subharmonic autoresonance

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(Received 12 August 1999)

Adiabatic passage through higher order resonances in a perturbatively driven dynamical system with a slow control parameter, yields persisting phase locking and a strong long time response. The phenomenon has a sharp threshold on the driving amplitude, which scales with the control parameter chirp rate A and resonance order n, as $A^{3/(4n)}$.

PACS number(s): 05.45.Xt, 05.45.Gg

Many physical applications need a strong, controllable response in a dynamical system by applying a weak perturbation. One can use a resonance for this purpose, i.e., tune the driving frequency ω_0 to the system's natural response frequency ω. Nevertheless, in most cases, the nonlinear frequency shift destroys the resonance, limiting the response amplitude to $O(\epsilon^{1/2})$, where ϵ is the characteristic strength of the drive [1]. This limitation can be removed by using a feedback. For example, one easily reaches large amplitudes of a swing by subconsciously decreasing the driving frequency, as the oscillation amplitude grows. There also exists another way of obtaining a strong response to a weak perturbation without the feedback and precise tuning. In this approach, one passes through the resonance by slowly varying some control parameter, $\lambda(t)$. Then, at certain conditions, on approaching the resonance, the system phase locks to the drive and enters an *autoresonant* evolution stage in which the phase locking continues despite the variation of λ. Typically, this means a slow increase of the response amplitude, which may become large over a long time. The system can be put back into a near equilibrium by reversing the direction of variation of λ. This simple control of excitation by a weak force is important in applications. The first known use of autoresonance was in particle accelerators [2]. Later, the idea was implemented in other physical problems, ranging from atomic physics [3] through nonlinear waves [4], and plasmas [5] to fluid dynamics [6]. Recently, it was noticed that passage through resonance yields autoresonance only if the driving amplitude exceeds a sharp threshold ϵ_{th} depending on the chirp rate $A = d\lambda/dt$. For instance, when the control parameter $\lambda(t)$ is the driving frequency ω_0 itself, one finds $\epsilon_{th} \sim A^{3/4}$ [5,6].

In this paper, we study a different type of autoresonance, taking place when one slowly passes a resonance $\omega_0 \approx \omega/n$ ($n = 2,3, \ldots$). This phenomenon will be referred to as *subharmonic* autoresonance (SHAR) in the following, in contrast to the *fundamental* autoresonance (FAR), corresponding to the $n = 1$ case. The physical mechanism leading to the SHAR differs from that in the FAR, namely, the SHAR is due to the ability of nth order nonlinearities in the driven system to generate an effective drive at the fundamental frequency. This effective drive, in turn, plays a role of an adiabatic forcing in transition to autoresonance in the system. Passage through $\omega_0 \approx \omega/n$ resonances is the only possibility for creating the internal forcing at fundamental frequency in the system, while other resonances (as in the second harmonic driving, for example) do not necessarily lead to phase locking and autoresonance in the system. We shall see below that the SHAR theory becomes increasingly complex at larger n, so we shall study $n = 2$ and 3 only. Nevertheless, recent experiments in driven pure electron plasmas [7] exhibit the SHAR for n up to 5.

We focus our analysis on finding thresholds for entering the SHAR in a simplest driven one-dimensional dynamical system with a slow parameter. Many other aspects, such as the details of the advanced SHAR stage, the effect of a weak dissipation, higher dimensionality, etc., will be left for future studies. Our system is

$$x_{tt} + \Lambda^2(t)x + \alpha x^2 + \beta x^3 = \epsilon \sin(t/n), \tag{1}$$

with $\Lambda^2 \equiv 1 + \lambda(t)$ a slow function of time (the variables x, t and parameters λ, α, β, $\epsilon \ll 1$, are set to be dimensionless). By starting in equilibrium $x = x_t = 0$, one crosses the resonance $\omega/n \approx \omega_0 = 1/n$ when $\lambda(t)$ passes zero. We proceed from numerical illustrations of the FAR and SHAR in this system. Figure 1 shows the evolution of the system's energy $E = \frac{1}{2}x_t^2 + \frac{1}{2}\Lambda^2 x^2 + \frac{1}{3}\alpha x^3 + \frac{1}{4}\beta x^4 - \epsilon x \sin(t/n)$ for $n = 1,2,3$ and driving amplitudes $\epsilon = 1.07 \times 10^{-3}$ [the upper curve in Fig. 1(a)], 0.05 and 0.12, respectively. We used $\alpha = 1$, $\beta = 0$ and initial conditions $x = x_t = 0$. The control parameter

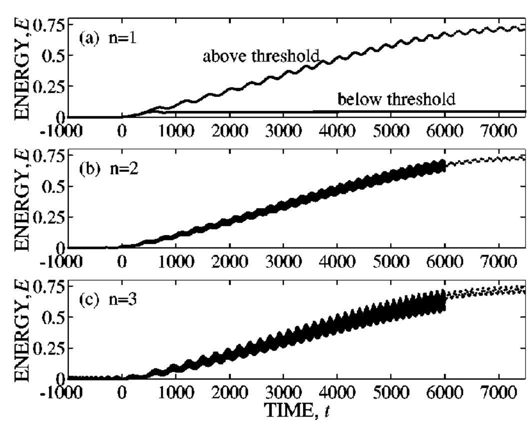

FIG. 1. Energy E versus time. (a) $n = 1$, $\epsilon = 1.07 \times 10^{-3}$ (just above the threshold, upper curve), $\epsilon = 1.03 \times 10^{-3}$ (just below the threshold, lower curve); (b) $n = 2$, $\epsilon = 0.05$; (c) $n = 3$, $\epsilon = 0.12$. Small time scale structures (dark areas) are seen for $n = 2$, 3. The mapping $E(k2\pi n)$, $k = 1,2, \ldots$ at $t > 6000$ (dots) illustrates the slow time scale phase locking in the system.

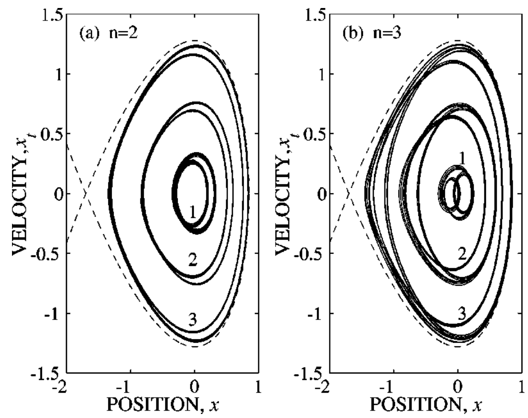

FIG. 2. Phase space portraits in autoresonance. (a) $n=2$; (b) $n=3$. Lines 1, 2, and 3 represent solutions in intervals $\Delta t=75$ in the vicinity of $t=360$, 2400, and 7400. The separatrix is shown by the dashed lines.

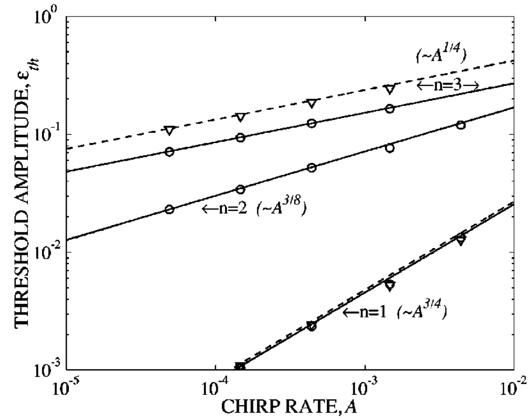

FIG. 3. Threshold drive amplitudes ϵ_{th} versus chirp rate A of the control parameter for $\alpha=1$, $\beta=0$ [numerical simulations (o), theory (full lines)] and $\alpha=0$, $\beta=1$ [numerical simulations (∇), theory (dashed lines)]. There is no trapping into $n=2$ SHAR in the latter case.

was varied as $\lambda(t)=D\sin(\frac{1}{2}\pi t/T)$ (with $D=0.7$ and $T=7500$) between the initial $t_0=-1000$ and final $t_1=7500$ integration times, so the desired resonances were encountered at $t=0$. One can see in the figure that, in all cases, the efficient excitation proceeds just beyond $t=0$ and the final energy reaches a substantial value of ~ 0.7, despite the relative smallness of the driving amplitude ϵ. In addition to the smooth averaged growth, the energy curves have *slow* (autoresonant) modulations, as well as, fast time scale structures, seen best for $n=2$ and 3 [dark areas in Figs. 1(b) and 1(c)]. Nevertheless, these structures disappear if one maps the energy at equal time intervals $2\pi n$ (the mapping is shown by dots in Fig. 1 for $t>6000$), demonstrating the continuing phase locking in the system on the slow time scale. Because of the phase locking, the energy is nearly $2\pi n$ periodic at all times. The phase locking of energy curves reflects the phase locking of autoresonant solutions. The latter are shown for $n=2$, 3 in Fig. 2, as portraits in the phase space in the vicinity ($\Delta t=75$) of three times beyond the linear resonance, $t=360$,2400, and 7400 (lines 1, 2, and 3 in the figure). We see that the portraits comprise a nested set of almost closed trajectories, each depending on *n*, and making two or three turns around the origin before nearly closing on itself. It takes a period of nearly $2\pi n$ to complete the closure, while fast rotations within this period yield the small time scale structures in Fig. 1. The slow expansion of the orbit itself (as moving from curves 1 to 2 and 3 in Fig. 2) reflects the slow time scale excitation to higher energies, necessary to preserve the phase locking with variation of λ. Note that, in all the examples, we closely approach the separatrix for $\Lambda=1+D$ (dashed lines in Fig. 2). If one further increases λ, the autoresonance is destroyed by overlap with other resonances, followed by stochastic escape from the potential well. The stochastic ionization of atoms or dissociation of molecules after a stage of autoresonant excitation [3] are among potential applications of this effect.

Next, we proceed to the problem of thresholds. Numerically, for entering autoresonances in Fig. 1, the driving amplitudes must exceed certain thresholds. We illustrate this phenomenon in Fig. 1(a), where the upper curve corresponds to $\epsilon=1.07\times10^{-3}$ just above the threshold ($\epsilon_{th}=1.05\times10^{-3}$, in this case), while the lower curve shows the result for $\epsilon=1.03\times10^{-3}$ just below the threshold. We see that for $\epsilon<\epsilon_{th}$, the phase locking is destroyed after passing $t=0$ and the final excitation energy is small. We have studied this threshold phenomenon numerically for other conditions and summarize our results in Fig. 3. The figure shows the dependence of the threshold drive amplitude on the chirp rate $A=\frac{1}{2}|D|\pi/T$ of the control parameter at $t=0$ for different n and two sets $\alpha=1$, $\beta=0$ (circles) and $\alpha=0$, $\beta=1$ (triangles). One finds, that for the second set, λ must decrease at $t=0$ for entering autoresonance, so D is negative. Furthermore, there is no $n=2$ autoresonance for this set. For a smooth initial phase locking in these calculations we used $\epsilon=\epsilon_0 f(t)$, where the slow switching on function was $f(t)=0.5\{1+\tanh[(t+2T_0)/T_0]\}$ and $T_0=T/25$. Thus, ϵ almost reached its constant value ϵ_0 at the resonance ($t=0$). For each set of parameters, we have taken 20 equally spaced over $[0,2\pi]$ values of the initial phase of the drive. The numerical threshold ϵ_{th} was defined as the average between two values of ϵ for which one obtains autoresonance for all ($\epsilon=\epsilon_1$) or none ($\epsilon=\epsilon_2$) of the initial driving phases. The width of the threshold, $(\epsilon_1-\epsilon_2)/\epsilon_{th}$, was typically less than 3%. One can see in Fig. 3 that our calculations yield the scaling $\epsilon_{th}\sim A^{3/(4n)}$ in agreement with the predictions of the theory (see below), shown by solid ($\alpha=1$, $\beta=0$) and dashed ($\alpha=0$, $\beta=1$) lines. Before proceeding to the theory, we give a qualitative explanation of the threshold scaling phenomenon in slow passage through higher order resonances. Suppose $n=2$. We drive the system at $\omega_0=1/2$ in this case, yielding a *nonresonant*, $O(\epsilon)$ linear response at this frequency. As the result, the quadratic nonlinearity of the oscillator yields a driven response of amplitude $\epsilon_{eff}\sim O(\epsilon^2)$ at frequency $\omega_{eff}=2\omega_0=1$. This nonlinear excitation plays a role of an *effective* drive, which passes the linear resonance, as the control parameter λ varies in time. This passage yields phase locking, and later the SHAR, as in the FAR case, but with the driving amplitude replaced by ϵ_{eff}. If the FAR threshold scales as $\epsilon_{th}^{n=1}\sim A^{3/4}$, then the $n=2$ SHAR threshold scales as $(\epsilon_{eff})_{th}\sim A^{3/4}$, or $\epsilon_{th}^{n=2}\sim A^{3/8}$. For $n=3$, the third order nonlinearity yields $\epsilon_{eff}\sim\epsilon^3$ and the corresponding threshold $\epsilon_{th}^{n=3}\sim A^{1/4}$. Finally, if α vanishes in Eq. (1), then there is no second order response in the system, so the trapping into $n=2$ SHAR is impossible. In order to confirm these qualitative predictions and find the proportionalities in the scaling relations, we develop the theory of *adiabatic passage* through higher order resonances.

Our theory uses Whitham's averaged variational principle [8] as a convenient tool in studying slow modulations in the system. We write the Lagrangian of the problem

$$L=\frac{1}{2}x_t^2-\frac{1}{2}\Lambda^2x^2-\frac{1}{3}\alpha x^3-\frac{1}{4}\beta x^4+\epsilon x\sin(t/n) \qquad (2)$$

and seek solution $x=u(\theta,t)$, which is $2\pi n$ periodic in θ and evolves on both slow and fast temporal scales. The explicit time dependence in $u(\theta,t)$ is assumed to be slow, but $\theta(t)$ is the *fast* angle variable, while the angular frequency $\Omega(t)=\theta_t$ is viewed as a slow function of time. We also assume a continuing phase locking in the system on the slow time scale, i.e., that the phase mismatch $\Phi\equiv(\theta-t)/n$ between the oscillator and the drive is bounded and slow. The Whitham's method allows to take advantage of a large difference between the time scales in the problem and, essentially, average out the fast scale. The analysis is greatly simplified in studying the threshold conditions, since the threshold is a *weakly* nonlinear effect. Indeed, in studying the threshold in the FAR case, one needs to consider oscillation amplitudes of $O(\epsilon^{1/3})$ (see below). Thus, as conjectured above, for finding the $n=2$ or 3 SHAR thresholds, one must deal with excitation amplitudes of $O(\epsilon^{2/3})$ or $O(\epsilon)$, respectively.

We proceed by constructing Whitham's averaged variational principle for the FAR case in our system. The solution is represented as $x=a_0+a_1\cos\theta+a_2\cos(2\theta+\eta_2)$, where $a_1(t)$ is slow and $O(\epsilon^{1/3})$ by assumption, while $a_{0,2}(t)$ are also slow, but scale (see below) as $a_{0,2}\sim a_1^2\sim O(\epsilon^{2/3})$. All higher harmonics in x are neglected to desired order. The amplitudes a_i and the phase η_2 are viewed as new unknown slow dependent variables. Next, we write the interaction term in the Lagrangian as $\epsilon x\sin(\theta-\Phi)$ and average L over the fast angular variable, i.e., calculate $\mathcal{L}_1=(2\pi)^{-1}\int_0^{2\pi}L(\theta,t)d\theta$. Here, the slow time is fixed and $L(\theta,t)$ is evaluated by substituting our three-term representation for x into Eq. (2) and neglecting the time derivatives (such as a_{it} and η_{2t}) of all *slow* objects. Formally, the averaged Lagrangian is a function of slow variables and $\lambda(t)$ only, $\mathcal{L}_1=\mathcal{L}_1[a_1,a_2,a_3,\eta_2,\Phi,\Phi_t;\lambda(t)]$ (recall that $\Phi(t)\equiv\theta-t$ is the phase mismatch, so the dependence on Φ_t enters $\mathcal{L}_1$ via $\Omega\equiv d\theta/dt=1+d\Phi/dt$). The main step of the Whitham's approach is to replace the variational principle $\delta\int L(x,x_t)dt=0$ in the original problem by the averaged variational principle $\delta\int\mathcal{L}_1[a_1,a_2,a_3,\eta_2,\Phi,\Phi_t;\lambda(t)]dt=0$. Then, variations with respect to a_i, η_2, and Φ yield the desired slow evolution equations in the problem. In our case, $\mathcal{L}_1=\mathcal{L}_0-(\epsilon/2)a_1\sin\Phi$, where near the resonance ($\Omega\approx\Lambda\approx1$), $\mathcal{L}_0=\frac{1}{4}(\Omega^2-\Lambda^2)a_1^2-\frac{1}{2}a_0^2+\frac{3}{4}a_2^2-\frac{1}{2}\alpha a_0a_1^2-\frac{3}{32}\beta a_1^4-\frac{\alpha}{4}a_2a_1^2\cos\eta_2$. Then, the variation with respect to a_0 yields $a_0=-\frac{1}{2}\alpha a_1^2$. Similarly, the variations with respect to a_2 and η_2, yield $a_2=\frac{1}{6}\alpha a_1^2\cos\eta_2$ and $\eta_2=0$, respectively. The substitution of these results back into $\mathcal{L}_1$ gives the following reduced, weakly nonlinear averaged Lagrangian: $4\mathcal{L}_1=(\Omega^2-\Lambda^2)a_1^2+\gamma a_1^4-2\epsilon a_1\sin\Phi$, where $\gamma=(5\alpha^2/12)-\frac{3}{8}\beta$. Now, $\mathcal{L}_1$ is used to write the variational evolution equations for the remaining slow variables a_1 and Φ (recall that $\Omega=1+d\Phi/dt$):

$$(\Omega a_1^2)_t+\epsilon a_1\cos\Phi=0, \qquad (3)$$

$$(\Omega^2-\Lambda^2)a_1+2\gamma a_1^3-\epsilon\sin\Phi=0. \qquad (4)$$

In the vicinity of the resonance, $\Lambda\approx1+\frac{1}{2}\lambda(t)$, so to lowest order, Eqs. (3) and (4) yield

$$a_{1t}=-\frac{1}{2}\epsilon\cos,\Phi,\quad \Phi_t=\frac{1}{2}\lambda(t)-\gamma a_1^2+\frac{\epsilon}{2a_1}\sin\Phi. \qquad (5)$$

Now, we assume $\gamma>0$ and seek solutions of Eq. (5) in the form $a_1=\overline{a}+p$ and $\Phi=\pi/2+\xi$, where $\overline{a}$ is defined via $\frac{1}{2}\lambda(t)-\gamma\overline{a}^2+(\epsilon/2\overline{a})\equiv0$, while $|p/\overline{a}|\ll1$ and $|\xi|\ll\pi$. Then, on linearization, Eqs. (5) yield $p_t=-(A/2G)+\frac{1}{2}\epsilon\sin\xi$, and $\xi_t=-Gp$, where $A=\lambda_t>0$ is the chirp rate and $G\equiv2\gamma+\epsilon/(2\overline{a}^2)$. The Hamiltonian for this system is $H(p,\xi)=-Gp^2/2+V(\xi)$, where $V(\xi)\equiv(A/2G)\xi+(\epsilon/2)\cos\xi$ is a tilted cosine potential. Only when the tilting is small enough, i.e., $A/G<\epsilon$ does this potential have equilibrium points, and we can have trapped solutions and phase locking in our system. But G has a minimum $G_m=3(2/\epsilon)^{1/3}\gamma^{-2/3}$ at $\overline{a}_m=(\epsilon/2\gamma)^{1/3}$, where the small tilting condition is most difficult to satisfy. This yields the threshold for FAR in our system

$$\epsilon_{th}^{n=1}=2|\gamma|^{-1/2}(A/6)^{3/4}. \qquad (6)$$

Here, we include the possibility of $\gamma<0$, in which case λ_t must be negative and $A\equiv|\lambda_t|$. We find Eq. (6) in excellent agreement with our numerical results (see the $A^{3/4}$ dependence for $n=1$ in Fig. 3). Note that these developments show that the critical amplitude $a_1\approx\overline{a}_m$ for having a continuing phase locking is of $O(\epsilon^{1/3})$, justifying the truncation of the series representation of our solution in calculating the averaged Lagrangian.

The threshold for entering the SHAR at $n=2$ can be also treated by using the averaged variational principle. One needs to choose a different lowest order representation of the solution, since we must extend the theory to $a_1\sim O(\epsilon^{2/3})$. The desired form is $x=a_0+a_1\cos\theta+a_2\cos(2\theta+\eta_2)+b\cos(\theta/2+\mu)$, while the interaction term in the Lagrangian (2) for this problem is $\epsilon x\sin(\theta/2-\Phi)$, where $\Phi\equiv(\theta-t)/2$ is the phase mismatch. Next, we average L over 4π, and arrive at the averaged Lagrangian $\mathcal{L}_2$ depending on the slow variables a_i, b, η_2, μ, and Φ:

$$\mathcal{L}_2=\mathcal{L}_0-\frac{\alpha}{4}a_1b^2\cos(2\mu)-\frac{3}{16}b^2-\frac{\epsilon b}{2}\sin(\Phi+\mu), \qquad (7)$$

where we approximate $\Lambda\approx\Omega\approx1$ in $\mathcal{L}_0$ and in the third term in the right hand side. Furthermore, one can again eliminate the slow variables $a_{0,2}$ and η_2 in $\mathcal{L}_0$, and obtain the expression, $4\mathcal{L}_0=(\Omega^2-\Lambda^2)a_1^2+\gamma a_1^4$, used already in deriving Eqs. (3) and (4). Next, we take variations with respect to Φ and a_1, yielding

$$(\Omega a_1^2)_t+\epsilon b\cos(\Phi+\mu)=0, \qquad (8)$$

$$(\Omega^2-\Lambda^2)a_1+2\gamma a_1^3-\frac{\alpha}{2}b^2\cos(2\mu)=0. \tag{9}$$

Finally, the variations with respect to μ, and b, yield $\epsilon\cos(\Phi+\mu)\approx(2\alpha b a_1/3)\sin(2\mu)$, $|b|\approx 4\epsilon/3$ and $|\Phi+\mu|\approx\pi/2$. Then, near the resonance, Eqs. (8) and (9) become

$$a_{1t}=-\frac{4\epsilon^2\alpha}{9}\sin(2\Phi), \tag{10}$$

$$(2\Phi)_t=\frac{1}{2}\lambda(t)-\gamma a_1^2-\frac{4\epsilon^2\alpha}{9a_1}\cos(2\Phi). \tag{11}$$

But this system is the same as Eqs. (5) for the FAR, with Φ replaced by $2\Phi-\pi/2$ and ϵ by $\epsilon_{eff}=8\epsilon^2\alpha/9$. This replacement yields the desired $n=2$ SHAR threshold (see the $A^{3/8}$ dependence in Fig. 3):

$$\epsilon_{th}^{n=2}=1.5|\alpha^2\gamma|^{-1/4}(A/6)^{3/8}. \tag{12}$$

Finally, we consider the $n=3$ SHAR case. We seek 6π-periodic solutions in θ, write, to desired order, $x=a_0+a_1\cos\theta+a_2\cos(2\theta+\eta_2)+c_1\cos(\theta/3+\kappa_1)+c_1\cos(2\theta/3+\kappa_2)$, and use this representation in calculating the averaged Lagrangian. Then, eliminating all the slow variables, but a_1 and $\Phi\equiv(\theta-t)/3$, one again arrives at the system of variational evolution equations having the same form as Eqs. (5), but Φ replaced by 3Φ and ϵ by $\epsilon_{eff}=(9\epsilon/8)^3(9\alpha^2/10-\beta/4)$. This similarity yields the following threshold for entering SHARat $n=3$ (see the $A^{1/4}$ dependence in Fig. 3):

$$\epsilon_{th}^{n=3}=(8/9)|9\alpha^2/20-\beta/8|^{-1/3}|\gamma|^{-1/6}(A/6)^{1/4}. \tag{13}$$

In conclusion, we have studied the adiabatic passage through subharmonic resonances in a driven nonlinear dynamical system. It was shown that, after starting in equilibrium, the system phase locks to the drive as one approaches the resonance by slowly varying the control parameter. Later, at certain conditions, the system enters the SHAR regime, where it self-adjusts its response continuously, to stay in resonance with the drive. The SHAR mechanism involves a different path to synchronization in driven systems with slow parameters via generating nth order driven response at the fundamental frequency, which, in turn, leads to phase locking and autoresonance in the system. By using a weak drive at a $1/n$ fraction of the fundamental resonant frequency, in SHAR, one reaches and controls large excitations in the system. The phenomenon has a sharp threshold on the drive's amplitude, which scales with the chirp rate of the control parameter as $\epsilon_{th}\sim A^{3/(4n)}$. While first experiments on the SHAR in pure electron plasmas demonstrate the effect for $n=2$, 3, 4, and 5 [7], the theoretical challenge remains to describe $n>3$ cases, multidimensional dynamical generalizations, and SHAR in extended systems, such as vortices and nonlinear waves.

This work was supported by the Israel Science Foundation of the Academy of Sciences and Humanities.

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