# Finite-size scaling and conformal anomaly of the Ising model in curved space

J. González
*Instituto de Estructura de la Materia, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain*
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We study the finite-size scaling of the free energy of the Ising model on lattices with the topology of the tetrahedron and the octahedron. Our construction allows us to perform changes in the length scale of the model without altering the distribution of the curvature in the space. We show that the subleading contribution to the free energy follows a logarithmic dependence, in agreement with the conformal field theory prediction. The conformal anomaly is given by the sum of the contributions computed at each of the conical singularities of the space, except when perfect order of the spins is precluded by frustration in the model.

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The introduction of conformal field theory about 15 years ago can be considered one of the most important developments towards the understanding of critical phenomena in two dimensions [1]. This subject added to the progress achieved ten years before by application of the renormalization group ideas to critical phenomena. Both the renormalization group and conformal field theory have in common the idea of scaling [2]. This plays a central role in the confrontation with experimental measurements near a critical point, as well as in numerical simulations where the critical behavior is approached by enlarging the size of the system [3].

The influence of the gravitational background on critical phenomena is largely unknown, though. This problem can be approached from the point of view of conformal field theory, which is able to deal with two-dimensional backgrounds related by conformal transformations to the plane. Although the kind of geometries one can handle in this way is restricted, we have learned about the interesting properties of conformal field theories on semiplanes [4], cylinders [5], and conical singularities [6].

In the case of a two-dimensional smooth manifold, it has been shown on general grounds that the free energy $F$ has corrections that depend directly on the central charge $c$ of the conformal field theory [2]. As a function of the length scale $L$ of the system, the free energy has to behave in the form

$$F(L)=aL^2+b\ln(L)+\ \dots, \tag{1}$$

where $b=-c\chi/12$ for a manifold with Euler characteristic $\chi$. A similar formula applies to the case of a conical singularity [6]. Logarithmic corrections to the free energy also arise associated to corners in higher-dimensional spaces [7]. Until now, though, no examples of statistical models have been considered where the logarithmic corrections due to the curvature have been tested. The question is actually nontrivial since, as we will see below, the simplest lattices that make feasible the construction of models with such scaling behavior do not give rise to smooth manifolds in the continuum limit.

In this paper we study the Ising model on lattices whose continuum limits have the topology of the tetrahedron and the octahedron. Our aim is to discern whether an expression like (1) applies, providing then a check of the conformal field theory description on the curved background. To accomplish this task we take the thermodynamic limit along a series of honeycomb lattices that are built by assembling triangular pieces like that in Fig. 1 as the facets of the given polyhedron. Our choice of this kind of lattice is determined by the feasibility of growing them up regularly while preserving the geometry of the polyhedron. From the point of view of simplistic geometry, the local curvature at each $n$-fold ring of the lattice is given by $R_i=\pi(6-n_i)/n_i$. Thus, our honeycomb lattices embedded on the tetrahedron, as well as in the octahedron, keep the same distribution of the curvature (nonvanishing only at the threefold and fourfold rings that encircle the vertices of the respective polyhedra), no matter the size of the lattice.

The critical behavior of the Ising model on the tetrahedron has been discussed in Ref. [8]. It has been shown there that the critical exponents $\alpha$, $\beta$, and $\gamma$ do not deviate in the curved geometry from the known values of the Ising model on a flat space. In the present paper we focus on the imprints that the curvature may leave in the scaling behavior of the statistical system. The ln($L$) correction to the free energy is known in the case of a conical singularity on a two-dimensional surface [6]. By measuring the ln($L$) scaling of the free energy we are then making a nontrivial check of the conformal field theory prediction, since we are dealing with geometries that are not small perturbations with respect to flat space. This may also validate our construction as an alternative procedure to the determination of the conformal anomaly in spaces with the topology of the cylinder [9–15].

We begin by analyzing the Ising model on the octahedron. Contrary to the case of the tetrahedron, where there is frus-

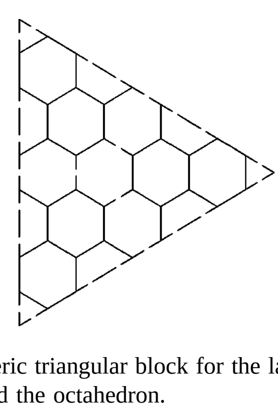

FIG. 1. A generic triangular block for the lattices embedded on the tetrahedron and the octahedron.

 

tration of the model, the present lattices are bipartite since they are built by assembling four of the pieces in Fig. 1 around each vertex. The partition function has to be then symmetric under the change of sign of the coupling constant $\beta \to -\beta$.

The thermodynamic limit has to be taken in order to approach the critical point of the model. This may pose a problem as long as we want to measure a scaling behavior like (1) that is supposed to be well defined at the critical point of the continuum model. On a lattice of finite length scale $L$, we may only define a ''pseudocritical'' coupling $\beta_L$ at which some of the observables, like, for instance, the specific heat, reaches a maximum value [3]. We will discern later what is the correct choice to extract the genuine scaling of the conformal field theory from the finite-size data.

From the technical point of view, high-precision measurements are needed in order to observe neatly a $\ln(L)$ dependence of the free energy, after subtraction of the leading contribution $\propto L^2$. The dimer approach affords such a possibility, by translating the computation of the partition function into the evaluation of the Pfaffian of an antisymmetric operator $A$ [16,17]. This is given by a coordination matrix of what is called the decorated lattice, which is obtained in our case by inserting a triangle in place of each of the points of the original lattice. A detailed example of how to build the coordination matrix for planar lattices similar to ours can be found in Ref. [8]. The partition function can be written in the form

$$Z = (\cos\beta)^l [\det(A)]^{1/2}, \tag{2}$$

where $l$ is the number of links of the original lattice. From (2) it is clear that the partition function and the free energy $F = -\ln(Z)$ can be computed with high precision on reasonably large lattices, as far as the evaluation of the corresponding determinant becomes feasible.

In the case of our lattices in curved space, the determination of the logarithmic correction to $F$ is facilitated by the fact that finite-size scaling sets in at very small lattice size. The honeycomb lattices embedded on the octahedron form a family with increasing number of sites $n = 24N^2, N = 1,2, \ldots$ . The pseudocritical couplings approach the critical coupling $\beta_\infty = \ln(2+\sqrt{3})/2 \approx 0.6585$ following the finite-size scaling law

$$|\beta_N - \beta_\infty| \sim 1/N^\lambda. \tag{3}$$

Usually $\lambda$ is related to the critical exponent $\nu$ of the correlation length $\lambda = 1/\nu$. One can check, however, that in the case of the octahedron $\lambda$ is sensibly higher than the expected value $\nu^{-1} = 1$. The values of $\beta_N$, which we have computed by looking at the maxima of the specific heat for $N = 1$ up to $N = 7$ (1176 lattice sites), are given in Table I. By carrying out a sequence of fits, taking four consecutive lattices for each of them, we obtain the respective estimates of the exponent $\lambda_{\text{octa}}$, in order of increasing lattice size: 1.825, 1.809, 1.798, 1.794.

We present in Fig. 2 a logarithmic plot of the values of $\beta_N - \beta_\infty$ vs $N$ and the linear fit for the last four points. It is remarkable the small deviation of the points from the law (3), even for the smaller lattices, which ensures that the estimates for $\lambda_{\text{octa}}$ are converging to a value different to that

TABLE I. Respective values of the pseudocritical couplings $\beta_N$ and the free energy $F_N(\beta_\infty)$ for the octahedron.

| $N$ | $\beta_N$ | $F_N(\beta_\infty)$ |
|---|---|---|
| 1 | 0.557109(1) | −8.69737867937620(1) |
| 2 | 0.629927(1) | −32.7353289357422(1) |
| 3 | 0.644793(1) | −72.6487548788477(1) |
| 4 | 0.650325(1) | −128.469770730314(1) |
| 5 | 0.653016(1) | −200.209189542789(1) |
| 6 | 0.654540(1) | −287.871899726545(1) |
| 7 | 0.655491(1) | −391.460522363036(1) |

expected in flat space. The exponent for the octahedron is very close to the exponent obtained in the case of the tetrahedron, for a wider range of lattice sizes, $\lambda_{\text{tetra}} \approx 1.745(2)$ [8]. We recall that these estimates do not point at a critical exponent of the correlation length different from $\nu = 1$, but rather at a violation of the Ferdinand-Fisher criterion for the determination of $\nu$ in the curved spaces.

We have computed the free energy $F_N$ for the members $N = 1$ up to $N = 7$ of the series of honeycomb lattices embedded on the octahedron. The values are given in Table I. We have observed a clear $\ln(N)$ correction to the leading behavior $\propto N^2$ of the free energy as a function of the lattice size, when the measurements are carried out at the critical coupling $\beta_\infty$. The task is facilitated by taking into account the precise value of the bulk free energy per site in the honeycomb lattice, $a/24 \approx -0.331\,912$ [18]. By computing at coupling constant $\beta = \beta_\infty$ and making a sequence of fits for sets of four consecutive lattices, we obtain the respective values of the $b$ coefficient in (1), in order of increasing lattice size: $-0.204\,86$, $-0.207\,63$, $-0.208\,07$, $-0.208\,20$. We observe a clear convergence towards a value $b \approx -0.208$. We have plotted in Fig. 3 the values of $F_N - aN^2$ and the best fit for the last four points in the plot. The sum of the squares of the deviations from the logarithmic dependence (for $N = 2$ up to

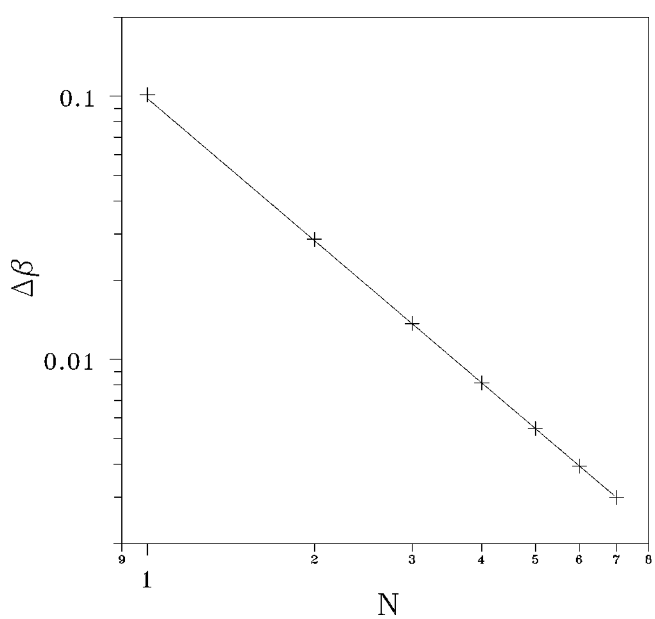


FIG. 2. Deviation of the pseudocritical couplings from $\beta_\infty$ vs the length scale $N$.

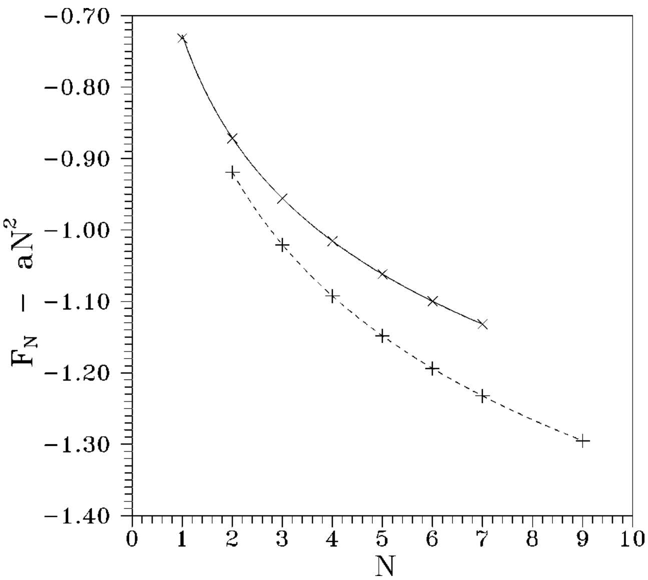


FIG. 3. Plot of the finite-size correction to the free energy of the Ising model at the ferromagnetic critical point, in lattices on the octahedron (points over the full line) and on the tetrahedron (points over the dashed line).

$N=7$) is $\approx 2.7\times 10^{-7}$. The accuracy of the fit is remarkable, given that it is achieved by adjusting only the coefficient $b$ and the constant term in Eq. (1).

The above results show that the hypothesis of finite-size scaling may be applied to the free energy to determine the conformal anomaly on a curved background. Let us now interpret the coefficient $b$ of the anomaly that we have obtained for the octahedron. We assume that the logarithmic correction can be computed as the sum of the corrections for each of the conical singularities $b=\Sigma_{i=1}^{6} b_i$. The coefficient $b_i$ for a conical singularity has been established in Ref. [6] in terms of the central charge $c$ and the angle $\theta$ enclosed by the cone:

$$b_i = c\,\frac{\theta}{24\pi}[1-(2\pi/\theta)^2]. \qquad (4)$$

This formula leads in the case of the octahedron to $b=-5c/12\approx -0.41666c$, which for $c=1/2$ corresponding to the Ising model is in very good agreement with our numerical result. This provides a clear indication that the continuum limit of the Ising model on the octahedron is given by a conformal field theory, with the same central charge as for the model in flat space.

We move now to the family of honeycomb lattices embedded on the tetrahedron. We may distinguish between the ferromagnetic and the antiferromagnetic regime, since the lattices are frustrated in this case. The finite-size scaling is actually different in the two regimes. The number of lattice sites is given now by the formula $n=12N^2$, where $N$ is the integer that labels the member in the family. At the ferromagnetic critical coupling $\beta_\infty > 0$, we have measured the free energy with the same precision as before, for $N=2$ up to $N=9$. The values that we have obtained are given in Table II. The accuracy of the fit to a $\ln(N)$ correction added to the leading behavior is again remarkable. By making a sequence

TABLE II. Respective values of the free energy $F_N$ at the critical couplings of the ferromagnetic and the antiferromagnetic regime of the tetrahedron.

| $N$ | $F_N(\beta_\infty)$ | $F_N(-\beta_\infty)$ |
|---|---|---|
| 2 | $-16.8509982571185(1)$ | $-14.2539387796425(1)$ |
| 3 | $-36.8670648949387(1)$ | $-33.8649750683343(1)$ |
| 4 | $-64.8195835913494(1)$ | $-61.5298831135922(1)$ |
| 5 | $-100.721855606886(1)$ | $-97.2090306933016(1)$ |
| 6 | $-144.579808605568(1)$ | $-140.884668912476(1)$ |
| 7 | $-196.396605031357(1)$ | $-192.547317539973(1)$ |
| 9 | $-323.913609253478(1)$ | $-319.813027662780(1)$ |

of fits, each of them for four consecutive lattices, we obtain the respective estimates for the $b$ coefficient, in order of increasing lattice size: $-0.249\,980\,1$, $-0.249\,994\,5$, $-0.249\,997\,8$, $-0.249\,999\,2$. The plot of the function $F_N - aN^2$ is given in Fig. 3, together with the logarithmic dependence from the last fit.

The result that we obtain for the coefficient $b$ is again in very good agreement with the value expected for a conformal field theory. The outcome of adding the effect of four conical singularities with enclosed angle $\theta=\pi$ yields the prediction $b=-c/2$. Therefore, we may conclude that the critical point in the ferromagnetic regime provides an example of a $c=1/2$ conformal field theory on the curved background.

The finite-size scaling works differently in the antiferromagnetic regime. The values of the free energy computed at $\beta=-\beta_\infty$ are given in Table II. The accuracy of the fits to determine the $\ln(N)$ correction is as good as in the former instances, but now the $b$ coefficient turns out to be positive. By carrying out the same sequence of fits as in the ferromagnetic regime, we find the convergent series for the estimates of $b$ : 0.749 44, 0.749 86, 0.749 95, 0.749 96. We have plot-

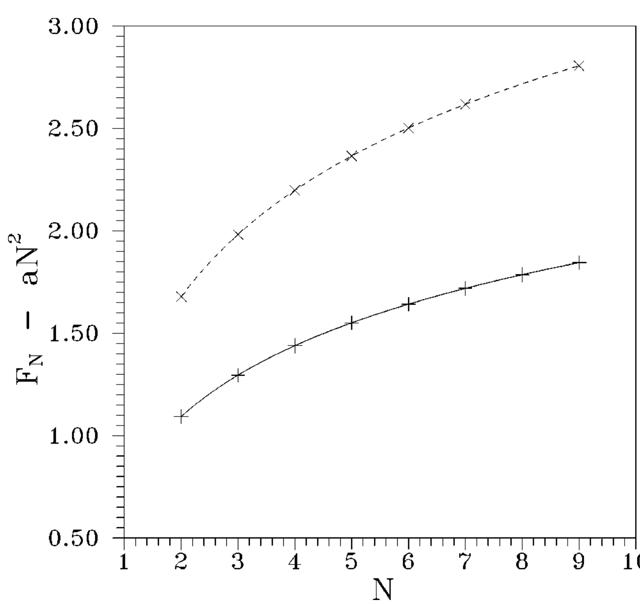


FIG. 4. Plot of the finite-size correction to the free energy of a free scalar field (points over the full line) and of the Ising model at the antiferromagnetic critical point (points over the dashed line), in lattices embedded on the tetrahedron.

ted in Fig. 4 the values of $F_N - aN^2$ and the logarithmic fit for the last four points. The sum of the squares of the deviations from the ln(*N*) dependence ($N=2$ to $N=9$) is $\approx 2.2 \times 10^{-7}$.

We can learn the correct interpretation of these results from a similar feature of the finite-size data of the free scalar field on the curved lattice. This can be described by a simple tight-binding model, whose spectrum reproduces that of the Laplacian on the lattice [19]. The partition function is computed through the determinant of the coordination matrix, but the zero mode has to be removed in order to obtain a nonsingular result. As a consequence of that, the coefficient of the ln(*N*) correction (fitted to data from $N=2$ to $N=9$ as shown in Fig. 4) turns out to be $\approx 0.49999$. The correct result in front of the logarithmic correction is obtained by adding the regularized contribution of the zero mode, which scales like $(1/2)\ln(L^{-2})$ after introducing the length dimensions of the Laplacian in the lattice.

The same effect operates in the antiferromagnetic regime of the lattices on the tetrahedron. These cannot be decomposed in two disjoint sublattices, so that there is an intrinsic frustration that rules out perfect antiferromagnetic order, irrespective of lattice size. The zero mode is missing in the spectrum, and the correct conformal anomaly of the $c=1/2$ conformal field theory is restablished adding ''by hand'' to the free energy the regularized zero mode contribution $\ln(L^{-1})$.

To summarize, we have checked that the continuum limit of the Ising model taken along lattices embedded on the tetrahedron and the octahedron corresponds to respective $c=1/2$ conformal field theories. We have seen that the convergence to the critical coupling is sensibly accelerated with respect to a flat geometry when performing the finite-size scaling in the curved lattices. Our construction may be useful to determine the central charge corresponding to other models whose underlying conformal field theory is not known.


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