Plasma oscillations and nonextensive statistics

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The dispersion relations for electrostatic plane-wave propagation in a collisionless thermal plasma are discussed in the context of the nonextensive statistics proposed by Tsallis. Analytic formulas both for the undamped (Bohm-Gross) and Landau damped waves are derived and compared with the standard results. In the extensive limiting case $(q=1)$, the classical dispersion relations based on the Maxwellian distribution are recovered. It is shown that the experimental results points to a class of Tsallis's velocity distribution described by a nonextensive q-parameter smaller than unity.

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Some years ago, heuristic arguments based on multifractals concepts inspired Tsallis to propose a generalization of the Gibbs-Jaynes-Shannon (GJS) entropy formula for statistical equilibrium [1–5]. Tsallis's nonextensive q-entropy is defined by the following expression:

$$S_q = k_B \frac{[1-\Sigma_i p_i^q]}{(q-1)}, \tag{1}$$

where k_B is the standard Boltzmann constant, p_i is the probability of the ith microstate, and q is a parameter quantifying the degree of nonextensivity. In the limit $q=1$ the celebrated GJS extensive formula, namely,

$$S = -k_B \sum_i p_i \ln p_i , \tag{2}$$

is recovered. One the most relevant properties of Tsallis's nonextensive entropy is its pseudoadditivity. Given a composite system $A+B$, constituted by two subsystems A and B, which are independent in the sense of factorizability of the microstate probabilities, the Tsallis measure verifies $S_q(A+B)=S_q(A)+S_q(B)+(1-q)S_q(A)S_q(B)$. In the limit $q\rightarrow 1$, S_q reduces to the standard logarithmic measure, and the usual additivity of the extensive statistical mechanics and thermodynamics is recovered. In other words, $|q-1|$ is a measure of the lack of extensivity of the system.

Several consequences of this starting basic assumption have already been investigated in the literature [1–14]. In this general framework, the nonextensive canonical ensemble, associated with the classical many body system, depends on the generalized velocities distribution function. The equilibrium velocity q-distribution may be obtained either through an adequate variational principle, that is, maximizing S_q under the normalization and kinetic mean energy constraints [1], or from a generalized version of the kinetic Boltzmann's H-theorem [7]. As a matter of fact, even Maxwell's first derivation of the equilibrium velocity distribution function for a dilute gas has consistently been generalized in the spirit of Tsallis's entropy [8]. In principle, the q-nonextensive formalism may be very important for systems endowed with long range interactions as usually happens in astrophysics and plasma physics. In this concern, the equilibrium velocity distribution has been successfully applied to stellar polytropes [9], two-dimensional Euler and drift turbulence in a pure-electron plasma column [10], as well as to the peculiar velocity function of galaxies clusters [11]. In particular, Liu *et al.* [12] showed a reasonable indication for the non-Maxwellian velocity distribution from plasma experiments. All these empirical evidences deal, directly or indirectly, with the q-distribution of velocities for a massive nonrelativistic gas. Even for massless particles (photons) some analyses have recently appeared in the literature. Restrictive bounds on the q-parameter were derived using data from the anisotropy of the cosmic background radiation [13], as well as from primordial nucleossintheses studies, using the present observed abundances of the light elements [14].

In the present work, we discuss a different application in the field of plasma physics. Our goal is to investigate the propagation of electrostatic waves in a collisionless, and magnetic-field-free thermal plasma in the q-nonextensive context. In principle, to check the validity of a theory (the standard dispersion relations for plasma oscillations), it is interesting to insert it in a more general framework, herein quantified by the fact that the q-parameter may differ from unity. As we shall see, analytic expressions for the dispersion relations in a collisionless plasma may rigorously be derived through the q-nonextensive velocity distribution function [8], thereby obtaining the q-generalized formulas both for undamped (Bohm-Gross) and Landau damped waves.

As is widely known, high frequency vibrations in a collisionless electronic plasma may be described in a highly simplified manner, where collisions of the electrons with the ions and with each other are unimportant, in such a way that the collisional integral term in the Boltzmann kinetic equation may be neglected [16,17]. In the first order of approximation, the distribution function of electrons is modified by a perturbative signal, while the distribution function of ions can be considered as an invariable quantity. Let $f(\vec{v},\vec{r},t)$ be the resulting electronic distribution function and $f_0(v)$ the corresponding q-nonextensive equilibrium unperturbed dis-

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tribution. If the plasma departs slightly from equilibrium, the electron distribution function may be approximated as

$$f=f_0(v)+f_1(\vec{v},\vec{r},t), \quad f_1 \ll f_0, \tag{3}$$

where f_1 is the corresponding perturbation in the distribution function. The oscillatory behavior of the plasma is governed by the Poisson and Vlasov's collisionless transport equations. Neglecting second-order terms in the expansion of the distribution function one finds

$$\frac{\partial f_1}{\partial t}+\vec{v}\cdot\frac{\partial f_1}{\partial \vec{r}}+\frac{e}{m}\nabla\phi\cdot\frac{\partial f_0}{\partial \vec{v}}=0, \tag{4}$$

$$\nabla^2\phi=-4\pi e\int f_1 d^3v, \tag{5}$$

where *e*, *m*, and ϕ denote, respectively, the electronic charge, electron mass, and electric field potential. In the *q*-nonextensive framework, the one-dimensional equilibrium distribution function, $f_0(v_x)$, is given by [7,8]

$$f_0(v_x)=A_q\left[1-(q-1)\frac{mv_x^2}{2k_BT}\right]^{1/(q-1)}, \tag{6}$$

where the normalization constant reads

$$A_q=\frac{n\Gamma\left(\frac{1}{1-q}\right)}{\Gamma\left(\frac{1}{1-q}-\frac{1}{2}\right)}\sqrt{\frac{m(1-q)}{2\pi k_BT}}, \quad \text{for } -1<q\leqslant 1 \tag{7}$$

and

$$A_q=n\left(\frac{1+q}{2}\right)\frac{\Gamma\left(\frac{1}{2}+\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}\right)}\sqrt{\frac{m(q-1)}{2\pi k_BT}}, \quad \text{for } q\geqslant 1, \tag{8}$$

where *n* is the number of electrons per unit volume of the plasma, k_B is the Boltzmann constant, and *T* is the temperature. As one may check, for $q<-1$, the *q*-distribution is unnormalizable. For $q>1$, the distribution function (6) exhibits a thermal cutoff on the maximum value allowed for the velocity of the particles, which is given by

$$v_{\max}=\sqrt{2k_BT/m(q-1)}. \tag{9}$$

We see that in the limit $q\to 1$, $v_{\max}$ goes to infinity and Eq. (8) reduces to $A_1=n\sqrt{m/2\pi k_BT}$, which is the standard one-dimensional Maxwell-Boltzmann normalization constant [8]. This thermal cutoff is absent when $q<1$, that is, $v_{\max}$ is also unbounded for these values of the *q*-parameter. It is also worth mentioning that the spirit of the *H*-theorem is preserved for this nonextensive velocity distribution. If one considers a generalized collisional term, $C_q(f)$, it is possible to show that the entropy source is positive, and does not vanish unless the *q*-distribution function assumes the above equilibrium form [7].

The set of coupled equations (4) and (5) may be worked either by the simplified derivation for electrostatic waves (longitudinal plasma waves), where the specific dispersion relation is derived by taking the constraint of null permittivity [15], or by using the more technical method of integral transform developed by Landau [16,17].

Let us consider the *x*-axis along the direction of the wave vector $\vec{k}$, with $v_x=u$. The dispersion relation is derived as a pole of the potential, being p_k the poles of ϕ, i.e., the roots of the equation

$$\frac{4\pi ie^2}{km}\int_L\frac{df_0}{du}\frac{du}{(p+iku)}=1, \tag{10}$$

where *L* stands for the Landau contour, and *p* is a complex variable appearing in the argument of the (time-dependent) Laplace transform.

We now consider the extreme case of long wavelengths ($\lambda\gg\lambda_D$), or equivalently, the limit of small wave numbers ($k\ll k_D$), where the subindex stands for Debye quantities. At this limit, the point $u=ip/k$ has very large absolute value, and since the generalized equilibrium function $f_0(u)$ decreases with increasing $|u|$, we can integrate Eq. (10) along the real axis. Thus, expanding the integrand in powers of *k*, one obtains

$$\frac{4\pi ie^2}{pkm}\int_{-v_{\max}}^{v_{\max}}\frac{df_0}{du}du\left[1-\frac{iku}{p}+\left(\frac{iku}{p}\right)^2-\left(\frac{iku}{p}\right)^3+\cdots\right]. \tag{11}$$

We call attention to the *q*-dependence on the limits of integration due to the *q*-dependent thermal cutoff inherent to the distribution function $f_0(u)$ for $q>1$. Naturally, as discussed before, the integration limits are $\pm\infty$ when the *q*-parameter is smaller than unity.

The first term appearing in the expansion of the above integral, namely,

$$\int_{-v_{\max}}^{v_{\max}}\frac{df_0}{du}du, \tag{12}$$

is identically zero regardless of the values assumed by the *q* parameter. This happens because the *q*-distribution is an even function of its argument. The next term in the expansion is the number of electrons per unit volume of the plasma

$$\int_{-v_{\max}}^{v_{\max}}u\frac{df_0}{du}du=uf_0\Big|_{-v_{\max}}^{v_{\max}}-\int_{-v_{\max}}^{v_{\max}}f_0\,du=-n. \tag{13}$$

Therefore, in this order of approximation, one finds

$$p_k=-i\omega, \quad \omega=\sqrt{\frac{4\pi ne^2}{m}}=\omega_0. \tag{14}$$

The disturbance corresponds to a plane wave propagating in the positive direction of the *x*-axis with natural oscillation plasma frequency, $\omega=\omega_0$. The next order yields the dispersion relation including the thermal correction from the *q*-nonextensive statistics

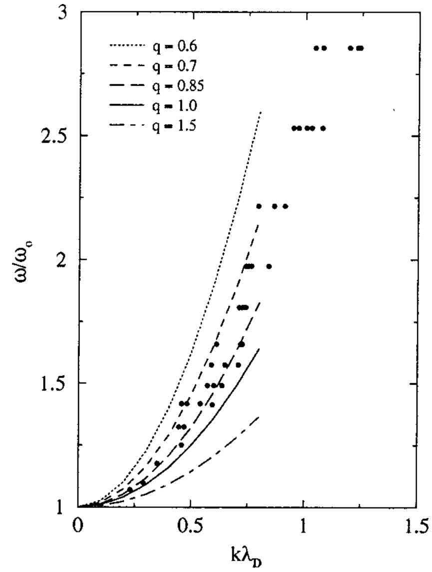

FIG. 1. Thermal dispersion relations for Tsallis velocity distribution. The selected values of the q-nonextensive parameter are shown in the picture. The data points are taken from Van Hoven [19]. The solid line is the extensive Bohm-Gross result based on the Maxwellian distribution ($q=1$). For $q>1$ the curves increase less rapidly than in the Maxwellian case, and clearly depart from the experimental results. We see that a nonextensive distribution with $q<1$ is strongly suggested by these results.

$$\omega^2=\omega_0^2+3(\lambda_D k)^2\omega_0^2\left(\frac{2}{3q-1}\right), \tag{15}$$

where $\lambda_D=\sqrt{k_BT/4\pi ne^2}$ is the electronic Debye-Hückel radius, and Eq. (15) has been calculated using the standard definition for the average value of u^2, that is,

$$\langle u^2\rangle=\int u^2 f_0\,du=\frac{2n}{3q-1}\frac{k_BT}{m}, \tag{16}$$

for all values of $q>1/3$. As expected, for $q=1$, Eq. (15) reduces to

$$\omega^2=\omega_0^2+3(\lambda_D k)^2\omega_0^2, \tag{17}$$

which is the standard Bohm-Gross relation [17]. In Fig. 1 we have plotted, for some selected values of the q, the ratio ω/ω_0 as a function of the dimensionless parameter $k\lambda_D$. Note that the standard Bohm-Gross relation ($q=1$) is only marginally compatible with the existing data. We see that a better fit is provided by a q-distribution with $q<1$.

On the other hand, as originally discovered by Landau, the vibrations are damped, and the damping coefficient is small for small wave numbers k. This means that in the limit $k\to 0$, the real part of p_k also goes to zero, with the imaginary part remaining finite (Landau's rule [16]). Following standard lines, we define

$$p_k=-i\omega-\gamma_q, \tag{18}$$

where γ_q is the extended damping coefficient ($0<\gamma_q\ll\omega$). In this approximation Eq. (10) can be put in the form

$$-\frac{4\pi ne^2}{mp^2}+i\frac{4\pi^2e^2}{mk^2}\frac{df_0(-p/ik)}{du}=1, \tag{19}$$

Inserting $p_k=-i\omega-\gamma_q$ in Eq. (19), and solving by means of successive approximations, we obtain the following generalization for the q-damping decrement

$$\gamma_q=\omega_0\sqrt{\frac{\pi}{8}}L_q\frac{1}{(k\lambda_D)^3}\times\left[1-(q-1)\left(-\frac{1}{2(k\lambda_D)^2}\right)\right]^{(2-q)/(q-1)}, \tag{20}$$

where

$$L_q=\frac{\Gamma\left(\frac{1}{1-q}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}\sqrt{1-q},\quad \text{for}\quad q\le 1 \tag{21}$$

and

$$L_q=\left(\frac{1+q}{2}\right)\frac{\Gamma\left(\frac{1}{2}+\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}\right)}\sqrt{q-1},\quad \text{for } q\ge 1. \tag{22}$$

Therefore, instead of an exponential decaying, the generalized q-damping decrement diminishes as a power law for decreasing values of k. In particular, taking the limit $q\to 1$, and using that $\lim_{|z|\to\infty} z^{-a}[\Gamma(a+z)/\Gamma(z)]=1$ (see [18]), we obtain the Landau expression for the damping decrement,

$$\gamma_1=\omega_0\sqrt{\frac{\pi}{8}}\frac{1}{(k\lambda_D)^3}e^{-1/2(k\lambda_D)^2}. \tag{23}$$

We recall that formulas (20)–(23) are valid for $\gamma\ll\omega$, which leads to $k\lambda_D\gg 1$. In Fig. 2 we show a plot of the damped q-dispersion relation (at the limit of long wavelengths) for some selected values of the nonextensive parameter.

Now we consider the inverse limiting case of short wavelengths ($\lambda\ll\lambda_D$), or equivalently the limit of long wave numbers ($k\gg k_D$). In this case, it is readily seen that the frequency and the damping decrement are given by

$$\omega=\sqrt{\frac{k_BT}{m}}\frac{k}{\xi}\frac{1}{(q-1)}\tan\left[\pi\frac{(q-1)}{(2-q)}\right], \tag{24}$$

$$\gamma=\sqrt{\frac{k_BT}{m}}k\xi, \tag{25}$$

with $\xi=\gamma/\omega_0 k\lambda_D$. This result means that in this limit the damping decrement of the vibrations is independent of the nonextensive parameter.

In conclusion, we stress that due to the long range of the Coulombian interaction, the standard Maxwell-Boltzmann distribution may provide only a very crude description in plasma physics, even in the collisionless limit. In the nonextensive formalism proposed by Tsallis, the longitudinal dispersion relations for a collisionless and magnetic-free-field

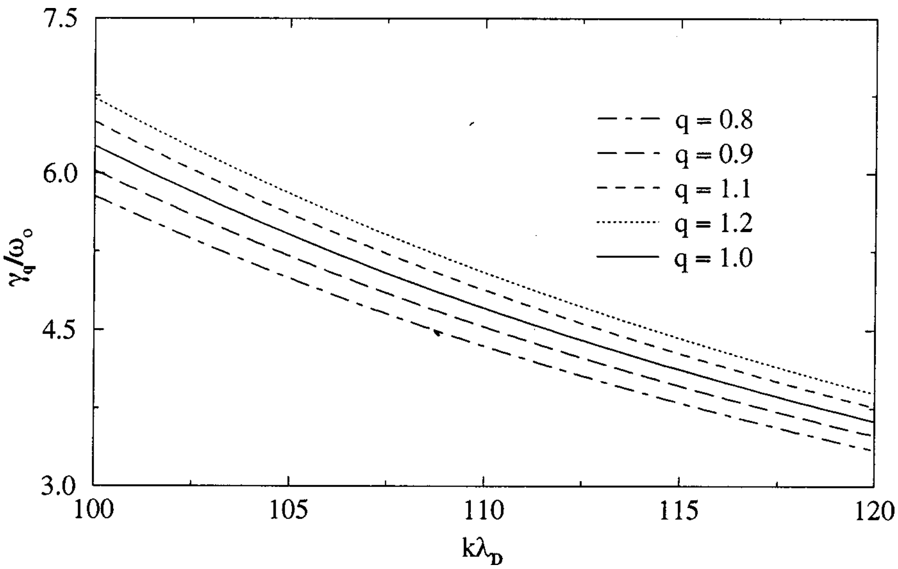

FIG. 2. Dispersion relations for damped electrostatic vibrations. The selected values of the q-nonextensive parameter are shown in the picture. The solid line is the standard Landau-damping result based on the Maxwellian equilibrium distribution ($q=1$). For a fixed value of the dimensionless parameter $k\lambda_D$ the nonextensive curves are slightly shifted relatively to the extensive case [see Eq. (20)].

electronic plasma, are significantly modified [see Eqs. (15) and (20)]. These extended wk relations may experimentally be verified using the standard technics designed to measure electrostatic wave excitations and detection [19–21]. As we have seen, the existing data provide a strong evidence in favor of a generalized Bohm-Gross nonextensive relation where the q-parameter is smaller than unity (see Fig. 1). We argue here that such experiments done more than 30 years ago, and whose original objectives were basically to demonstrate the reality of the standard Landau damping, should carefully be repeated as a crucial test for the validity of the standard Maxwellian distribution in collisionless plasmas. Naturally, independent probes of the q-nonextensive statistics, including another physical effect based on Vlasov equation applied to collisionless plasmas or to a system of particles interacting gravitationally take on even greater interest. A more detailed account for vibrations in this enlarged framework will be presented elsewhere.

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