# Turbulence of nonlocally coupled oscillators in the Benjamin-Feir stable regime

Dorjsuren Battogtokh and Yoshiki Kuramoto
*Department of Physics, Graduate School of Sciences, Kyoto University, Kyoto 606-8502, Japan*
(Received 19 March 1999)

The small-amplitude equation appropriate for self-oscillatory fields with nonlocal coupling is studied numerically in the Benjamin-Feir stable regime. Depending on the system size, two characteristic forms of turbulent behavior are observed. For relatively small system size $L$, the whole space splits into two domains with distinct dynamics, while for larger $L$ the turbulent fluctuations become better characterized by spatial intermittency with power-law scaling.

PACS number(s): 05.45.−a, 47.54.+r, 82.20.Fd

The dynamics of coupled limit-cycle oscillators is relevant to a wide variety of fields including physics, chemistry, biology, and brain sciences, thus representing itself as a subject of central importance in nonlinear dynamics far from equilibrium. Large ensembles of oscillators in fact exhibit various forms of collective dynamics, and such behavior depends crucially on how the oscillators are mutually coupled. In particular, it has recently been realized that the change in the range of coupling has some drastic effects on the system dynamics. The coupling range may formally be classified into three types, i.e., local, global, and intermediate. For simplicity, the last case will be called *nonlocal*. Over the last few decades, a large amount of work has been devoted to the first two types of coupling. Locally coupled limit-cycle oscillators have predominantly been studied through the complex Ginzburg-Landau equation (CGLE) [1,2], while most studies on globally coupled oscillators have used the phase oscillator model [1,3] with some exceptions [4]. Some people also studied the case where both local and global couplings coexist [5]. In contrast, studies on nonlocally coupled oscillators started only recently [6–8], and there remain a lot of questions yet to be answered.

Recent studies on nonlocally coupled oscillators [6–8] revealed that the system can display a type of turbulence bearing a strong resemblance to fully developed Navier-Stokes turbulence [9]. More recently, it was found that the same type of turbulence can also arise in a certain class of reaction-diffusion systems where the system reduces in practice to a nonlocally coupled system with reduced degrees of freedom [10]. The type of turbulent behavior thus discovered was interpreted in terms of an interplay between the nonlocality of the coupling and strong Benjamin-Feir (BF) instability (i.e., instability of the uniform oscillation).

In this report, we present some results of our numerical study on the nonlocally coupled complex Ginzburg-Landau equation, but in the BF *stable* regime, contrary to the foregoing works which concentrated on the BF unstable regime. Our numerical results suggest that in the BF stable regime nonlocal coupling can cause a nonlinear instability that is stronger where the spatial variation of the amplitude disturbances is stronger. Specifically, the instability occurs around minima of the amplitude profile, and this can initiate localized high-frequency oscillation causing repeated formation of holelike objects. This mechanism, combined with the nonlocality of the coupling, can give rise to two different types of turbulence depending on the system size. For smaller system size, the whole space splits into domains with distinct dynamics, while for larger system size, such splitting is not seen, but the system comes to exhibit spatial intermittency characterized by power-law scaling.

In previous papers [6], it was argued that a complex Ginzburg-Landau type equation with nonlocal coupling naturally arises in large assemblies of oscillatory elements with indirect coupling mediated by a diffusive scalar field. We start with this type of equation appropriate for a one-dimensional regular array of $N$ oscillators with spacing $\delta$:

$$\frac{\partial W_j}{\partial t'} = W_j - (1+ic_2)|W_j|^2 W_j + k(1+ic_1) \times \sum_{j'=1}^{N} \sigma(j-j')(W_{j'} - W_j). \tag{1}$$

Here $W_j$ is a complex variable, $\sigma$ is a coupling function, and $k$, $c_1$, and $c_2$ are real parameters. $N$ is supposed to be sufficiently large. A systematic derivation of the space-continuous limit of Eq. (1) can be found in [1]. We assume periodic boundary conditions and work with an exponential coupling function, i.e., $\sigma(j-j') = C\exp[-\gamma(|j-j'|)\delta]$, where $C$ is the normalization constant ensuring $\Sigma_{j=1}^{N}\sigma(j) = 1$, and $\gamma^{-1}$ gives the coupling range. As shown in [6], the last term in Eq. (1) reduces to the usual diffusion coupling when $W$ is sufficiently long-waved that the local-coupling approximation is valid. In addition to the coupling strength $k$, we have the important parameter the reduced system size $L \equiv \gamma N \delta$; $\delta$ is irrelevant because it simply scales the length. Numerical integration of Eq. (1) was carried out with use of the fourth-order Runge-Kutta method with the elementary time step of 0.05. The parameters are fixed as $c_1 = -0.3$ and $c_2 = 3.0$, which satisfies the condition for BF stability $1 + c_1 c_2 > 0$.

Under the conditions specified above, the system behavior obtained numerically from Eq. (1) turned out nontrivial. When the perturbation given initially to the uniform oscillation is not too small, the system goes directly into a state of spatio-temporal intermittency, which is also the case for the standard (i.e., locally coupled) CGLE [11]. Still, the nature of the spatio-temporal intermittency in the present case seems rather unusual. This is illustrated in Fig. 1, where a

 

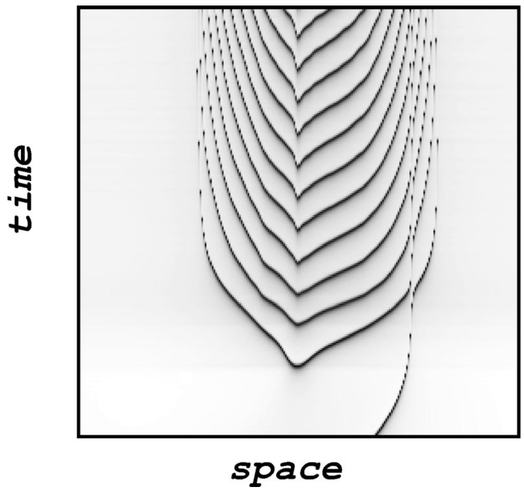


FIG. 1. Space-time pattern of the amplitude $|W|$ of 256 oscillators with time span $T=50$, where time advances upward. The values of $|W|$ are indicated in gray scale, where the darker (lighter) regions correspond to larger (smaller) $|W|$. Parameter values are $L=5$, $k=0.3$.

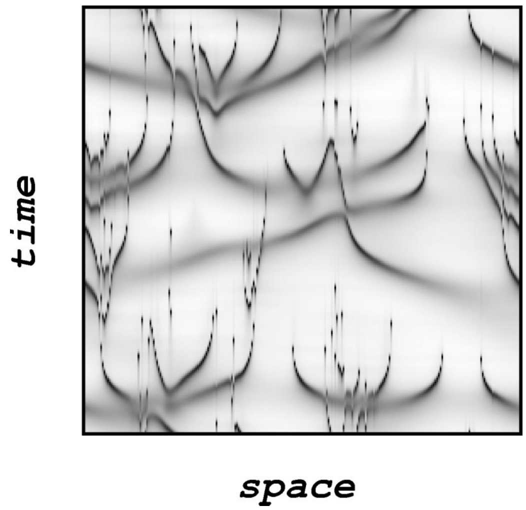


FIG. 2. Space-time pattern of the amplitude $|W|$ of 256 oscillators with time span $T=50$. $L=8$, $k=0.45$.

typical space-time pattern of $|W|$ for the system size $L=5$ is shown in gray scale. It is seen that a pair of low-amplitude (dark) holelike objects emanate periodically from the central part of the system. Across each hole, the phase jumps by $2\pi$. Our interpretation for such behavior is the following. As a result of a nonlinear instability, an amplitude dip is formed around the middle of the system. Since the oscillators with smaller amplitude oscillate faster, this amplitude dip plays the role of a high-frequency pacemaker. The pacemaker then starts to entrain the surrounding medium through periodic production of a pair of holelike objects which propagate over a certain distance but decelerate and finally vanish. In this way, the system is divided into two domains with distinct dynamics. The inner domain is characterized by successive production and propagation of holelike objects each accompanied by a $2\pi$ phase slip. Note that this is similar to what occurs in excitable media with a source of pulse production. In contrast, the outer domain, which stays unentrained by the inner domain, is characterized by large-amplitude slow oscillations with relatively smooth spatial variation. We also found that for larger $k$ multiple wave sources can appear, and in that case the dynamics in the inner domain becomes a little more complicated.

We confirmed that similar behavior to the above remains practically unchanged when we work with a nonlocally coupled phase equation as a reduction of Eq. (1) in the limit of weak coupling. Such robustness of the peculiar dynamics described above suggests that its principal cause could be the nonlocality of the coupling combined with the localized nonlinear instability.

When the scaled system size $L$ becomes larger, a number of wave sources appear and their locations are no longer fixed. Figure 2 shows a typical space-time image of $|W|$ for such a situation. Unlike the case of small system size, no division of the system into different space-time patterns can be seen, implying a recovery of spatial translational symmetry in a statistical sense. Our numerical results show that the system then comes to exhihit power-law scaling in various moments of the amplitude increments $y(x)=|W(x_0+x)-W(x_0)|$. Power-law behavior is clearly observed in the parameter range of $0.45\leqslant k\leqslant 0.6$. We introduce the spatial correlation $G(x)=\langle \overline{W}(0)W(x)\rangle$. Figure 3 shows the difference $G(0)-G(x)$ vs $x$, which is equivalent to the second moment of $y$ as a function of $x$, obtained numerically for $L=16$, i.e., a case giving a good fit to the power law. The corresponding exponent is nontrivial and estimated to be 0.59. We then expect that in nonlocally coupled oscillators power-law behavior in the moments of the amplitude increments is not limited to the BF unstable regime. It should be noted, however, that the intermittent nature of the pattern, which is responsible for the power-law behavior, seems even stronger in the BF stable regime than in the BF unstable regime. Such a difference in the nature of the pattern is reflected in the difference in the probability distribution $P(y)$ in the two regimes, which is shown in Fig. 4. The saturation of $P$ at small $y$ values is less complete in the BF stable case. This seems to come from the fact that stronger spatial intermittency, which implies persistence of almost constant amplitude over large domains in space, makes smaller amplitude increments more probable, practically down to the zero value. In contrast, in the BF unstable regime, fluctuations of various wavelengths coexist, forming a background of occa-

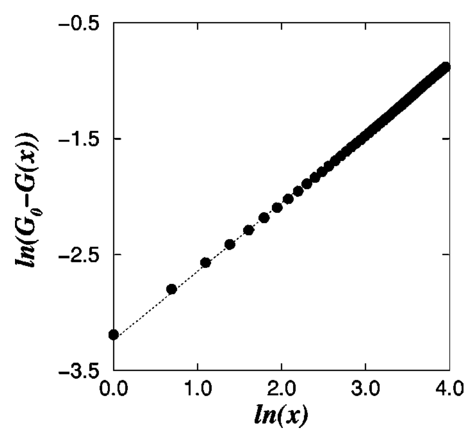


FIG. 3. Ln-ln plot of the correlation gap $G(0)-G(x)$ for $L=16$, $k=0.45$, $N=512$.

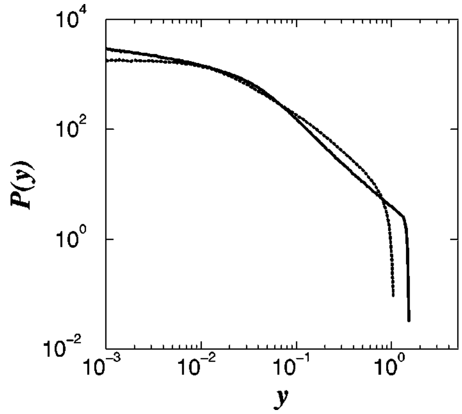


FIG. 4. Log-log plot of probability distributions $P(y)$. Solid line corresponds to $k=0.45$, $c_1=-0.3$, $c_2=3$. Dotted line corresponds to $k=0.85$, $c_1=-2$, $c_2=2$. $L=16$ and $N=512$.

sional sharp changes in the amplitude, so that the probability of occurrence of small values of amplitude increments is more or less the same as long as they are below the level of such ''normal'' fluctuations. In the large $y$ regime, a sudden drop of $P$ occurs for the BF unstable case around $y\sim 1$, while $P$ persists for the BF stable case up to $y\sim 1.5$. Such persistence of large $y$ is possibly associated with the formation of holelike objects with abrupt amplitude change.

Another feature of the power-law behavior characteristic of the BF stable regime is that the dependence of the scaling exponent on the coupling strength $k$ is much weaker than in the BF unstable regime. The reason is not clear, but this might have something to do with the fact that in the BF unstable regime the seeming randomness of the pattern becomes stronger with decreasing $k$, which is natural because of the stronger BF instability, while in the BF stable regime the degree of randomness seems insensitive to $k$ once the nonlinear instability occurs. For intermediate values of $y$, $P$ obeys a power law $P(y)\sim y^{-(1+\beta)}$ in both the BF stable and unstable cases, and its origin seems to be common. For the particular parameter values assumed, $\beta$ is estimated to be 0.58 and 0.45 in the BF stable and unstable cases, respectively.

In summary, for the sake of deeper understanding of the effects of nonlocal coupling, we studied numerically the nonlocally coupled complex Ginzburg-Landau equation in the BF stable regime. For relatively small system size, the whole space was found to split into inner and outer domains with distinct dynamics, the inner domain being characterized by generation of holelike objects. We gave a qualitative description of how such behavior changes with the system size and is connected smoothly to the power-law regime of turbulent fluctuations. Some unique features of the power-law behavior discovered in the BF stable regime were pointed out, in comparison with the similar behavior already known for the BF unstable regime.

D.B. was supported by a Monbusho Grant-in-Aid. He also thanks members of the Nonlinear Dynamics Group of Kyoto University for their warm hospitality.

[1] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer-Verlag, Berlin, 1984).
[2] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[3] Y. Kuramoto and I. Nishikawa, J. Stat. Phys. **49**, 569 (1987); H. Sakaguchi, S. Shinomoto, and Y. Kuramoto, Prog. Theor. Phys. **77**, 1005 (1987); S.H. Strogatz and R.E. Mirollo, J. Phys. A **21**, L699 (1988); D. Golomb, D. Hansel, B. Shraiman, and H. Sompolinsky, Phys. Rev. A **45**, 3516 (1992).
[4] V. Hakim and J. Rappel, Phys. Rev. A **46**, R7347 (1992); N. Nakagawa and Y. Kuramoto, Prog. Theor. Phys. **89**, 313 (1993); Physica D **75**, 74 (1994); **80**, 307 (1995).
[5] D. Battogtokh, A. Preusser, and A. Mikhailov, Physica D **106**, 327 (1997); D. Lima, D. Battogtokh, A. Mikhailov, P. Borkmans, and G. Dewel, Europhys. Lett. **42**, 631 (1998); D. Battogtokh and A. Mikhailov, Physica D **90**, 84 (1996).
[6] Y. Kuramoto, Prog. Theor. Phys. **94**, 321 (1995); Int. J. Bifurcation Chaos Appl. Sci. Eng. **7**, 789 (1997).
[7] Y. Kuramoto and H. Nakao, Phys. Rev. Lett. **76**, 4352 (1996); **78**, 4039 (1997).
[8] Y. Kuramoto and H. Nakao, Physica D **103**, 294 (1997).
[9] U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985).
[10] Y. Kuramoto, D. Battogtokh, and H. Nakao, Phys. Rev. Lett. **81**, 3543 (1998).
[11] H. Chate, Nonlinearity **7**, 185 (1994).