Dynamics of a one-dimensional inelastic particle system

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The dynamical behavior of a one-dimensional inelastic particle system is investigated. By the means of map and spatial-temporal pattern we find the chaotic motion and the periodic motion in this simple system. We characterize several kinds of transitions and introduce the idea of a small collision chain to explain the universal relation $n=N^2$ between the number of collisions in a cycle n and the number of the particles N of the system for period-1 behavior.

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Granular materials appear broadly in nature (sand dunes, powders, etc.) and are of great technological importance (handling and transporting of, e.g., seeds and pharmaceuticals). In general, they consist of macroscopic grains which do not interact with each other except that they collide inelastically. The behaviors of granular materials are greatly influenced by energy injection. Without energy injection, granular materials will cool down gradually and the motion of grains will eventually stop. For example, in vertically periodically vibrating granular materials, the system may manifest fluidlike or solidlike behaviors depending on the vibrating amplitude and driving frequency. In the past few years, the investigation of one-dimensional (1D) inelastic gas has attracted much attention in the hope that the origin of phenomena that appear in models for two or three dimensions can be illuminated by results in a much simpler 1D case.

In Ref. [1], Du, Li, and Kadanoff stated that the hydrodynamic description failed in the case of 1D inelastic gas. Since then, several papers discussing this kind of system have been published [2–7]. However, all of these papers were only concerned with the macroscopic aspect, such as hydrodynamic properties and spatial and temporal average properties, etc. Little attention was paid to the dynamic properties at a microscopic scale. However, the dynamic behavior of the system is very important in understanding the behavior of the granular materials not only at a microscopic scale but also at a macroscopic scale. For example, it is the basis for us to understand the pattern formation of the granular materials [8,9].

In this paper, we investigate the dynamic behavior of 1D inelastic gas. The model is introduced in Ref. [1]. The system consists of many inelastic particles in which the particles interact via inelastic collisions which conserve momentum but dissipate kinetic energy. Let us consider a horizontal column of N sizeless identical inelastic particles confined by two walls of infinite mass between whom the distance is L. When two particles collide, the velocities after the collision v_1' and v_2' are expressed in terms of the velocities before the collision v_1 and v_2 as

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} \frac{1-r_0}{2} & \frac{1+r_0}{2} \\ \frac{1+r_0}{2} & \frac{1-r_0}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Here r_0 is the restitution coefficient defined by $v_2'-v_1' = -r_0(v_2-v_1)$. Energy is pumped in from the left wall. The leftmost particle will return with a velocity v_0 when it hits the left wall (hot wall). In this paper, we take this velocity to be a constant $v_0=1$. The other wall is a reflecting wall, that is, the collisions between the rightmost particle and reflecting wall only change the sign of the velocity of the rightmost particle. For convenience, we order the particles 1, 2, . . . , N according to their spatial order. For example, the leftmost particle is particle 1. In contrast to the previous works on 1D inelastic gas, we let r_0 be a function of $|v_2-v_1|$ [10],

$$r_0 = \begin{cases} r & (|v_2-v_1| > v_s) \\ 1-(1-r)(|v_2-v_1|/v_s)^{1/2} & (|v_2-v_1| \leq v_s). \end{cases}$$

Experiments [11] have shown some velocity-dependent behaviors. The increase in r_0 for low $|v_2-v_1|$ avoids inelastic collapse [12,13] and constant r_0 at higher $|v_2-v_1|$ is more computationally efficient.

To investigate the dynamic behaviors at microscope scale, we first define the event to be the collision between particle 1 and the hot wall, then we define the cycle as the process which begins with one *event* and ends at the next one. We measure two quantities: one is the number of the collisions among all particles which happen in one cycle, n, the other is the duration time of one cycle T. In the beginning, the particles are uniformly distributed between two walls and only particle 1 has nonzero velocity. In Fig. 1, we show the results about n and T when r is varied. Here, $L=210$, $N=20$, $v_s=1$ and we keep these parameters unchanged throughout the paper. We record the results after the system reaches its steady state. T has been shown in Fig. 1(a) where it follows the exponential form $T=(L/v_0)\{1+\exp[2N(1-r)]\}$, when $r_0>0.55$. We may take n as a dynamical variable $n(k)$ (k means the kth cycle) and consider the map $n(k+1) = f(n(k))$. Then to characterize the dynamic behavior of the system, we just need consider the solution of map f [see Fig. 1(b)]. We find that the dynamic behavior of the system is dependent on r. For example, we find an irregular motion (including two-band irregular motion) for $n(k)$ when $r>0.935$, a period-2 solution $[n(k+2)=n(k)]$ when $r\in(0.89,0.935)$ and a period-1 solution for a large range when $r\in(0.62,0.87)$. To learn the nature of the irregular motion occuring for $r>0.935$, we use a method designed for experimental data to calculate the Lyapunov exponent and

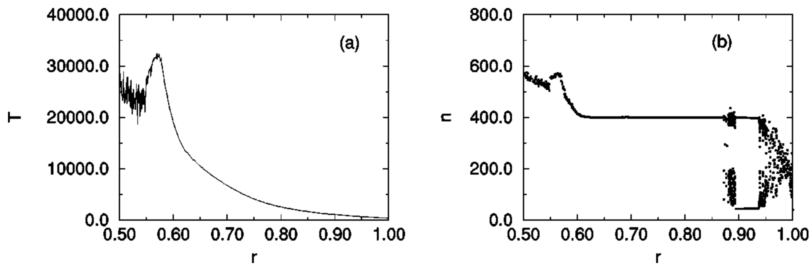

FIG. 1. (a) The period of a cycle T versus the restitution coefficient r. (b) The bifurcation diagram of $n(k)$.

find that the maximum Lyapunov exponent is positive when the system is in irregular motion. Actually, the bifurcation diagram of Fig. 1(b) is quite robust. When we change N, v_0, v_s, and L, we still observe a similar picture. The consequences of different sets of parameters are to vary the location of the transition (or bifurcation) point and perhaps to make period-2 solution disappear in some cases. All of the results show that there exist quite rich dynamical behaviors in this weakly nonlinear system.

The chaotic nature of the system near elastic situation ($r \approx 1$) is important. It provides the evidence on the molecular chaos in this simple system and certain possibility of the statistical description of macroscopic behavior of the system.

It is necessary to discuss the bifurcations between different dynamic behaviors. Due to the fact that $n(k)$ is integral and discontinuous, it is not convenient to use it to discuss bifurcation. We replace it as our dynamical variable with $v(2)$, which is the velocity of particle 2 when particle 1 collides with the hot wall. The bifurcation diagrams are shown in Fig. 2. In Fig. 2(a), we decrease r from 0.95 to 0.86 and let the initial condition for each r be the ending results of the previous r, and we observe chaos, period-2 solution, two-band chaos and period-1 solution in turn. In Fig. 2(b), we reverse the process to increase r from 0.86 to 0.95. The initial condition is the same as for (a). We find period-1, period-2 and chaotic motion, and the locations of different bifurcations are changed. In Fig. 2(c), we adopt the same initial condition scheme and vary the parameter r in a small region from 0.89 to 0.95, and now two band chaos, period-2 solution and chaos appear in turn. Finally, we choose random initial condition for each r in Fig. 2(d). In this plot, we find that bifurcations are not clear. From these plots, we can draw several conclusions: (1) The bifurcation is hysteretic. For a large range of r, there will be a coexistence of multistate, i.e., the coexistence of chaos, period-2, and period-1 solutions. Depending on different parameter regions, the attraction regions of different dynamic behaviors are different. (2) The period-2 solution exists only in a small region of r, and it will be replaced by chaotic motion beyond this region. When $0.55 < r < 0.62$, the dynamic behavior is different from that for $r \in (0.62, 0.87)$ though $n(k)$ continues to have a period-1 solution. The transition at $r \approx 0.62$ is dramatic and cannot be explained by bifurcation of the map directly. We will discuss it below.

We know that there will be a cluster consisting of $N-1$ particles near reflecting wall when r is near 1 [1]. How does the cluster change when r varies? To characterize the cluster, we may discuss its size and location. In numerical simulation, the maximum and minimum of $x(N)-x(2)$ are the good candidates for characterizing the size of the cluster and the distance of particle N away from the reflecting wall may account for the position of the cluster. In Fig. 3, we show our results. When $r > 0.62$, we find (1) The size of the cluster and its fluctuations increase as r decreases. The observation agrees with the thermodynamic argument. Both the kinetic energy of the cluster transfered from hot wall via particle 1 and the dissipation of energy in the cluster increase with the decrease of r. The former fact results in the increase of the granular temperature in the cluster. Considering the fact that the injection velocity of the leftmost particle is constant and its velocity leaving the cluster is very small, we know that the pressure exerted on cluster from the hot wall via particle 1 is nearly a constant. As a result of the thermodynamic laws, we know the size of the cluster will increase with the

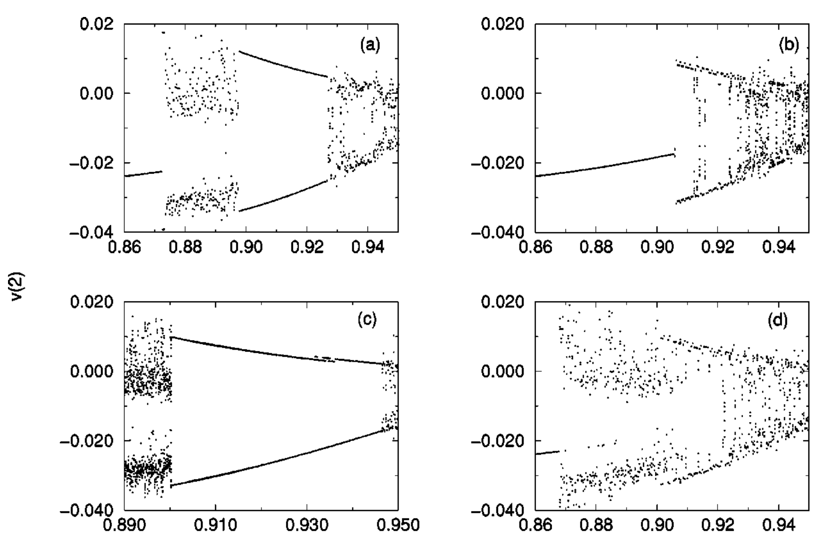

FIG. 2. The bifurcation diagram of $v(2)$, the velocity of the particle 2 when particle 1 collides with the hot wall, versus restitution coefficient r. (a) Decreasing r and the ending state of the system for every r is adopted as the initial condition of next r. (b) Increasing r from 0.86 to 0.95. The initial condition is the same as (a). (c) Increasing r from 0.9 to 0.95, the initial condition is the same as (a). (d) Random initial condition.

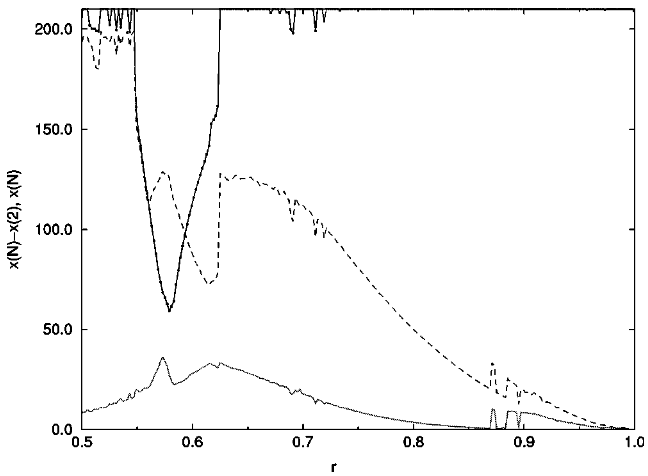

FIG. 3. The maximum size (dashed line), the minimum size (solid line) and the location of the cluster (line with solid circles) versus restitution coefficient *r*.

decrease of *r*. Then decrease of *r* will also induce a large fluctuation of the temperature of the cluster. Similarly, we know that the fluctuation of the size of the cluster will increase with the decrease of *r*. (2) Particle *N* remains almost at the reflecting wall for a range of *r*, it means that the cluster sticks to the reflecting wall regardless of the size of the cluster and its fluctuations. But when *r* decreases below about 0.62, we find that the particle *N* may move away from the reflecting wall, so that the cluster moves back and forth between two walls. This transition is very different from the bifurcations at large *r*. It reflects the transition between the different macroscopic behaviors of the system. In this sense, it looks rather similar to a phase transition in statistical physics.

We can get information about the dynamic behavior of the system from the spatial-temporal pattern of the system. We plot the pattern by recording the positions of all particles at different time. For convenience, we use the number of collisions as time, *t*. In Fig. 4, we show the results under the same parameter sets as those in Fig. 1. In Fig. 4(a), $r = 0.99$. We find that particle 1 moves fast between two walls and the other $N-1$ particles form a cluster located at the reflecting wall. In agreement with the discussion above, the cluster changes its size a little and has a small volume. In this plot, if we use real time, the trajectory of particle 1 will be a straight line. Furthermore, we decrease *r* to 0.92 with the results shown in Fig. 4(b). We find that the cluster has enlarged its size greatly, and the fluctuation of its size is obvious. The figure shows a clear period-2 solution. At point *A*, particle 1 hits the expanding cluster and makes it shrink. But the impact is not strong enough to clamp all $N-1$ particles together. When particle 1 hits the loose cluster again at *B*, the impact of particle 1 eventually makes the outer particles move to the reflecting wall and form a dense cluster including $N-1$ particles. Then the cluster expands again after lots of collisions among it. The similar process is repeated to form a period-2 structure. Figure 4(c) where we let $r = 0.78$ shows a clear period-1 structure. Comparing with Fig. 4(b), the strong impact of particle 1 on the cluster and the slow energy injection make the cluster have enough time to reach its minimum of the size. The expansion of the cluster is observed also after that until the next impact of the particle 1. It is worth mentioning that the cluster does not change its location in these plots. However when we let $r = 0.6$, the scenario is different. The location of the cluster varies with the evolution of the system and it moves back and forth between the hot and reflecting wall.

Now we come back to Fig. 1(a) again. The period-one solution obeys $n = N^2$, when $r > 0.62$. Actually, we can construct a simple collision sequence to fulfill the relation. To perform the construction, we let *W* represent the collision between the particle 1 and the hot wall *i* ($i = 1, 2, \ldots, N-1$) represent the collision between the *i*th particle and the $i+1$th particle and *N* represent the one between the rightmost particle and the reflecting wall. When $N = 5$, the simple collision sequence is shown in Fig. 5(a). In this simple chain, after the collision 1 at *A*, 2 at *B*, 3 at *C*, and 4 at *D*, respectively, particles 1, 2, 3 and 4 will not take part in the collisions in the cycle any more. These particles in effect leave the cluster after the corresponding collision. We may dissemble the chain into several small chains. For example, {*W*, 1, 2, 3, 4, 5, 4, 3, 2, 1}, {2, 3, 4, 5, 4, 3, 2}, {3, 4, 5, 4, 3}, and {4, 5, 4}. Each small chain includes parts moving toward the reflecting wall and moving toward the hot wall. In fact, these

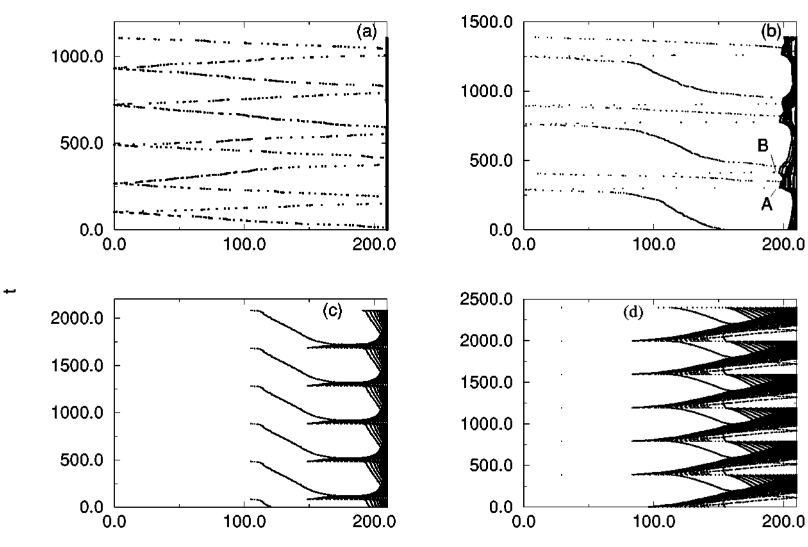

FIG. 4. The spatial-temporal pattern of the system at different restitution coefficient *r*. (a) $r = 0.99$, (b) $r = 0.92$, (c) $r = 0.8$, (d) $r = 0.60$.

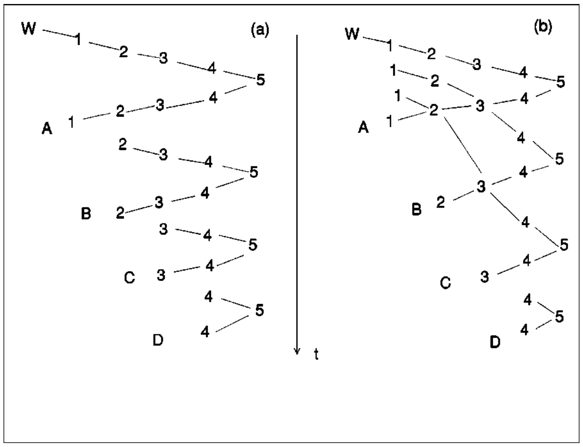

FIG. 5. (a) The simple collision sequence when $N=5$. (b) A possible collision sequence observed in the numerical simulation.

small chains may be classified according to the velocities of the small chains which are characterized by the order of $1-r$, and the small chain with higher order ends with larger i. For example, the small chain $\{W,1,2,3,4,5,4,3,2,1\}$ is $(1-r)^0$ order which ends at collision 1, $\{2,3,4,5,4,3,2\}$ is of $(1-r)$ order and ends at 2, and so on.

Unfortunately, we only observe the simple collision chain in certain range of r at small N. In general case, the simple collision chain will be blurred by the intersection of the small chains. Here, the intersection of the small chains means that two small chains have a common collision as their junction. However, no matter how these small chains arrange themselves, they are complete in any cycle in the sense that they have the same order and the same ending collision as those in the simple chain. The example for $N=5$ is shown in Fig. 5(b). Here the $1-r$ order and the $(1-r)^2$ order small chains appear before the $(1-r)^0$ order one ends. Another difference of these small chains from those in simple chain is that they may have more collisions that those in the simple chain if they intersect with other small chains. In Fig. 5(b), compared with the corresponding small chains in the simple case, we may find that there are extra collision 1 in the $(1-r)^1$ order chain and extra collisions 1 and 2 in the $(1-r)^2$ order one. It is the balance between the intersection of different order small chains and the extra collisions in the small chains that maintains the relation $n=N^2$. When N is large, the number of possible arrangement of the small chains is very large. This may be proven from the measure of collisions performed by different particles in one cycle. However, outside of the period-1 region, the complete small chain cannot exist since the collision W interrupts the collision sequence. This interruption destroys the relation $n=N^2$.

In summary, we have investigated the dynamic behaviors in one-dimensional inelastic particle system at microscope scale. The map and the spatial-temporal pattern are introduced to characterize and classify the dynamic behaviors. We have also introduced the small collision chain to explain the relation $n=N^2$ in the period-1 solution region. In fact, the investigation of the dynamic behavior perhaps provides some insight into the pattern formation in granular system. During these years, some models have been introduced to explain the pattern forming of the vibrating vertically granular system [14,15]. However, we know that the pattern formation may appear in many kinds of systems, i.e., fluid, chemical systems, and biological systems. It is well known that the phenomena observed may be the same for different systems though the underlying mechanisms are different. So we cannot say lots about models even if they can generate patterns similar to those in the real systems until we have an insight into the dynamic behaviors of the granular materials at microscope scale. The paper takes a first step to understand the dynamic behaviors at microscope scale.

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