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# Critical dynamics of Heisenberg fluids at the gas-liquid transition

R. Folk[1] and G. Moser[2]
[1]*Institute for Theoretical Physics, University of Linz, Linz, Austria*
[2]*Institute for Physics and Biophysics, University of Salzburg, Salzburg, Austria*

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We derive the static and dynamical model describing the critical statics and dynamics of a Heisenberg fluid in a magnetic field near its gas-liquid critical point. It turns out that the model is equivalent to the one describing a liquid mixture near its plait point. Using the results known for that case we predict the critical behavior of the liquid and magnetic transport coefficients and the velocity and absorption of the sound mode.

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## I. INTRODUCTION

We consider a fluid whose molecules have both transnational and spin degrees of freedom. We assume that the interaction of the spins is described by a Heisenberg model. Such a magnetic fluid in zero magnetic field manifests a line of magnetic transitions of second order and a gas-liquid phase transition. The topology of the phase diagram depends on the ratio of the strength of the magnetic to the nonmagnetic interaction (for example, see Ref. [1] for the phase diagram in mean field theory and Refs. [2,3] for Monte Carlo calculations at the magnetic phase transition). In a finite external field the magnetic transition is absent but the gas-liquid transition remains. There might be another liquid-liquid transition [4] but this transition is of no interest here. Due to the finite field the fluid has a finite magnetization and therefore the spin and transnational degrees of freedom are coupled. This coupling in a finite field makes the investigation of the gas-liquid transition interesting [5–7]. Until now, investigations of the critical behavior have been restricted to static phenomena. In this paper we consider the dynamical critical behavior of the system. We set up a dynamical model suitable for the description of the dynamical critical behavior.

The hydrodynamics of such fluids are well known and have been derived using several methods (see, e.g., Refs. [8], [9]). The dynamical critical model is compatible with linearized hydrodynamics in the region where the wave vector is smaller than the inverse correlation length. By methods of nonequilibrium thermodynamics [10,11] we derive the reversible and irreversible terms in the equations of motion for the conserved quantities containing the order parameter at the gas-liquid critical point. Further it is shown that the model for the critical dynamic behavior is equivalent to the model *H* describing the critical dynamics of a mixture near the plait point [12]. Thus, asymptotically a Heisenberg magnetic liquid in a *homogenous* field lies in the universality class of model *H* (pure fluid). This correspondence allows us to use the results obtained in Ref. [12] by renormalization group theory and to predict the critical temperature dependence of hydrodynamic transport coefficients such as thermal conductivity, magnetic diffusion, and the thermomagnetic diffusion ratio. In addition, we consider sound propagation and predict the frequency and temperature dependence of the sound velocity and sound absorption as well. As one knows from the critical behavior at a plait point the asymptotic behavior might be masked by strong nonasymptotic effects. Thus, we expect the situation in magnetic fluids to be similar; however, this has to await computer simulations, experimental realizations of such a magnetic fluid are not yet available.

## II. HYDRODYNAMICS OF HEISENBERG FLUIDS

### A. Continuity equations and dissipation

In order to obtain the hydrodynamic equations for a Heisenberg fluid we have to consider the corresponding conserved densities. Of course, all conserved local densities (per volume) of a normal fluid [namely, the entropy density $s(x)$, the mass density $\rho(x)$, and the momentum density $\vec{J}(x)$] have to be taken into account. In addition, a classical spin density $\vec{m}(x)$ enters the hydrodynamics. The corresponding intensive local fields are then the temperature $T(x)$, the chemical potential $\mu(x)$, the velocity $\vec{v}(x)$, and the magnetic field $\vec{H}(x)$. The reversible continuity equations can be expressed by generalized Poisson bracket relations and a corresponding Hamiltonian $\mathcal{H}$:

$$\frac{\partial \alpha}{\partial t}=\{a,\mathcal{H}\}, \tag{2.1}$$

where $\alpha$ denotes $s$, $\rho$, $\vec{j}$, or $\vec{m}$. The Hamiltonian is the total energy of the system,

$$\mathcal{H}=\int d^3x e(s(x),\rho(x),\vec{j}(x),\vec{m}(x)) \tag{2.2}$$

where $e$ is the total energy density of the system. In the presence of an external magnetic field $\vec{H}^{\rm ex}$, the energy density separates into three parts,

$$e=u+e_{\rm kin}-\vec{H}^{\rm ex}\cdot\vec{m} \tag{2.3}$$

with the internal energy $u$ and the kinetic energy $e_{\rm kin}$ defined by

$$du=Tds+\mu d\rho+\vec{H}\cdot d\vec{m}, \quad e_{\rm kin}=\frac{1}{2}\rho\vec{v}^2. \tag{2.4}$$

The magnetic field in Eq. (2.4) is defined by $\vec{H}=(\partial u/\partial\vec{m})_{s,\rho,\vec{j}}$, which represents the internal magnetic field produced by the spin density. Following Ref. [11], the generalized Poisson brackets $\{\alpha,\beta\}$ between the conserved densities are determined from the group generators of operations in coordinate and spin space. The momentum density, as the generator of translations in coordinate space, determines the Poisson brackets,

$$\{s(x),\vec{j}(x')\}=s(x')\vec{\nabla}'\delta(x-x'), \tag{2.5}$$

$$\{\rho(x),\vec{j}(x')\}=\rho(x')\vec{\nabla}'\delta(x-x'), \tag{2.6}$$

$$\{j_i(x),j_k(x')\}=j_i(x')\nabla'_k\delta(x-x')-j_k(x)\nabla_i\delta(x-x'). \tag{2.7}$$

In quantum mechanics, the spin operator is the generator of rotations in spin space, acting only on the spin and not quantities in coordinate space. The commutator between the spin components corresponds to a classical Poisson bracket between the classical spin density components,

$$\{m_i(x),m_k(x')\}=-\epsilon_{ikl}m_l(x)\delta(x-x'). \tag{2.8}$$

We use the Einstein summation convention throughout. The spin density is given by $\vec{m}(x)=\Sigma_i\vec{S}_i\delta(x-x_i)$, where in a coarse grained theory the $\vec{S}_i$ represents the mean spin in a small volume around $x_i$. The spin vector $\vec{S}_i$ itself is not affected by transformations in coordinate space such as translation or rotation. This is in contrast to a classical magnetization density in the Maxwell theory. Thus, the translation operator acts only on the scalar $\delta$ function; and the Poisson bracket between the components of $\vec{m}$ and the momentum density are the same as for the scalar fields in Eqs. (2.5) and (2.6):

$$\{m_i(x),\vec{j}(x')\}=m_i(x')\vec{\nabla}'\delta(x-x'). \tag{2.9}$$

Inserting Eqs. (2.5)–(2.8) and (2.2) into Eq. (2.1), the following continuity equations are obtained:

$$\frac{\partial s}{\partial t}+\vec{\nabla}\cdot(s\vec{v})=0, \tag{2.10}$$

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0, \tag{2.11}$$

$$\frac{\partial j_i}{\partial t}+\nabla_i P+\nabla_l v_l j_i=0, \tag{2.12}$$

$$\frac{\partial m_i}{\partial t}+\epsilon_{ikl}(H_k-H_k^{ex})m_l+\nabla_l v_l m_i=0. \tag{2.13}$$

In Eqs. (2.10) and (2.11) the symbol ‘‘$\cdot$’’ denotes the scalar product of vectors. So far, Eqs. (2.10)–(2.13) do not contain dissipative processes. We have made use of the fact that the external field is homogeneous, otherwise an additional term proportional to the scalar product between the magnetization and the gradient of the field appears. This would change the resulting critical model drastically and is not considered here. In fact, it would lead to anisotropies in the fluid system [9]. Thus, the total energy $\mathcal{H}$ has to remain constant in time, which can be immediately verified by inserting Eqs. (2.10)–(2.13) into

$$\frac{\partial\mathcal{H}}{\partial t}=\int d^3x\,\frac{\partial e}{\partial t}=0. \tag{2.14}$$

Defining $\vec{j}(x)=\rho(x)\vec{v}(x)$, and introducing the tensor

$$[\mathbf{P}]_{ki}=P_{ki}\equiv P\,\delta_{ki}, \tag{2.15}$$

we rewrite Eq. (2.12) as

$$\frac{\partial j_i}{\partial t}+\nabla_k[\rho v_k v_i+P_{ki}]=0. \tag{2.16}$$

From the conservation equations for the mass density (2.11) and the momentum density (2.12), we immediately obtain the equation

$$\frac{\partial v_i}{\partial t}+v_k\nabla_k v_i+\frac{1}{\rho}\nabla_k P_{ki}=0 \tag{2.17}$$

for the velocity. The hydrodynamic equation for the kinetic energy is completely determined by Eqs. (2.12) and (2.17). We get

$$\frac{\partial e_{\text{kin}}}{\partial t}=-\vec{\nabla}\cdot[e_{\text{kin}}\vec{v}+\mathbf{P}\cdot\vec{v}]+P_{ki}\nabla_k v_i. \tag{2.18}$$

The first term on the right hand side in Eq. (2.18) represents the divergence of the kinetic energy current $e_{\text{kin}}\vec{v}+\mathbf{P}\cdot\vec{v}$. In addition, an energy source term appears in Eq. (2.18) because the kinetic energy is not a conserved density. The total energy density $e$ is conserved and therefore obeys the continuity equation,

$$\frac{\partial e}{\partial t}=-\vec{\nabla}\cdot\vec{J}_e, \tag{2.19}$$

where $\vec{J}_e$ is the total energy current. Without dissipative effects, the total energy current is the sum of the convective flow $e\vec{v}$ and the flow of mechanical work $\mathbf{P}\cdot\vec{v}$.

In order to treat dissipation we have to amend the total energy current in order to include two dissipative processes [10]. The first of these accounts for the finite viscosity of the fluid. Viscosity causes additional mechanical work when different fluid layers shift against each other, and is taken into account using an extension of the pressure tensor,

$$P_{ki}=P\,\delta_{ki}+\Pi_{ki}, \tag{2.20}$$

where $\Pi_{ki}$ is a symmetric tensor. The second dissipative process takes into account the energy transport by heat conduction, which may be included by adding a heat current $\vec{J}_q$ to the total energy current. Thus we get

$$\vec{J}_e=e\vec{v}+\mathbf{P}\cdot\vec{v}+\vec{J}_q \tag{2.21}$$

for the total energy current with dissipation. From Eq. (2.3) we obtain the hydrodynamic equation for the internal energy $u$. It is given by

$$\frac{\partial u}{\partial t}=\frac{\partial e}{\partial t}-\frac{\partial e_{\mathrm{kin}}}{\partial t}+\vec{H}^{\mathrm{ex}}\cdot\frac{\partial \vec{m}}{\partial t}. \tag{2.22}$$

Inserting Eqs. (2.13), (2.18), and (2.19) into Eq. (2.21) we obtain for the internal energy the equation

$$\frac{\partial u}{\partial t}=-\vec{\nabla}\cdot[u\vec{v}+\vec{J}_q]-P_{ki}\nabla_k v_i-\vec{H}^{\mathrm{ex}}\cdot(\vec{H}\times\vec{m}). \tag{2.23}$$

From Eq. (2.4) we also determine the equation for the entropy density conservation

$$T\frac{\partial s}{\partial t}=\frac{\partial u}{\partial t}-\vec{H}\cdot\frac{\partial \vec{m}}{\partial t}-\mu\frac{\partial \rho}{\partial t}. \tag{2.24}$$

The final equation for the entropy density is found by inserting Eqs. (2.23) and (2.11) into Eq. (2.24). The Heisenberg energy of a spin system depends only on the orientation of the spins. So we assume that the spin density does not contribute to the energy transport, which is expressed in Eq. (2.21) for the case of the absence of a magnetic current. As a consequence, only the continuity equation (2.13) has been used to obtain Eq. (2.23). Changes in the spin orientation cause magnetic spin currents that affect the structure of the system, and therefore, the density of entropy. In order to obtain an equation for the entropy density $s$ we have to include the dissipative processes in Eq. (2.13). These processes depend on the external magnetic field. In the following we will consider two cases. First, the case of zero external magnetic field and, second, the case of a homogeneous magnetic field.

### B. Zero external magnetic field

In the case of a zero external magnetic field $\vec{H}_{\mathrm{ex}}=\vec{0}$ all spin components are conserved, and Eq. (2.13) is extended to

$$\frac{\partial m_i}{\partial t}+\epsilon_{ikl}H_k m_l+\nabla_l v_l m_i=-\nabla_l J^{(m)}_{li}, \tag{2.25}$$

where $J^{(m)}_{li}$ is the spin current tensor. The resulting equation for the energy density is

$$\frac{\partial s}{\partial t}+\vec{\nabla}\cdot(s\vec{v})=-\vec{\nabla}\cdot\left(\frac{\vec{J}_q-\mathbf{J}^{(m)}\cdot\vec{H}}{T}\right)+q_s, \tag{2.26}$$

where the first term on the right hand side is the divergence of the entropy current, and $q_s$ denotes the source term for the entropy, which is given by

$$q_s=-(\vec{J}_q-\mathbf{J}^{(m)}\cdot\vec{H})\cdot\frac{\vec{\nabla}T}{T^2}-\mathbf{J}^{(m)}:\frac{\boldsymbol{\nabla}\mathbf{H}}{T}-\boldsymbol{\Pi}:\frac{\boldsymbol{\nabla}\mathbf{v}}{T}. \tag{2.27}$$

TABLE I. Currents $J_i$ and conjugated thermodynamic forces $X_i$ in a Heisenberg ferrofluid at zero external field.

| $J_i$ | $X_i$ |
|---|---|
| $\Pi$ | $\frac{1}{T}\vec{\nabla}\cdot\vec{v}$ |
| $\vec{J}_q-\mathbf{J}^{(m)}\cdot\vec{H}$ | $\frac{1}{T^2}\vec{\nabla}T$ |
| $\mathbf{J}^{(m)}$ | $\frac{1}{T}\boldsymbol{\nabla}\mathbf{H}^{(s)}$ |
| $\boldsymbol{\Pi}$ | $\frac{1}{T}\boldsymbol{\nabla}\mathbf{v}^{(s)}$ |

In the above equation $\mathbf{ab}$ denotes the dyadic product $[\mathbf{ab}]_{ij}=a_i b_j$ of two vectors $\vec{a}$ and $\vec{b}$. The symbol ":" expresses the total contraction, $\mathbf{A}:\mathbf{B}=A_{ij}B_{ij}$, of two second rank tensors. Separating the tensors into traceless symmetric parts, e.g., into $\Pi^{(s)}$, together with the traces $\Pi=1/3\Pi_{ll}$, and using Eq. (2.12) together with (2.20), we get the following equation:

$$\frac{\partial j_i}{\partial t}+\nabla_i P+\nabla_l v_l j_i=-\nabla_l\Pi^{(s)}_{li}-\nabla_{i,}\Pi \tag{2.28}$$

for the momentum density when dissipative processes are included. The entropy source term (2.27) is given explicitly by

$$q_s=-(\vec{J}_q-\mathbf{J}^{(m)}\cdot\vec{H})\cdot\frac{\vec{\nabla}T}{T^2}-\mathbf{J}^{(m)}:\frac{\boldsymbol{\nabla}\mathbf{H}^{(s)}}{T}$$
$$-\boldsymbol{\Pi}^{(s)}:\frac{\boldsymbol{\nabla}\mathbf{v}^{(s)}}{T}-\Pi\frac{\vec{\nabla}\cdot\vec{v}}{T}, \tag{2.29}$$

where $\mathbf{J}^{(m)}$ and $\boldsymbol{\Pi}^{(s)}$ are symmetric tensors. The total contraction with the antisymmetric part of $\boldsymbol{\nabla}\mathbf{H}$ and, respectively, with $\boldsymbol{\nabla}\mathbf{v}$ vanishes, leading to the absence of antisymmetric contributions in Eq. (2.29). In the absence of boundaries, the relation $\vec{B}=\vec{H}-4\pi\vec{M}$ between the vacuum field $\vec{B}$ (which obeys the Maxwell equation $\vec{\nabla}\cdot\vec{B}=0$), the macroscopic magnetic field $\vec{H}$, and the magnetization $\vec{M}$, reduces to $\vec{B}=\mu\vec{H}$ ($\mu$ is the permeability). In this case the macroscopic magnetic field $\vec{H}$ also obeys the Maxwell equation $\vec{\nabla}\cdot\vec{H}=0$. Therefore $\boldsymbol{\nabla}\mathbf{H}$ is a traceless tensor and no term proportional to the divergence of the magnetic field appears in Eq. (2.29). From Eq. (2.29) we can see that the entropy source has the structure

$$q_s=-\sum_i J_i X_i, \tag{2.30}$$

with the currents $J_i$ and corresponding thermodynamic forces $X_i$ listed in Table I. The thermodynamic forces have different tensor characters. The first one in Table I is a scalar, the second is a polar vector, the third is a polar tensor of the second rank, and the last one is an axial tensor of the second rank. The currents in Table I are functions of the thermodynamic forces $J_i=J_i(\{X_j\})$. For small gradients we can assume a linear dependence $J_i=\Sigma_j L_{ij}X_j$, where $L_{ij}$ are the Onsager coefficients. Generally, all currents may depend on all thermodynamic forces that define a set of Onsager coefficients, which, of course, have different tensor character depending on the tensor character of the connected current and force. But the interaction symmetry and the tensor character of the thermodynamic forces restrict the form of the thermodynamic forces in a current, thus reducing the number of nonvanishing Onsager coefficients. For an isotropic system, the form of the thermodynamic forces follows from invariance under coordinate inversion and arbitrary rotation, so that the four different tensor characters in Table I cannot have any cross coefficients. The currents decouple to

$$\vec{J}_q - \mathbf{J}^{(m)}\cdot\vec{H} = -\kappa_T\vec{\nabla}T, \tag{2.31}$$

$$\mathbf{J}^{(m)} = -2\lambda_s\boldsymbol{\nabla}\mathbf{H}^{(s)}, \tag{2.32}$$

$$\boldsymbol{\Pi} = -2\,\overline{\eta}\boldsymbol{\nabla}\mathbf{v}^{(s)}, \tag{2.33}$$

$$\Pi = -\zeta\vec{\nabla}\cdot\vec{v}, \tag{2.34}$$

with the thermal conductivity $\kappa_T$, the coefficient for the spin diffusion $\lambda_s$, the shear viscosity $\overline{\eta}$, and the bulk viscosity $\zeta$. Equations (2.25), (2.26), (2.29), (2.28), and (2.11) together with the currents (2.31), constitute the hydrodynamic equation for the Heisenberg fluid at zero external magnetic field. For the hydrodynamic equations linearized about the mean values $\overline{s}$, $\overline{\rho}$, $\overline{\vec{m}}=0$ and $\overline{\vec{j}}=0$ we obtain

$$\frac{\partial s}{\partial t} + \overline{s}\vec{\nabla}\cdot\vec{v} = \frac{\kappa_T}{T}\nabla^2 T, \tag{2.35}$$

$$\frac{\partial\vec{m}}{\partial t} = \lambda_s\nabla^2\vec{H}, \tag{2.36}$$

$$\frac{\partial\rho}{\partial t} + \overline{\rho}\vec{\nabla}\cdot\vec{v} = 0, \tag{2.37}$$

$$\frac{\partial\vec{j}}{\partial t} + \vec{\nabla}P = \overline{\eta}\nabla^2\vec{v} + \left(\zeta + \frac{4}{3}\overline{\eta}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{v}). \tag{2.38}$$

### C. Homogeneous external magnetic field

In the following, we will consider a fluid in an external magnetic field $\vec{H}^{\text{ex}} = H^{\text{ex}}\vec{e}_z$, with $\vec{e}_z$ the unit vector in the $z$ direction. In this case only the $z$ component $m_z(x)$ of the spin density is conserved. As a consequence spin diffusion exists only for this component. The dissipation in the $x$-$y$ plane takes place via relaxation. Hence Eq. (2.25) splits into the two equations,

$$\frac{\partial m_z}{\partial t} + [\vec{H}\times\vec{m}]_z + \nabla_l v_l m_z = -\vec{\nabla}\vec{J}_z^{(m)}, \tag{2.39}$$

$$\frac{\partial m_\alpha}{\partial t} + [(\vec{H}-\vec{H}^{\text{ex}})\times\vec{m}]_\alpha + \nabla_l v_l m_\alpha = -r_\alpha, \tag{2.40}$$

where $\alpha$ denotes $x$ or $y$ and $r_\alpha$ a source term. Repeating the steps of the previous subsection we obtain the continuity equation for the entropy density,

$$\frac{\partial s}{\partial t} + \vec{\nabla}\cdot(s\vec{v}) = -\vec{\nabla}\cdot\left(\frac{\vec{J}_q - \vec{J}_z^{(m)}H_z}{T}\right) + q_s, \tag{2.41}$$

with the entropy source term,

$$q_s = -(\vec{J}_q - \vec{J}_z^{(m)}H_z)\cdot\frac{\vec{\nabla}T}{T^2} - \vec{J}_z^{(m)}\cdot\frac{\vec{\nabla}H_z}{T} - \boldsymbol{\Pi}^{(s)}:\frac{\nabla v^{(s)}}{T} - \Pi\frac{\vec{\nabla}\cdot\vec{v}}{T}. \tag{2.42}$$

TABLE II. Currents $J_i$ and conjugated thermodynamic forces $X_i$ in a Heisenberg ferrofluid in a fixed external field.

| $J_i$ | $X_i$ |
|---|---|
| $\Pi$ | $\frac{1}{T}\vec{\nabla}\cdot\vec{v}$ |
| $\vec{J}_q - \vec{J}_z^{(m)}H_z$ | $\frac{1}{T^2}\vec{\nabla}T$ |
| $\vec{J}_z^{(m)}$ | $\frac{1}{T}\vec{\nabla}H_z$ |
| $\boldsymbol{\Pi}$ | $\frac{1}{T}\boldsymbol{\nabla}\mathbf{v}^{(s)}$ |

In Eq. (2.42) we have neglected the contributions of the fast modes, which are related to the relaxation process. The entropy source term (2.42) includes the currents and the corresponding thermodynamic forces listed in Table II. The tensor character of the third thermodynamic force in Table II is different from that in the case of zero magnetic field given in Table I. The magnetic field in Eq. (2.42) is an axial vector, therefore the gradient of the $z$ component is a polar vector. The interaction between the molecules of the liquid remains isotropic in the case of a finite external magnetic field. The same symmetry arguments as before can be applied now in order to determine the nonvanishing Onsager coefficients. The main difference to the zero magnetic field case is that we now have the polar vector $\vec{\nabla}T$ as well as $\vec{\nabla}H_z$. The symmetry of the system permits a cross coefficient that couples the gradient of the temperature and the gradient of the magnetic field component. Thus, the resulting currents are

$$\vec{J}_q - \vec{J}_z^{(m)}H_z = -\gamma\vec{\nabla}T - T\beta\vec{\nabla}H_z, \tag{2.43}$$

$$\vec{J}_z^{(m)} = -\beta\vec{\nabla}T - \alpha\vec{\nabla}H_z, \tag{2.44}$$

$$\boldsymbol{\Pi} = -2\,\overline{\eta}\boldsymbol{\nabla}\mathbf{v}^{(s)}, \tag{2.45}$$

$$\Pi = -\zeta\vec{\nabla}\cdot\vec{v}. \tag{2.46}$$

In Eqs. (2.43) and (2.44) we have already used the symmetry properties of the Onsager coefficients. The resulting three Onsager coefficients $\alpha$, $\beta$, and $\gamma$ determine the spin diffusion, the thermomagnetic diffusion ratio, and the thermal conductivity. At vanishing external magnetic field, the system is described by the case of the previous subsection, where the heat and spin modes decouple. Thus, the cross coefficient $\beta$ is a function of the external magnetic field, with the property,

$$\beta(\vec{H}^{\text{ex}} = \vec{0}) = 0. \tag{2.47}$$

As in the case of zero magnetic field, the coefficients $\overline{\eta}$ and $\zeta$ represent the shear and bulk viscosity.

Linearizing the hydrodynamic equations about the mean values $\overline{s}$, $\overline{m}$, $\overline{\rho}$ and $\overline{\vec{j}}=0$ we obtain

$$\frac{\partial s}{\partial t}+\overline{s}\vec{\nabla}\cdot\vec{v}=\frac{\gamma}{T}\nabla^2 T+\beta\nabla^2 H_z\,, \tag{2.48}$$

$$\frac{\partial \rho}{\partial t}+\overline{\rho}\vec{\nabla}\cdot\vec{v}=0, \tag{2.49}$$

$$\frac{\partial m_z}{\partial t}+\overline{m}\vec{\nabla}\cdot\vec{v}=\beta\nabla^2 T+\alpha\nabla^2 H_z\,, \tag{2.50}$$

$$\frac{\partial \vec{m}_\perp}{\partial t}+\vec{H}\times\vec{\overline{m}}-\vec{H}^{(\mathrm{ex})}\times\vec{m}=-\alpha_R\vec{m}_\perp\,, \tag{2.51}$$

$$\frac{\partial \vec{j}}{\partial t}+\vec{\nabla}P=\overline{\eta}\nabla^2\vec{v}+\left(\zeta+\frac{4}{3}\overline{\eta}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{v}). \tag{2.52}$$

In the equations above we have explicitely inserted an external homogeneous magnetic field in $z$ direction, causing a mean magnetization $\overline{m}(H_z)=\overline{m}\vec{e}_z$. The transverse magnetization $\vec{m}_\perp$ is not conserved and does not belong to the set of slow variables. However, it couples via the nonlinear reversible terms considered below to the equations of the other densities. Therefore, we take into account the dynamical equation for the transverse components in the form of a relaxation equation. The relaxation coefficient $\alpha_R$ is connected to the source term in Eq. (2.40) by $\vec{r}_\perp=\alpha_R\vec{m}_\perp$.

### D. The hydrodynamic transport coefficients

The Onsager coefficients introduced in Eqs. (2.43) and (2.44) are related to the thermal conductivity, thermomagnetic diffusion ratio, and spin diffusion coefficient in a way that is quite analogous to their relationship to the thermal conductivity, the thermodiffusion ratio, and the mass diffusion in binary liquid mixtures. Assuming a vanishing spin current (2.44) ($\vec{J}_z^{(m)}=0$), the gradients of the temperature and magnetic field are related by $\vec{\nabla}H_z=-(\beta/\alpha)\vec{\nabla}T$. Inserting this relation into Eq. (2.43), we obtain for the heat current

$$\vec{J}_q=-\left(\gamma-\frac{T\beta^2}{\alpha}\right)\vec{\nabla}T, \tag{2.53}$$

from which we identify the thermal conductivity at vanishing spin current to be

$$\kappa_T=\gamma-\frac{T\beta^2}{\alpha}\qquad(\vec{J}_z^{(m)}=0). \tag{2.54}$$

At constant spin density and pressure, the temperature gradient is proportional to the gradient of the entropy density per unit mass $\sigma=s/\rho$,

$$\vec{\nabla}T=\left(\frac{\partial T}{\partial\sigma}\right)_{\mu_z,P}\vec{\nabla}\sigma. \tag{2.55}$$

We have also introduced the spin density per unit mass $\mu_z=m_z/\rho$. The equation for $\sigma$ is easily derived from Eqs. (2.48) and (2.49),

$$\frac{\partial\sigma}{\partial t}=\frac{\kappa_T}{\rho T}\left(\frac{\partial T}{\partial\sigma}\right)_{\mu_z,P}\nabla^2\sigma, \tag{2.56}$$

and this describes heat conduction in the magnetic fluid. From this equation we identify the thermal diffusion constant to be

$$D_T=\frac{\kappa_T}{\rho T}\left(\frac{\partial T}{\partial\sigma}\right)_{\mu_z,P}. \tag{2.57}$$

In general, at local thermodynamic equilibrium we have for the gradient of the internal magnetic field

$$\vec{\nabla}H_z=\left(\frac{\partial H_z}{\partial T}\right)_{m_zP}\vec{\nabla}T+\left(\frac{\partial H_z}{\partial m_z}\right)_{T,P}\vec{\nabla}m_z+\left(\frac{\partial H_z}{\partial P}\right)_{T,m_z}\vec{\nabla}P. \tag{2.58}$$

Inserting this into the equation for the spin current (2.44), we obtain

$$\vec{J}_z^{(m)}=-\alpha\left(\frac{\partial H_z}{\partial m_z}\right)_{T,P}\left\{\vec{\nabla}m_z+\left(\frac{\partial m_z}{\partial H_z}\right)_{T,P}\left[\left(\frac{\partial H_z}{\partial T}\right)_{m_z,P}+\frac{\beta}{\alpha}\right]\right.$$
$$\left.\times\vec{\nabla}T+\left(\frac{\partial m_z}{\partial H_z}\right)_{T,P}\left(\frac{\partial H_z}{\partial P}\right)_{T,m_z}\vec{\nabla}P\right\}. \tag{2.59}$$

Considering a magnetic fluid at rest ($\vec{v}=\vec{0}$), at constant temperature and at constant pressure we derive from Eq. (2.50) the following spin diffusion equation:

$$\frac{\partial\mu_z}{\partial t}=\frac{\alpha}{\rho}\left(\frac{\partial H_z}{\partial\mu_z}\right)_{T,P}\nabla^2\mu_z\,. \tag{2.60}$$

Thus, the spin diffusion coefficient reads

$$D_m=\frac{\alpha}{\rho}\left(\frac{\partial H_z}{\partial\mu_z}\right)_{T,P}\qquad(\vec{J}_z^{(m)}=0). \tag{2.61}$$

At vanishing spin current and at constant pressure, a temperature gradient causes a spin density gradient. From Eq. (2.59) we immediately obtain

$$\vec{\nabla}\mu_z=-\left(\frac{\partial\mu_z}{\partial H_z}\right)_{T,P}\left[\left(\frac{\partial H_z}{\partial T}\right)_{\mu_z,P}+\frac{\beta}{\alpha}\right]\vec{\nabla}T, \tag{2.62}$$

which leads to the identification of the thermomagnetic diffusion ratio,

$$\frac{k_T}{T}=\left(\frac{\partial\mu_z}{\partial H_z}\right)_{T,P}\left[\left(\frac{\partial H_z}{\partial T}\right)_{\mu_z,P}+\frac{\beta}{\alpha}\right]. \tag{2.63}$$

This transport coefficient corresponds to the thermodiffusion ratio in binary mixtures. In the same way, we obtain for the magnetic pressure diffusion coefficient the expression,

$$\frac{k_P}{P}=\left(\frac{\partial\mu_z}{\partial H_z}\right)_{T,P}\left(\frac{\partial H_z}{\partial P}\right)_{T,\mu_z}. \tag{2.64}$$

With Eqs. (2.61)–(2.64), the spin current retains the same structure as the mass current in binary mixtures [13]:

$$\frac{\vec{J}_z^{(m)}}{\rho}=-D_m\left[\vec{\nabla}\mu_z+\frac{k_T}{T}\vec{\nabla}T+\frac{k_P}{P}\vec{\nabla}P\right]. \tag{2.65}$$

## III. STATIC FUNCTIONAL OF THE HEISENBERG FLUID

### Static functional of the Heisenberg fluid at the gas-liquid transition

The general steps of the derivation of the static functional describing the critical behavior of a system is presented in the Appendix. In the case of the Heisenberg fluid, the local densities $\alpha_i(x)$ introduced in the Appendix are the entropy density $s(x)$, the mass density $\rho(x)$, the momentum density $\vec{j}(x)$, and the spin density $\vec{m}(x)$. The external fields $F_i(x)$ are the temperature $T(x)$, the chemical potential $\mu(x)$, the velocity $\vec{v}(x)$, and the magnetic field $\vec{H}(x)$. In respect to the gas-liquid transition, the entropy density represents the order parameter implying that the fluctuation terms of this density have to be taken into account up to fourth order. The remaining terms in the static functional (A20) are the quadratic terms of all the densities mentioned, $\Delta\alpha_i(x)\Delta\alpha_j(x)$, the gradient term $[\nabla\Delta s(x)]^2$, and the non-Gaussian terms $[\Delta s(x)]^3$, $[\Delta s(x)]^4$, $\Delta\vec{m}(x)[\Delta s(x)]^2$, and $\Delta\rho(x)[\Delta s(x)]^2$. The total energy density $e(s,\rho,\vec{m},\vec{j})$ is the sum of the internal energy density $u(s,\rho,\vec{m})$ and the kinetic energy $e_{\mathrm{kin}}=\vec{j}^2/2\rho$. Thus, the momentum density appears in Eq. (A20) up to quadratic order and no $\Delta\vec{j}(x)[\Delta s(x)]^2$ term exists (this follows from the different behavior of the densities under time inversion). In order to change from the chemical potential to the experimentally accessible pressure it is convenient to introduce the entropy per unit mass $\sigma(x)=s(x)/\rho(x)$ and the spin density per unit mass $\vec{\mu}_m(x)=\vec{m}(x)/\rho(x)$. The structure of the static functional remains invariant, but $s$ and $\vec{m}$ are replaced by $\sigma$ and $\vec{\mu}_m$. The third-order term of the order parameter $[\Delta\sigma(x)]^3$ and the quadratic couplings of the secondary densities to the order parameter, $\Delta\sigma(x)\Delta\rho(x)$ and $\Delta\sigma(x)\Delta\vec{\mu}_m(x)$, are eliminated by introducing the densities,

$$\mathring{\phi}(x)=\sqrt{N_A}[\Delta\sigma(x)-\langle\sigma(x)\rangle], \tag{3.1}$$

$$\mathring{\vec{q}}_1(x)=\sqrt{N_A}\left[\Delta\vec{\mu}_m(x)-\left(\frac{\partial\vec{\mu}_m}{\partial\sigma}\right)_{\vec{H},P}[\Delta\sigma(x)-\langle\sigma(x)\rangle]\right], \tag{3.2}$$

$$\mathring{q}_2(x)=\sqrt{N_A}\left[\Delta\rho(x)-\left(\frac{\partial\rho}{\partial\sigma}\right)_{\vec{H},P}[\Delta\sigma(x)-\langle\sigma(x)\rangle]\right]. \tag{3.3}$$

$N_A$ denotes the Avogadro number. The momentum density fluctuations $\vec{j}=\sqrt{N_A}\Delta\vec{j}$ are also rescaled by the square root of the Avogadro number. The overcircle denotes that these quantities are unrenormalized. In the calculation of the transport coefficients their renormalized counterparts may appear (then without the overcircle). Then the static functional (A20) reads

$$\mathcal{H}=\int d^3x\left\{\frac{1}{2}\mathring{\tau}\mathring{\phi}^2+\frac{1}{2}(\vec{\nabla}\mathring{\phi})^2+\frac{1}{2}\mathring{\vec{q}}_1\cdot\mathbf{a}_{11}\cdot\mathring{\vec{q}}_1+\vec{a}_{12}\cdot\mathring{\vec{q}}_1\mathring{q}_2+\frac{1}{2}a_{22}\mathring{q}_2^2+\frac{1}{2}a_j\vec{j}\cdot\vec{j}+\frac{1}{2}\mathring{\vec{\gamma}}_1\cdot\mathring{\vec{q}}_1\mathring{\phi}^2+\frac{1}{2}\mathring{\gamma}_2\mathring{q}_2\mathring{\phi}^2+\frac{\mathring{\tilde{u}}}{4!}\mathring{\phi}^4-\mathring{\vec{h}}_1\cdot\mathring{\vec{q}}_1-\mathring{h}_2\mathring{q}_2\right\}. \tag{3.4}$$

The coefficients $a_{ij}$ are related to the thermodynamic derivatives,

$$[\mathbf{a}_{11}]_{ij}=\frac{\rho}{RT}\left(\frac{\partial H_i}{\partial\mu_j}\right)^{(0)}_{\sigma,\rho},\quad a_{22}=\frac{1}{RT\rho}\left(\frac{\partial P}{\partial\rho}\right)^{(0)}_{\sigma,\vec{\mu}_m}, \tag{3.5}$$

$$\vec{a}_{12}=\frac{\rho}{RT}\left(\frac{\partial\vec{H}}{\partial\rho}\right)^{(0)}_{\sigma,\vec{\mu}_m}=\frac{1}{RT\rho}\left(\frac{\partial P}{\partial\vec{\mu}_m}\right)^{(0)}_{\sigma,\rho}, \tag{3.6}$$

where the superscript (0) indicates that the derivatives are uncritical background quantities. The components of the spin density are abbreviated as $[\vec{\mu}_m]_i\equiv\mu_i$. The coefficients (3.5) and (3.6) represent the zeroth order contribution to the static vertex functions $\mathring{\Gamma}_{q_iq_j}$. The fields $\mathring{\vec{h}}_1=\mathring{\vec{\gamma}}_1\langle 1/2\mathring{\phi}\rangle$ and $\mathring{h}_2=\mathring{\gamma}_2\langle 1/2\mathring{\phi}\rangle$ are chosen so as to compensate the finite expectation values of $\mathring{\vec{q}}_1$ and $\mathring{q}_2$. The spin density and $\mathring{\vec{q}}_1$ are odd quantities with respect to time inversion. The static functional has to be invariant under time inversion and under the change $\vec{H}^{\mathrm{ex}}\rightarrow-\vec{H}^{\mathrm{ex}}$. This means that the coupling $\mathring{\vec{\gamma}}_1(\vec{H}^{\mathrm{ex}})$ obeys $\mathring{\vec{\gamma}}_1(-\vec{H}^{\mathrm{ex}})=-\mathring{\vec{\gamma}}_1(\vec{H}^{\mathrm{ex}})$. At vanishing external magnetic field we have $\mathring{\vec{\gamma}}_1(\vec{H}^{\mathrm{ex}}=\vec{0})=\vec{0}$, and the static functional (3.4) simplifies to

$$\mathcal{H}=\int d^3x\left\{\frac{1}{2}\mathring{\tau}\mathring{\phi}^2+\frac{1}{2}(\vec{\nabla}\mathring{\phi})^2+\frac{1}{2}\mathring{\vec{q}}_1\cdot\mathbf{a}_{11}\cdot\mathring{\vec{q}}_1+\vec{a}_{12}\cdot\mathring{\vec{q}}_1\mathring{q}_2+\frac{1}{2}a_{22}\mathring{q}_2^2+\frac{1}{2}a_j\vec{j}\cdot\vec{j}+\frac{1}{2}\mathring{\gamma}_2\mathring{q}_2\mathring{\phi}^2+\frac{\mathring{\tilde{u}}}{4!}\mathring{\phi}^4-\mathring{h}_2\mathring{q}_2\right\}. \tag{3.7}$$

Then the spin density completely decouples from the order parameter.

The order parameter correlation function, calculated from Eq. (3.4) [for the cummulant $\langle\rangle_c$, see Eq. (A25)],

$$\langle\mathring{\phi}\mathring{\phi}\rangle_c=\frac{RT}{\rho}\left(\frac{\partial\sigma}{\partial T}\right)_{\vec{H},P} \tag{3.8}$$

is related to the specific heat at fixed pressure and magnetic field, which is strongly divergent with exponent $\gamma$, as expected. The correlations of the secondary densities,

$$\langle\mathring{q}_{1i}\mathring{q}_{1j}\rangle_c=\frac{RT}{\rho}\left(\frac{\partial\mu_i}{\partial H_j}\right)_{\sigma,P},\quad\langle\mathring{q}_2\mathring{q}_2\rangle_c=RT\rho\left(\frac{\partial\rho}{\partial P}\right)_{\sigma,\vec{H}}, \tag{3.9}$$

$$\langle \mathring{q}_{1i}\mathring{q}_2\rangle_c = \frac{RT}{\rho}\left(\frac{\partial\rho}{\partial H_i}\right)_{\sigma,P} = RT\rho\left(\frac{\partial\mu_i}{\partial P}\right)_{\sigma,\hat{H}}, \tag{3.10}$$

are weakly divergent (with exponent $\alpha$) [14]. The vertex functions are calculated from the one-particle irreducible graphs in a perturbative expansion in the couplings. The vertex functions are related to the correlations by

$$\begin{pmatrix} \mathring{\mathbf{\Gamma}}_{q_1q_1} & \mathring{\vec{\Gamma}}_{q_1q_2} \\ \mathring{\vec{\Gamma}}^T_{q_1q_2} & \mathring{\Gamma}_{q_2q_2} \end{pmatrix} = \begin{pmatrix} \langle \mathring{\mathbf{q}}_1\mathring{\mathbf{q}}_1\rangle_c & \langle \mathring{\vec{q}}_1\mathring{q}_2\rangle_c \\ \langle \mathring{q}_2\mathring{\vec{q}}_1\rangle_c & \langle \mathring{q}_2\mathring{q}_2\rangle_c \end{pmatrix}^{-1}, \tag{3.11}$$

where superscript "$T$" denotes the transpose of a vector. From the perturbation expansion one can see that the static vertex functions have the structure

$$\mathring{\mathbf{\Gamma}}_{q_1q_1} = \mathbf{a}_{11} - \frac{1}{2}\mathring{\vec{\gamma}}_1\mathring{\vec{\gamma}}_1 G(\mathring{u}), \tag{3.12}$$

$$\mathring{\vec{\Gamma}}_{q_1q_2} = \vec{a}_{12} - \frac{1}{2}\mathring{\vec{\gamma}}_1\mathring{\gamma}_2 G(\mathring{u}), \tag{3.13}$$

$$\mathring{\Gamma}_{q_2q_2} = a_{22} - \frac{1}{2}\mathring{\gamma}_2^2 G(\mathring{u}). \tag{3.14}$$

The function $G(\mathring{u})$ contains the contributions of the perturbation expansion in the static coupling $\mathring{u}$. This coupling is the fourth-order coupling of the $\phi^4$ model that is obtained when the secondary densities in Eq. (3.4) are eliminated in the partition function [15,12]. In Eqs. (3.12)–(3.14), $G$ is eliminated by inserting one of the equations into the two remaining equations. This leads to two relations between the background parameters $a_{ij}$ and the static vertex functions $\mathring{\Gamma}_{q_iq_j}$. Using the thermodynamic expressions (3.5)–(3.10), these may be expressed as

$$\left[\mathbf{a}_{11} - \frac{\mathbf{a}_{12}\mathring{\gamma}_1}{\mathring{\gamma}_2}\right]_{ij} = \frac{\rho}{RT}\left(\frac{\partial H_i}{\partial\mu_j}\right)_{\sigma,t}, \tag{3.15}$$

$$\vec{a}_{12} - a_{22}\frac{\mathring{\vec{\gamma}}}{\mathring{\gamma}_2} = \frac{1}{RT\rho}\left(\frac{\partial P}{\partial\vec{\mu}_m}\right)_{\sigma,t}, \tag{3.16}$$

$$\frac{\mathring{\vec{\gamma}}_1}{\mathring{\gamma}_2} = -\left(\frac{\partial\rho}{\partial\vec{\mu}_m}\right)_{\sigma,t}, \tag{3.17}$$

The index $t$ in Eqs. (3.15)–(3.17) denotes derivatives parallel to the critical line. These themodynamic derivatives remain finite at the critical point, and are treated as constants in the critical region.

Time inversion symmetry of the external magnetic field implies that the coupling between the order parameter and the secondary density vector $\mathring{\vec{q}}_1$ must have the structure $\mathring{\vec{\gamma}}_1 = \vec{H}^{\mathrm{ex}} f(|\vec{H}^{\mathrm{ex}}|^2)$ with a regular function $f$. Applying an external magnetic field in the $z$ direction, the only nonzero component of the vector $\mathring{\vec{\gamma}}_1$ is the $z$ component $\mathring{\vec{\gamma}}_1^T = (0,0,\gamma_z)$. From Eq. (3.17), it immediately follows that $(\partial\rho/\partial\mu_x)_{\sigma,t} = (\partial\rho/\partial\mu_y)_{\sigma,t} = 0$. Thus, along the critical line, the density and the pressure (which is the conjugated field) only depend on the $z$ component of the magnetization per unit mass $\mu_z$. Then Eq. (3.17) reduces to

$$\frac{\mathring{\gamma}_z}{\mathring{\gamma}_2} = -\left(\frac{\partial\rho}{\partial\mu_z}\right)_{\sigma,t}. \tag{3.18}$$

Inserting Eq. (3.18) into Eq. (3.16) shows that the vector $\vec{a}_{12}^T$ in Eq. (3.6) contains only a $z$ component. The relations (3.16) and (3.15) simplify to

$$a_{z2} - a_{22}\frac{\mathring{\gamma}_z}{\mathring{\gamma}_2} = \frac{1}{RT\rho}\left(\frac{\partial P}{\partial\mu_z}\right)_{\sigma,t} \tag{3.19}$$

and

$$a_{zz} - a_{z2}\frac{\mathring{\gamma}_z}{\mathring{\gamma}_2} = \frac{\rho}{RT}\left(\frac{\partial H_z}{\partial\mu_z}\right)_{\sigma,t}. \tag{3.20}$$

The coefficient $a_{22}$ is the $z$ component of the vector $\vec{a}_{12}^T$, and $a_{zz}$ is the $zz$ element of the matrix $a_{ij} = [\mathbf{a}_{11}]_{ij}$ in Eq. (3.5). Nondiagonal elements of this matrix vanish. Thus, for an external magnetic field in the $z$ direction, the static functional (3.4) reduces to

$$\begin{aligned}\mathcal{H} = \int d^3x \Bigg\{ & \frac{1}{2}\mathring{\tau}\mathring{\phi}^2 + \frac{1}{2}(\vec{\nabla}\mathring{\phi})^2 + \frac{1}{2}a_\perp\vec{q}_\perp\cdot\vec{q}_\perp + \frac{1}{2}a_{zz}\mathring{q}_{1z}^2 \\ & + a_{z2}\mathring{q}_{1z}\mathring{q}_2 + \frac{1}{2}a_{22}\mathring{q}_2^2 + \frac{1}{2}a_j\vec{j}\cdot\vec{j} + \frac{1}{2}\mathring{\gamma}_z\mathring{q}_{1z}\mathring{\phi}^2 \\ & + \frac{1}{2}\mathring{\gamma}_2\mathring{q}_2\mathring{\phi}^2 + \frac{\mathring{\tilde{u}}}{4!}\mathring{\phi}^4 - \mathring{h}_z\mathring{q}_{1z} - \mathring{h}_2\mathring{q}_2\Bigg\}.\end{aligned} \tag{3.21}$$

$\vec{q}_\perp^T = (q_{1x}, q_{1y})$ denotes the secondary density vector in the $x$-$y$ subspace. The coefficient,

$$a_\perp = \frac{\rho}{RT}\left(\frac{\partial H_x}{\partial\mu_x}\right)_\sigma = \frac{\rho}{RT}\left(\frac{\partial H_y}{\partial\mu_y}\right)_\sigma, \tag{3.22}$$

is the inverse magnetic susceptibility in the $x$-$y$ subspace, which thermodynamically decouples completely from the density-pressure subspace. Therefore no distinction is necessary between the derivative at constant density in Eq. (3.5) and the derivative along the critical line in Eq. (3.15). Thus there is only a single index $\sigma$ in Eq. (3.22). Due to the absence of a term in Eq. (3.21) which contains $\vec{q}_\perp$, as well as other densities, the corresponding static correlations are constant,

$$\langle q_{1x}q_{1x}\rangle_c = \langle q_{1y}q_{1y}\rangle_c = \frac{1}{a_\perp}. \tag{3.23}$$

The remaining nonzero correlations are those of the densities $q_{1z}$ and $q_2$ in Eqs. (3.9) and (3.10). As in binary mixtures, the secondary densities can be thermodynamically orthogonalized by introducing proper linear combinations

$$\begin{pmatrix}\mathring{q}_{1z}\\ \mathring{q}_2\end{pmatrix} = \begin{pmatrix}\bar{M}_{11} & \bar{M}_{12}\\ \bar{M}_{21} & \bar{M}_{22}\end{pmatrix}\begin{pmatrix}\mathring{m}_1\\ \mathring{m}_2\end{pmatrix}. \tag{3.24}$$

The elements of the transformation matrix are determined by two conditions: (i) the thermodynamic correlations of the secondary densities are diagonal; and (ii) the $\gamma$ coupling of $m_1$ with the order parameter vanishes. This leads to

$$\overline{M}_{11}=\frac{a_{zz}-a_{z2}\dfrac{\mathring{\gamma}_z}{\mathring{\gamma}_2}}{a_{zz}-2a_{z2}\dfrac{\mathring{\gamma}_z}{\mathring{\gamma}_2}+a_{z2}\left(\dfrac{\mathring{\gamma}_z}{\mathring{\gamma}_2}\right)^2}=\chi_{\sigma,t}^{-1}\left(\frac{\partial\mu_z}{\partial H_z}\right)_{\sigma,t}, \tag{3.25}$$

$$\overline{M}_{12}=-\frac{a_{z2}-a_{22}\dfrac{\mathring{\gamma}_z}{\mathring{\gamma}_2}}{a_{zz}-a_{z2}\dfrac{\mathring{\gamma}_z}{\mathring{\gamma}_2}}=-\frac{1}{\rho^2}\left(\frac{\partial P}{\partial H_z}\right)_{\sigma,t}, \tag{3.26}$$

$$\overline{M}_{21}=-\frac{\mathring{\gamma}_z}{\mathring{\gamma}_2}=\chi_{\sigma,t}^{-1}\left(\frac{\partial\rho}{\partial H_z}\right)_{\sigma,t}, \tag{3.27}$$

$$\overline{M}_{22}=1. \tag{3.28}$$

The thermodynamic derivatives on the right hand side in the above equations are obtained by inserting relations (3.15)–(3.17) into (3.25)–(3.27). We have also introduced

$$\chi_{\sigma,t}=\left(\frac{\partial\mu_z}{\partial H_z}\right)_{\sigma,t}+\frac{1}{\rho^2}\left(\frac{\partial P}{\partial H_z}\right)_{\sigma,t}\left(\frac{\partial\rho}{\partial H_z}\right)_{\sigma,t}. \tag{3.29}$$

The resulting static functional is then

$$\mathcal{H}=\int d^3x\left\{\frac{1}{2}\mathring{\tau}\mathring{\phi}+\frac{1}{2}(\vec{\nabla}\mathring{\phi})^2+\frac{1}{2}a_{m_1}\mathring{m}_1^2+\frac{1}{2}a_{m_2}\mathring{m}_2^2+\frac{1}{2}a_j\vec{j}\cdot\vec{j}+\frac{1}{2}a_\perp\vec{q}_\perp\cdot\vec{q}_\perp+\frac{1}{2}\mathring{\gamma}_{m_2}\mathring{m}_2\mathring{\phi}^2+\frac{\mathring{\tilde{u}}}{4!}\mathring{\phi}^4-\mathring{h}_{m_2}\mathring{m}_2\right\}. \tag{3.30}$$

Since $\mathring{m}_1$ appears to quadratic order only, the coefficient $a_{m_1}$ represents the complete static vertex function $\mathring{\Gamma}_{m_1m_1}$, and we have

$$\langle\mathring{m}_1\mathring{m}_1\rangle_c=\frac{1}{\mathring{\Gamma}_{m_1m_1}}=\frac{1}{a_{m_1}}=\frac{RT}{\rho}\chi_{\sigma,t}, \tag{3.31}$$

which does not contain a singular temperature dependence. The coefficient of the quadratic term of the secondary density $a_{m_2}=\mathring{\Gamma}^{(0)}_{m_2m_2}$ represents the zeroth order of the corresponding vertex function. The complete vertex function and the corresponding correlation are related to the thermodynamic derivatives by

$$\langle\mathring{m}_2\mathring{m}_2\rangle_c=\frac{1}{\mathring{\Gamma}_{m_2m_2}}=RT\rho^3\left(\frac{\partial H_z}{\partial P}\right)^2_{\sigma,t}\left[\left(\frac{\partial\mu_z}{\partial H_z}\right)_{\sigma,P}-\left(\frac{\partial\mu_z}{\partial H_z}\right)^2_{\sigma,t}\chi_{\sigma,t}^{-1}\right]. \tag{3.32}$$

Thus, the vertex function contains the magnetic susceptibility, which behaves like $(\partial\mu_z/\partial H_z)_{\sigma,P}\sim t^{-\alpha}$ in the asymptotic critical region.

Apart from the Gaussian term of the transverse spin components, which do not contribute to the critical behavior, the static functionals (3.4) and (3.30) are the same as those of binary liquid mixtures at the plait point [12]. All thermodynamic relations can be taken over from this system if one replaces the concentration $c$ and the chemical potential $\Delta$ of the liquid mixture by the spin density $\mu_z$ and the magnetic field $H_z$ of the magnetic liquid. Therefore the static critical behavior of the quantities in the magnetic liquid is the same as the static critical behavior of the corresponding quantities in the liquid mixture at the plait point.

## IV. DYNAMIC MODEL FOR HEISENBERG FLUIDS

### A. General relations

Using Zwanzig's projection operator method, one can derive dynamic equations that describe the time development of the slow densities (see Ref. [16]). Considering a set $\{\alpha_k(x,t)\}$ of slow densities, one obtains

$$\frac{\partial\alpha_i(x,t)}{\partial t}=V_i(\{\alpha_k(x,t)\})-\sum_j\Lambda_{ij}\frac{\delta\mathcal{H}(\{\alpha_k(x,t)\})}{\delta\alpha_j(x,t)}+\theta_i(x,t) \tag{4.1}$$

with $\Lambda_{ij}=-L_{ij}\nabla^2$ for conserved densities and $\Lambda_{ij}=L_{ij}$ for nonconserved densities. The term

$$V_i(\{\alpha_k(x,t)\})=\int dx'\sum_j\left[Q_{ij}(x,t;x',t')\frac{\delta\mathcal{H}(\{\alpha_k(x,t)\})}{\delta\alpha_j(x,t)}-\frac{\delta Q_{ij}(x,t;x',t')}{\delta\alpha_j(x,t)}\right] \tag{4.2}$$

constitutes the reversible part of the equations. The functions $Q_{ij}$ are determined by

$$Q_{ij}(x,t;x',t')=k_BT\{\alpha_i(x,t),\alpha_k(x',t')\}, \tag{4.3}$$

where the curly brackets {.,.} denote generalized Poisson brackets. $\mathcal{H}$ represents a static functional of the type discussed in Sec. II. The coefficients $\Lambda_{ij}$ are related to the Onsager coefficients of hydrodynamic theory. The functions $\theta_i(x,t)$ include the contributions of the fast variables, which will be considered as stochastic forces. Assuming that these stochastic forces $\theta_i(x,t)$ are determined by a Markov process, the coefficients $\Lambda_{ij}$ fulfill the relations

$$\langle\theta_i(x,t)\theta_j(x',t')\rangle=\Lambda_{ij}\delta(x-x')\delta(t-t'). \tag{4.4}$$

### B. Dynamic model equations

In the case of Heisenberg fluids the densities $\{\alpha_k(x,t)\}$ are represented by the hydrodynamic densities $s(x,t)$, $\rho(x,t)$, $\vec{m}(x,t)$, and $\vec{j}(x,t)$ considered in Sec. II. The generalized Poisson brackets (2.5)–(2.9) determine the reversible part (4.2), which is independent of the external magnetic field. The dissipative part described by the coefficients $\Lambda_{ij}$ is different depending on whether the external magnetic field is zero or nonzero. The coefficients $\Lambda_{ij}$ are related to the Onsager coefficients introduced in Sec. II. They may be identified by comparing the general structure of the dynamic equations with the linearized hydrodynamic equations. These are given in Eqs. (2.35) and (2.37) for the zero external magnetic field, and in Eqs. (2.48)–(2.52) for a homogeneous external magnetic field in the $z$ direction.

In the case of a homogeneous external magnetic field in $z$ direction we obtain

$$\frac{\partial s}{\partial t}=k_B\gamma\nabla^2\frac{\delta\mathcal{H}}{\delta s}+k_BT\beta\nabla^2\frac{\delta\mathcal{H}}{\delta m_z}+V_s(\{\alpha_k\})+\theta_s\,, \tag{4.5}$$

$$\frac{\partial m_z}{\partial t}=k_BT\beta\nabla^2\frac{\delta\mathcal{H}}{\delta s}+k_BT\alpha\nabla^2\frac{\delta\mathcal{H}}{\delta m_z}+V_{m_z}(\{\alpha_k\})+\theta_{m_z}, \tag{4.6}$$

$$\frac{\partial\rho}{\partial t}=V_\rho(\{\alpha_k\}), \tag{4.7}$$

$$\frac{\partial\vec{j}}{\partial t}=k_BT\left(\zeta^{(0)}+\frac{4}{3}\,\overline{\eta}^{(0)}\right)\nabla^2\frac{\delta\mathcal{H}}{\delta\vec{j}_l}+k_BT\,\overline{\eta}^{(0)}\nabla^2\frac{\delta\mathcal{H}}{\delta\vec{j}_t}+\vec{V}_j'(\{\alpha_k\})+\vec{\theta}_j\,, \tag{4.8}$$

$$\frac{\partial\vec{m}_\perp}{\partial t}=-k_BT\alpha_R\frac{\delta\mathcal{H}}{\delta\vec{m}_\perp}+\vec{V}_{m_\perp}(\{\alpha_k\})+\vec{\theta}_\perp\,. \tag{4.9}$$

The Onsager coefficients $\alpha$, $\beta$, $\gamma$, $\zeta^{(0)}$, and $\overline{\eta}^{(0)}$ have been introduced in Eqs. (2.43)–(2.46). The superscript (0) indicates uncritical background values. The equation of mass conservation (4.7) does not contain dissipative terms (Onsager coefficients) or stochastic forces. In Eq. (4.8), the momentum density has been separated into a longitudinal component $\vec{j}_l$ defined by $\vec{\nabla}\times\vec{j}_l=\vec{0}$, which is necessary for the description of the sound mode, and a transverse component $\vec{j}_t$ defined by $\vec{\nabla}\cdot\vec{j}_t=0$, which is necessary for the description of the shear mode. Formally, we have added a relaxation equation for the transverse components of the magnetization, even though it does not belong to the set of densities of conserved variables. It turns out that the reversible couplings to the other densities are irrelevant for the critical behavior.

In the case of zero external magnetic field, the Onsager coefficient $\beta=0$ and the heat conduction mode and the spin diffusion mode decouple. Equations (4.5), (4.6), and (4.9) reduce to

$$\frac{\partial s}{\partial t}=k_{B\gamma}\nabla^2\frac{\delta\mathcal{H}}{\delta s}+V_s(\{\alpha_k\})+\theta_s\,, \tag{4.10}$$

$$\frac{\partial\vec{m}}{\partial t}=k_BT\alpha\nabla^2\frac{\delta\mathcal{H}}{\delta\vec{m}}+\vec{V}_m(\{\alpha_k\})+\vec{\theta}_m\,, \tag{4.11}$$

while the equations for the mass density (4.7) and the momentum density (4.8) remain unchanged. The reversible contributions determined by the generalized Poisson brackets are

$$V_s(\{\alpha_k\})=-k_BT\vec{\nabla}\cdot\left(s\frac{\delta\mathcal{H}}{\delta\vec{j}}\right), \tag{4.12}$$

$$\vec{V}_m(\{\alpha_k\})=-k_BT\sum_{i=x,y,z}\nabla_i\vec{m}\frac{\delta\mathcal{H}}{\delta j_i}+k_BT\vec{m}\times\frac{\delta\mathcal{H}}{\delta\vec{m}}, \tag{4.13}$$

$$V_\rho(\{\alpha_k\})=-k_BT\vec{\nabla}\cdot\left(\rho\frac{\delta\mathcal{H}}{\delta\vec{j}}\right), \tag{4.14}$$

$$\begin{aligned}\vec{V}_j'(\{\alpha_k\})=&-k_BT\left(s\vec{\nabla}\frac{\delta\mathcal{H}}{\delta s}+\rho\vec{\nabla}\frac{\delta\mathcal{H}}{\delta\rho}+\sum_{i=x,y,z}m_i\vec{\nabla}\frac{\delta\mathcal{H}}{\delta m_i}\right)\\&-k_BT\sum_{i=x,y,z}\left(j_i\vec{\nabla}\frac{\delta\mathcal{H}}{\delta j_i}+\nabla_i\vec{j}\frac{\delta\mathcal{H}}{\delta j_i}\right).\end{aligned} \tag{4.15}$$

Introducing the order parameter (3.1) and the secondary densities (3.2) and (3.3), we obtain the final dynamic equations for the case of a homogeneous external magnetic field in the $z$ direction,

$$\frac{\partial\mathring{\phi}}{\partial t}=\mathring{\Gamma}\nabla^2\frac{\delta\mathcal{H}}{\delta\mathring{\phi}}+\mathring{L}\nabla^2\frac{\delta\mathcal{H}}{\delta\mathring{q}_1}+\mathring{L}_\phi\nabla^2\frac{\delta\mathcal{H}}{\delta\mathring{q}_2}-\mathring{g}\frac{\delta\mathcal{H}}{\delta\vec{J}}\cdot\vec{\nabla}\mathring{\phi}+\theta_\phi\,, \tag{4.16}$$

$$\frac{\partial\mathring{q}_1}{\partial t}=\mathring{L}\nabla^2\frac{\delta\mathcal{H}}{\delta\mathring{\phi}}+\mathring{\mu}\nabla^2\frac{\delta\mathcal{H}}{\delta\mathring{q}_1}+\mathring{L}_{12}\nabla^2\frac{\delta\mathcal{H}}{\delta\mathring{q}_2}-\mathring{g}\frac{\delta\mathcal{H}}{\delta\vec{J}}\cdot\vec{\nabla}\mathring{q}_1+\theta_{q1}\,, \tag{4.17}$$

$$\begin{aligned}\frac{\partial\mathring{q}_2}{\partial t}=&\mathring{L}_\phi\nabla^2\frac{\delta\mathcal{H}}{\delta\mathring{\phi}}+\mathring{L}_{12}\nabla^2\frac{\delta\mathcal{H}}{\delta\mathring{q}_1}+\mathring{\lambda}\nabla^2\frac{\delta\mathcal{H}}{\delta\mathring{q}_2}-\mathring{c}\vec{\nabla}\cdot\frac{\delta\mathcal{H}}{\delta\vec{j}_l}\\&-\mathring{g}\vec{\nabla}\cdot\left(\mathring{q}_2\frac{\delta\mathcal{H}}{\delta\vec{j}}\right)-\mathring{g}_l\mathring{\phi}\vec{\nabla}\cdot\frac{\delta\mathcal{H}}{\delta\vec{J}_l}+\theta_{q2}\,,\end{aligned} \tag{4.18}$$

$$\begin{aligned}\frac{\partial\vec{j}}{\partial t}=&\mathring{\lambda}_l\nabla^2\frac{\delta\mathcal{H}}{\delta\vec{j}_l}+\mathring{\lambda}_t\nabla^2\frac{\delta\mathcal{H}}{\delta\vec{j}_t}-\mathring{c}\vec{\nabla}\frac{\delta\mathcal{H}}{\delta\mathring{q}_2}\\&-\mathring{g}\left[\mathring{q}_2\vec{\nabla}\frac{\delta\mathcal{H}}{\delta\mathring{q}_2}-\frac{\delta\mathcal{H}}{\delta\mathring{\phi}}\vec{\nabla}\mathring{\phi}-\frac{\delta\mathcal{H}}{\delta\mathring{q}_1}\vec{\nabla}\mathring{q}_1-\sum_{i=x,y}\frac{\delta\mathcal{H}}{\delta\mathring{q}_i}\vec{\nabla}\mathring{q}_i\right]\\&-\mathring{g}\sum_{i=x,y,z}\left(j_i\vec{\nabla}\frac{\delta\mathcal{H}}{\delta j_i}+\nabla_i\vec{J}\frac{\delta\mathcal{H}}{\delta j_i}\right)-\mathring{g}_l\vec{\nabla}\cdot\left(\mathring{\phi}\frac{\delta\mathcal{H}}{\delta\vec{J}_l}\right)+\vec{\theta}_j\,,\end{aligned} \tag{4.19}$$

TABLE III. Cutoff dimensions of the kinetic coefficients.c

| Cutoff dim. | Kinetic coeff. |
|---|---|
| $\Lambda^0$ | $\mathring{\Gamma}$ |
| $\Lambda^1$ | $\mathring{L},\mathring{L}_\phi$ |
| $\Lambda^2$ | $\mathring{L}_{12},\mathring{\lambda},\mathring{\lambda}_l,\mathring{\lambda}_t$ |
| $\Lambda^4$ | $\alpha_\perp$ |

$$\frac{\partial \vec{q}_\perp}{\partial t} = -\alpha_\perp \frac{\delta \mathcal{H}}{\delta \vec{q}_\perp} - \mathring{g}\left(\frac{\delta \mathcal{H}}{\delta \vec{J}}\cdot\vec{\nabla}\right)\mathring{\vec{q}}_\perp + \omega_L \vec{e}_z \times \frac{\delta \mathcal{H}}{\delta \mathring{\vec{q}}_1} + \mathring{g}_L \mathring{\vec{q}}_1$$

$$\times \frac{\delta \mathcal{H}}{\delta \mathring{\vec{q}}_1} + \mathring{g}_L' \mathring{\phi} \vec{e}_z \times \frac{\delta \mathcal{H}}{\delta \mathring{\vec{q}}_1} + \vec{\theta}_{q\perp} . \tag{4.20}$$

In the last equation we have introduced the three-dimensional vector $\mathring{\vec{q}}_1^{\,T} \equiv (\vec{q}_\perp^{\,T}, \mathring{q}_1)$. The corresponding static functional $\mathcal{H}$ is given by Eq. (3.21). Only three out of the six kinetic coefficients appearing in Eqs. (4.16)–(4.20) are independent. They are related to the Onsager coefficients introduced in Eqs. (2.43) and (2.44) by

$$\mathring{\Gamma} = \frac{R\gamma}{\rho^2}, \quad \mathring{L} = \frac{RT}{\rho^2}\left[\beta + \left(\frac{\partial \mu_z}{\partial \sigma}\right)_{\vec{H},P} \frac{\gamma}{T}\right], \tag{4.21}$$

$$\mathring{\mu} = \frac{RT}{\rho^2}\left[\alpha - 2\left(\frac{\partial \mu_z}{\partial \sigma}\right)_{\vec{H},P} \beta + \left(\frac{\partial \mu_z}{\partial \sigma}\right)^2_{\vec{H},P} \frac{\gamma}{T}\right]. \tag{4.22}$$

The remaining three coefficients are expressed by these three coefficients. We have

$$\mathring{L}_\phi = -\left(\frac{\partial \rho}{\partial \sigma}\right)_{\vec{H},P} \mathring{\Gamma}, \quad \mathring{\lambda} = \left(\frac{\partial \rho}{\partial \sigma}\right)^2_{\vec{H},P} \mathring{\Gamma}, \quad \mathring{L}_{12} = -\left(\frac{\partial \rho}{\partial \sigma}\right)_{\vec{H},P} \mathring{L}. \tag{4.23}$$

The relaxation coefficient in Eq. (4.20) is $\alpha_\perp = RT\alpha_R/\rho$. The linear mode coupling parameters are

$$\mathring{c} = RT\rho, \quad \omega_L = \frac{RT}{\rho}\mu_z, \tag{4.24}$$

while the nonlinear mode couplings are defined as

$$\mathring{g} = \frac{RT}{\sqrt{N_A}}, \quad \mathring{g}_l = \frac{RT}{\sqrt{N_A}}\left(\frac{\partial \rho}{\partial \sigma}\right)_{\vec{H},P}, \tag{4.25}$$

$$\mathring{g}_L = \frac{RT}{\rho\sqrt{N_A}}, \quad \mathring{g}_L' = \frac{RT}{\rho\sqrt{N_A}}\left(\frac{\partial \mu_z}{\partial \sigma}\right)_{\vec{H},P}. \tag{4.26}$$

From power counting arguments we obtain the cutoff ($\Lambda$) dimensions of the kinetic coefficients, as shown in Table III. Within the perturbation expansion the kinetic coefficients appear in the form of time scale ratios, which determine which contributions to the critical behavior are relevant and irrelevant. The time scale ratios and their cutoff dimension are shown in Table IV. From this table it is seen that the set of time scale ratios in the second row has a cutoff dimension of $\Lambda^{-2}$, which means that contributions proportional to these time scale ratios are irrelevant. The same is true for the time scale ratios in the last row which is proportional to $\Lambda^{-4}$. Only the ratios with a cutoff dimension of $\Lambda^0$, in the first row in Table IV, are relevant in the critical region. In order to examine the influence of the transverse spin components to the remaining densities in the perturbation expansion, we have to consider the mode coupling terms, which include the transverse spin component and other densities. The cutoff dimensions of the mode couplings are listed in Table V.

TABLE IV. Cutoff dimensions of the time scale ratios.

| Cutoff dim. | Time scale ratio |
|---|---|
| $\Lambda^0$ | $\frac{\mathring{L}}{\sqrt{\mathring{\Gamma}\mathring{\mu}}}, \frac{\mathring{L}_\phi}{\sqrt{\mathring{\Gamma}\mathring{\lambda}}}, \frac{\mathring{L}_{12}}{\sqrt{\mathring{\lambda}\mathring{\mu}}}$ |
| $\Lambda^{-2}$ | $\frac{\mathring{\Gamma}}{\mathring{\mu}}, \frac{\mathring{\Gamma}}{\mathring{\lambda}}, \frac{\mathring{\Gamma}}{\mathring{\lambda}_l}, \frac{\mathring{\Gamma}}{\mathring{\lambda}_t}$ |
| $\Lambda^{-4}$ | $\frac{\mathring{\Gamma}}{\alpha_\perp}$ |

Within perturbation theory, the vertex functions can be expressed in terms of the time scale ratios introduced in Table IV and mode coupling parameters introduced in Eqs. (4.25) and (4.26), namely,

$$\frac{\mathring{g}^2}{\mathring{\Gamma}\mathring{\lambda}_t}, \quad \frac{\mathring{g}_l^2}{\mathring{\lambda}\mathring{\lambda}_l}, \quad \frac{\mathring{g}_L^2}{\alpha_\perp}, \tag{4.27}$$

which are all proportional to $\Lambda^{\epsilon/2}$. Only those terms which include relevant time scale ratios contribute to the critical behavior.

In the dynamic equations (4.16)–(4.20) some further terms do not lead to relevant contributions. First, Eq. (4.19) for the momentum density and Eq. (4.20) for the transverse spin components are coupled by $\mathring{g}\Sigma_{i=x,y}(\delta\mathcal{H}/\delta\mathring{q}_i)\vec{\nabla}\mathring{q}_i$ and $\mathring{g}[(\delta\mathcal{H}/\delta\vec{J})\cdot\vec{\nabla}]\mathring{\vec{q}}_\perp$. These terms lead to vertices with two transverse spin component lines in the dynamic functional. Thus, contributions to each of the two two-point-vertex functions always must have at least one internal transverse spin line. As a result, one obtains a coupling $\mathring{g}^2/\alpha_\perp \sim \Lambda^{-2+\epsilon}$ that is irrelevant. Combinations of these two vertices only produce irrelevant contributions. Second, inserting for $\mathcal{H}$ the terms $\mathring{g}_L\mathring{\vec{q}}_1 \times (\delta\mathcal{H}/\mathring{\delta}\vec{q}_1)$ and $\mathring{g}_L'\mathring{\phi}\vec{e}_z \times (\delta\mathcal{H}/\mathring{\delta}\vec{q}_1)$ in Eq. (4.20), leads to expression $1/2\mathring{g}_L\mathring{\gamma}_z\mathring{\phi}^2(\mathring{q}_y, -\mathring{q}_x, 0)^T$ and

TABLE V. Cutoff dimensions of the mode couplings. We have introduced $\epsilon = 4-d$.

| Mode coupling | Cutoff dim. |
|---|---|
| $\mathring{c}$ | $\Lambda^3$ |
| $\mathring{\omega}_L$ | $\Lambda^4$ |
| $\mathring{g}$ | $\Lambda^{1+\epsilon/2}$ |
| $\mathring{g}_l,\mathring{g}_L$ | $\Lambda^{2+\epsilon/2}$ |
| $\mathring{g}_L'$ | $\Lambda^{3+\epsilon/2}$ |

TABLE VI. One loop expressions of the dynamic $\zeta$ functions and amplitude functions.

| Function | One loop expression |
|---|---|
| $\zeta_{\Gamma}^{(d)}$ | $-\frac{3}{4}f_t^2$ |
| $\zeta_{\lambda t}$ | $-\frac{f_t^2}{24(1-w_3^2)}$ |
| $G$ | $-\frac{f_t^2}{16}$ |
| $E_t$ | $-\frac{f_t^2}{36(1-w_3^2)}$ |

$a_{\perp}\mathring{g}_L'\mathring{\phi}(\mathring{q}_y,-\mathring{q}_x,0)^T$. These axial vectors do not give perturbational contributions to the vertex functions with the polar vectors from above. Therefore, these two vertices contribute to the $q_{\perp}-q_{\perp}$-two-point function only. Thus we can conclude that in the perturbation expansion, the transverse spin densities completely decouple from the remaining densities, and it is sufficient to consider only the equations for $\mathring{\phi}$, $\mathring{q}_1$, $\mathring{q}_2$, and $\vec{j}$. Neglecting the $\mathring{g}\Sigma_{i=x,y}(\delta\mathcal{H}/\delta\mathring{q}_i)\vec{\nabla}\mathring{q}_i$ in Eq. (4.19) and $\mathring{g}[(\delta\mathcal{H}/\delta\vec{j})\cdot\vec{\nabla}]\mathring{\vec{q}}_{\perp}$ in Eq. (4.20) the dynamic equations (4.16)–(4.19) have the same form as in the case of binary liquid mixtures at the plait point [12]. Having considered the dynamical critical phenomena in the Heisenberg liquid, one can now proceed along the lines developed in Refs. [17,18]. From the dynamical functional one calculates the dynamical correlations by a perturbation expansion in the mode couplings and the static couplings.

## V. HYDRODYNAMIC TRANSPORT COEFFICIENTS

In the previous section we have shown that the dynamic model for the Heisenberg fluid in an external homogeneous magnetic field reduces to the model known from binary liquid mixtures. The relations of the hydrodynamic transport coefficients to the model parameters and their explicit calculation within renormalization group theory for a liquid mixture have been treated in Ref. [12]. Therefore we shall not repeat these steps in the present paper, and only summarize the results we obtained.

### A. Critical heat and spin diffusion

From calculations analogous to those for binary liquid mixtures at the plait point [12], we obtain, at zero frequency, the thermal conductivity,

$$\frac{\kappa_T(l)}{\rho T}=\frac{1}{D_m(l)}\left(\frac{\rho}{RT}\right)^2\left(\frac{\partial H_z}{\partial\mu_z}\right)_{T,P}\mathring{\mu}\Gamma^{(d)}(l)[1-w_3^2(l)+G(w_3^2(l),f_t(l),u(l))], \tag{5.1}$$

where $G(w_3,f_t,u)$ contains the contributions of the perturbation expansion of the amplitude function for the dynamic order parameter two-point function. The corresponding one loop expression is listed in Table VI. The quantity $w_3$ denotes the time scale ratio defined by the renormalized model kinetic coefficients $\Gamma$, $L$, and $\mu$,

$$w_3^2(l)=\frac{L^2(l)}{\Gamma(l)\mu(l)}. \tag{5.2}$$

The flow of the kinetic coefficient $\Gamma$ (the kinetic coefficient $\Gamma^{(d)}$ follows from $\Gamma$ by separating the static contributions), and of the time scale ratio $w_3$, are determined by the flow equations,

$$l\frac{d\Gamma^{(d)}(l)}{dl}=\Gamma^{(d)}(l)\zeta_{\Gamma}^{(d)},\quad l\frac{dw_3(l)}{dl}=-\frac{1}{2}w_3(l)\zeta_{\Gamma}^{(d)}, \tag{5.3}$$

with proper initial conditions. The functions $\zeta_{\phi}$ and $\zeta_{\Gamma}^{(d)}$ are calculated from the static and dynamic renormalization factors within the perturbation expansion, and their one-loop expressions are given in Table VI. The static $\zeta$ function, $\zeta_{\phi}(u)$, depends only on the fourth-order coupling of the $\phi^4$ model $u(l)$. It is obtained from the renormalization of the $k^2$ terms of the static two-point order parameter vertex function. The dynamic $\zeta$ function, $\zeta_{\Gamma}^{(d)}(w_3,f_t)$, which is obtained from the renormalization of the $k^2$ terms of the dynamic order parameter two-point function, depends on the time scale ratio $w_3$ and the mode coupling parameter

$$f_t^2(l)=\frac{g^2(l)}{\Gamma(l)\lambda_t(l)}. \tag{5.4}$$

This mode coupling is determined by the flow equation

$$l\frac{df_t(l)}{dl}=-\frac{1}{2}f_t(l)(\epsilon+\zeta_{\phi}+\zeta_{\Gamma}^{(d)}+\zeta_{\lambda_t}). \tag{5.5}$$

The function $\zeta_{\lambda_t}(w_3,f_t)$ is obtained from the renormalization of the $k^2$ terms of the transverse momentum current density two-point function. The reduced temperature $t$ enters the transport coefficients through the matching condition,

$$\frac{\xi^{-2}(t)}{(\xi_0^{-1}l)^2}=1, \tag{5.6}$$

with the correlation length $\xi(t)$. The spin diffusion coefficient $D_m$ appearing in Eq. (5.1) is given by

$$D_m(l)=\frac{\rho}{RT}\left(\frac{\partial H_z}{\partial\mu_z}\right)_{T,P}\{\mathring{\mu}+2a\mathring{L}+a^2\Gamma^{(d)}(l)\times[1+G(w_3(l),f_t(l),u(l))]\}. \tag{5.7}$$

The parameter $a$ in the above equation is determined by the thermodynamic derivative,

$$a=\left(\frac{\partial\mu_z}{\partial\sigma}\right)_{H_z,P}, \tag{5.8}$$

which depends only weakly on the temperature and therefore can be considered as a constant in the critical region. The thermomagnetic diffusion ratio related to the cross effect between heat conduction and spin diffusion is

TABLE VII. Asymptotic temperature dependence of the hydrodynamic transport coefficients of a Heisenberg fluid in a homogeneous magnetic field.

| TC | Crit. Beh. | Remark |
|---|---|---|
| $\kappa_T$<br>thermal conductivity | Finite<br>enhancement | Constant magnetization current<br>depend. on background par. |
| $D_m$<br>magnetic diffusion | Goes to zero<br>$\xi^{-\gamma/\nu+x_\lambda}$ | $\gamma/\nu\simeq 1.8$<br>$x_\lambda=0.916$ (2 loop) |
| $k_T$<br>thermal magnetic<br>diffusion ratio | Diverging<br>$\xi^{\gamma/\nu-x_\lambda}$ | $\sim D_m^{-1}$ |
| $\overline{\eta}$<br>viscosity | Diverging<br>$\xi^{-x_\eta}$ | $x_\eta=0.065$ (2 loop) |
| $c_s$ sound velocity | Finite | frequency zero |
| $\alpha$ sound absorption | $\omega^2\xi^{z-\alpha\nu}$ | small frequency; $z\sim 3$ |

$$\frac{k_T(l)}{T}=-\frac{\rho}{RT}\frac{1}{D_m(l)}\left[\mathring{L}+\frac{\mathring{\mu}}{a}\right]-\left(\frac{\partial\mu_z}{\partial T}\right)_{\sigma,P}. \tag{5.9}$$

The asymptotic critical behavior resulting from these expressions for the transport coefficients (5.1), (5.7), and (5.9) is listed in Table VII.

### B. Critical viscosity

The shear viscosity can be calculated as in pure fluids and liquid mixtures. At zero frequency the resulting expression for the shear viscosity is

$$\overline{\eta}(t)=\frac{1}{RT}(\kappa l)^2\lambda_t(l)[1+E_t(f_t(l),w_3(l))]. \tag{5.10}$$

The amplitude function $E_t(f_t(l),w_3(l))$ contains the contributions of the perturbation expansion in the couplings (static and dynamic). In one loop order this function is given in Table VI and depends only on the mode coupling and the time ratio $w_3$. The kinetic coefficient $\lambda_t(l)$ is determined by the flow equation,

$$l\frac{d\lambda_t}{dl}=\lambda_t(l)(-2+\zeta_{\lambda t}), \tag{5.11}$$

while $w_3(l)$ and $f_t(l)$ are calculated from Eqs. (5.3) and (5.5).

### C. Critical sound propagation

Neglecting the contribution of the viscosity and thermal conductivity the sound velocity and the sound attenuation can be written simply as

$$c_s^2(t,\omega)=\mathrm{Re}[\mathcal{C}_s^2(t,\omega)], \tag{5.12}$$

$$D_s(t,\omega)=-\frac{1}{\omega}\mathrm{Im}[\mathcal{C}_s^2(t,\omega)], \tag{5.13}$$

where $t$ denotes the reduced temperature, and $\mathcal{C}_s$ is given by

$$\mathcal{C}_s^2(t,\omega)=a_j[a_1\mathring{c}_1^2+\mathring{c}_2^2\mathring{\Gamma}^{(s)}_{m_2m_2}\mathcal{V}_s(v(t,l),\tilde{w}(l))]. \tag{5.14}$$

The thermodynamic expressions for the static vertex functions $a_1$ and $\mathring{\Gamma}^{(s)}_{m_2m_2}$ have been given in Eqs. (3.31) and (3.32). The parameters $\mathring{c}_1^2$ and $\mathring{c}_2^2$ are related to the linear mode coupling parameter $\mathring{c}^2$ [Eq. (4.24)] by

$$\mathring{c}_1^2=\frac{1}{\rho^2}\left(\frac{\partial P}{\partial H_z}\right)_{\sigma,t}\mathring{c}^2,\quad \mathring{c}_2^2=\frac{1}{\chi_{\sigma,t}}\left(\frac{\partial\mu_z}{\partial H_z}\right)_{\sigma,t}\mathring{c}^2. \tag{5.15}$$

The complex function $\mathcal{V}_s$ can be written as

$$\mathcal{V}_s(v(t,l),\tilde{w}(l))=\frac{1+\gamma^2(l)F_+^{(s)}(u(l))}{1+\gamma^2(l)F_+(v(t,l),\tilde{w}(l))}. \tag{5.16}$$

The static parameter $\gamma^2$ is related to the corresponding parameter in the static functional (3.30) by

$$\gamma^2=\frac{\gamma_{m_2}^2}{a_2}. \tag{5.17}$$

The amplitude function of the specific heat $F_+^{(s)}$ and the dynamic amplitude function $F_+$ can be calculated within perturbation expansion. At vanishing frequency, $F_+$ turns into the amplitude functions of the specific heat. Thus we have

$$\lim_{\omega\to 0}F^+(v,\tilde{w})=F_+^{(s)}(u), \tag{5.18}$$

with $u$ being the static coupling of the $\phi^4$ model. The unrenormalized counterpart $\mathring{u}$ is related to parameters of Eq. (3.30) via

$$\mathring{u}=\mathring{\tilde{u}}-3\frac{\mathring{\gamma}_{m_2}^2}{a_2}. \tag{5.19}$$

The parameter $v$, which depends explicitly on the correlation length $\xi(t)$, and the parameter $\tilde{w}$, which depends explicitly on the frequency $\omega$, are defined by

$$v(t,l)=\frac{\xi^{-2}(t)}{(\xi_0^{-1}l)^2}, \quad \tilde{w}(l)=\frac{\omega}{2\Gamma(l)(1-w_3^2(l))(\xi_0^{-1}l)^4}, \tag{5.20}$$

$$l\frac{d\Gamma(l)}{dl}=\Gamma(l)(\zeta_\phi+\zeta_\Gamma^{(d)}). \tag{5.21}$$

The flow parameter $l$ is connected to the reduced temperature by a matching condition of the form

$$\left|\left(\frac{v(t,l)}{2}\right)^2+i\tilde{w}(l)\right|=\frac{1}{4}, \tag{5.22}$$

which implicitly defines a function $l(t,\omega)$. In the limit $\omega\to 0$, the relation reduces to the static matching condition Eq. (5.6), which is equivalent to setting $v=1$.

## VI. CONCLUSION

The dynamical critical properties near the gas-liquid critical line (as a function of the magnetic field) of the Heisenberg liquid in a magnetic field turns out to remain in the same universality class as the gas-liquid critical point in a pure nonmagnetic liquid. The nonuniversal properties, however, can be identified with those of a liquid mixture near the plait point. This means that the thermal conductivity of the liquid at constant magnetization current is nondivergent but enhanced, while the magnetic diffusion coefficient goes to zero and the thermomagnetic ratio diverges (see Table VII). This critical behavior of the dynamic magnetic properties is due to the coupling of magnetization to the order parameter at the gas-liquid transition in the finite static magnetic field. In the case when the magnetic field goes to zero, the amplitude of the divergence of the Onsager coefficient corresponding to the spin diffusion goes to zero, and the magnetic susceptibility becomes uncritical. In this limit the thermomagnetic diffusion ratio goes to zero, whereas the enhancement of thermal conductivity increases and crosses over into the divergent behavior of a pure nonmagnetic fluid.

Regarding the sound mode, the sound velocity at zero frequency is finite at $T_c$. At the plait point in liquid mixtures, the magnitude of the sound velocity at $T_c$ might be very small, whereas at the consolute point it is very large, and this influences the observable behavior near the critical point [12]. Therefore the observable critical behavior of the Heisenberg liquid depends on the (so far unknown) magnitude of the sound velocity at $T_c$. In the limit of zero external magnetic field, the sound velocity goes to zero at $T_c$.

All of these predictions may be tested in computer simulations. The results presented here apply to magnetic liquids irrespective of the existence of a ferromagnetic phase in the field free case. Thus, there may be a wider class of suitable liquids where real experiments can test our findings. So far, no experiments on the dynamical critical behavior near the gas-liquid phase transition for magnetic fluid systems [19–21] interacting via short range Heisenberg force have been performed.

Quite different critical behavior is to be expected at the magnetic phase transition in zero field. In this case one expects the critical magnetic transport properties to belong to the universality class of a solid ferromagnet, while the transport properties of the liquid are not affected by the magnetic transition. However, the situation may become more complicated depending on the strength of the magnetic relative to the nonmagnetic interactions. Multicritical behavior might also be possible and further work has to be done to consider this possibility.

So far, only the situation in a *homogeneous* external field has been considered. In the case of an *inhomogeneous* external field, additional dynamic couplings are present, involving the gradient of the external field. In consequence, transport coefficients such as viscosity, sound velocity, sound absorption, etc., may become anisotropic (see, e.g., Ref. [9]). This is also expected to influence fluctuation effects.

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## APPENDIX: GENERAL THEORY

The static critical behavior of the Heisenberg fluid is obtained from the static functional, from which the static correlation functions are calculated. These determine the critical behavior of the thermodynamic derivatives. The derivation of the static functional follows the same steps as in the case of binary liquid mixtures [15,12] and normal fluids [22]. In the following, we sketch the general steps used to obtain such a static functional for the description of the critical behavior.

Let the system be characterized by $N+1$ extensive local densities per volume $\alpha_0(x),\ldots,\alpha_N$. The function $\alpha_0(x)$ represents the entropy density, while the remaining densities, which depend on the specific system are the mass density, the momentum density, and the magnetization density for the Heisenberg liquid. The intensive external fields conjugate to the densities are denoted by $F_0(x),\ldots,F_N(x)$. $F_0(x)$ represents the local temperature, the remaining fields correspond to the local chemical potential, the local velocity, and the magnetic field for the Heisenberg liquid. In the case of local thermodynamic equilibrium, a local probability function

$$w_{\text{loc}}=\frac{1}{\mathcal{N}}\exp\left[-\int d^3x\,\frac{\Omega(x)}{k_BT(x)}\right] \tag{A1}$$

is introduced, which is found from the microscopic Hamiltonian by means of nonequilibrium thermodynamics [23]. $\mathcal{N}$ is a suitable normalization constant. The local thermodynamic potential $\Omega(x)$ is determined by the Legendre transform,

$$\Omega(x)=e(\{\alpha_k(x)\})-\sum_{i=0}^{N}\alpha_i(x)F_i(x), \tag{A2}$$

where $e(\{\alpha_k(x)\})$ is the local total energy density, which is a function of the extensive densities. The function $\Omega(x)$ in Eq. (A2) is interpreted as the local Gibb's free energy density, which is a function of the conjugate fields $F_i$. In the following, we assume that the extensive densities fluctuate around a spatially constant mean value. Thus, we can write

$$\alpha_i(x)=\alpha_i+\Delta\alpha_i(x), \quad i=0,\ldots,N \tag{A3}$$

$$e(\{\alpha_k(x)\})=e+\Delta e(\{\Delta\alpha_k(x)\}). \tag{A4}$$

Let the external fields vary as

$$F_i(x)=F_i+\delta F_i(x), \quad i=0,\ldots,N. \tag{A5}$$

The local thermodynamic potential is then expanded in powers of the variations $\delta F_i(x)$. Inserting Eqs. (A3)–(A5) into Eq. (A2) and expanding up to first order in the variations of the field, the Gibb's free energy splits up into the following three terms,

$$\frac{\Omega(x)}{k_BT(x)}=\frac{\Omega^{(0)}}{k_BT}+\mathsf{H}(\{\delta\alpha_k(x)\})-\sum_{i=o}^{N} r_i(\{\alpha_k(x)\})\delta F_i(x). \tag{A6}$$

The first term is the equilibrium Gibb's free energy density,

$$\Omega^{(0)}=e-\sum_{i=0}^{N}\alpha_iF_i\,. \tag{A7}$$

The second term contains the contributions due to fluctuations in the densities,

$$\mathsf{H}=\frac{1}{k_BT}\left[\Delta e(\{\Delta\alpha_k(x)\})-\sum_{i=0}^{N}F_i\Delta\alpha_i(x)\right], \tag{A8}$$

and this term defines the static functional for the critical theory. The last term in Eq. (A6) describes the linear response of the system to small variations in the external fields. The functions $r_i(\{\alpha_k(x)\})$ are

$$r_0(\{a_k(x)\})=\frac{-P+F_0\alpha_0(x)}{k_BT^2}, \tag{A9}$$

$$r_i(\{a_k(x)\})=\frac{F_i\alpha_i(x)}{k_BT}, \quad i=1,\ldots,N. \tag{A10}$$

In Eq. (A9) we have introduced the mean pressure $P$. In the case of local thermodynamic equilibrium, a local pressure $P(x)$ may be defined from a local Gibbs-Duhem relation,

$$e(\{\alpha_k(x)\})+P(x)=\sum_{i=0}^{N}F_i(x)\alpha_i(x). \tag{A11}$$

The local pressure varies around the mean value; thus we have $P(x)=P+\delta P(x)$. Inserting (A3) and (A4) into (A11), the equilibrium Gibbs-Duhem relation defines the equilibrium pressure,

$$e+P=\sum_{i=0}^{N}F_i\alpha_i\,, \tag{A12}$$

while the pressure variation is obtained,

$$\delta P(x)=\sum_{i=0}^{N}\alpha_i(x)\,\delta F_i(x). \tag{A13}$$

Using Eq. (A6), the local probability distribution may be written

$$w_{\mathrm{loc}}=\frac{1}{\mathcal{N}}e^{-\mathcal{H}}e^{\delta\mathcal{H}}, \tag{A14}$$

with the functionals,

$$\mathcal{H}=\int d^3x\,\mathsf{H}(\{\Delta\alpha_k(x)\}), \tag{A15}$$

$$\delta\mathcal{H}=\int d^3x\sum_{i=0}^{N}r_i(\{\alpha_k(x)\})\delta F_i(x), \tag{A16}$$

and the normalization factor

$$\mathcal{N}=\int\mathcal{D}(\{\Delta\alpha_k\})e^{-\mathcal{H}}e^{\delta\mathcal{H}}. \tag{A17}$$

At fixed external fields, which means $\delta F_i(x)=0$, we obtain the expectation values at local thermodynamic equilibrium. From Eq. (A14), we have

$$\langle A(\{\alpha_k(x)\})\rangle=\frac{1}{\mathcal{N}_0}\int\mathcal{D}(\{\Delta\alpha_k\})A(\{\alpha_k(x)\})e^{-\mathcal{H}}, \tag{A18}$$

with the equilibrium normalization constant $\mathcal{N}_0=\mathcal{N}(\delta\mathcal{H}=0)$. Expanding the fluctuation of the local energy density $\Delta e(\{\Delta\alpha_k(x)\})$ into powers of $\Delta\alpha_k(x)$ and its gradients, we obtain

$$\Delta e(x)=\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i_1,\ldots,i_n=0}^{N}\left[\left(\frac{\partial^n e}{\partial\alpha_{i_1}\cdots\partial\alpha_{i_n}}\right)\Delta\alpha_{i_1}(x)\cdots\Delta\alpha_{i_n}(x)+\left(\frac{\partial^n e}{\partial\nabla\alpha_{i_1}\cdots\partial\nabla\alpha_{i_n}}\right)\nabla\Delta\alpha_{i_1}(x)\cdots\nabla\Delta\alpha_{i_n}(x)\right]. \tag{A19}$$

For the description of the critical behavior it is sufficient to consider powers of $\Delta\alpha$ up to fourth order and powers of the gradients up to quadratic order. Thus we obtain

$$\begin{aligned}\mathcal{H}=\int d^3x\Bigg\{&\frac{1}{2}\sum_{i,j=0}^{N}a_{ij}\Delta\alpha_i(x)\Delta\alpha_j(x)\\&+\frac{1}{2}\sum_{i,j=0}^{N}c_{ij}(\nabla\Delta\alpha_i(x))(\nabla\Delta\alpha_j(x))\\&+\frac{1}{3!}\sum_{i,j,k=0}^{N}v_{ijk}\Delta\alpha_i(x)\Delta\alpha_j(x)\Delta\alpha_k(x)\\&+\frac{1}{4!}\sum_{i,j,k,l=0}^{N}u_{ijkl}\Delta\alpha_i(x)\Delta\alpha_j(x)\Delta\alpha_k(x)\Delta\alpha_l(x)\Bigg\},\end{aligned} \tag{A20}$$

after inserting Eqs. (A8) and (A19) into (A15). In the equation above, no linear terms appear because they cancel, since,

$$F_i=\left(\frac{\partial e}{\partial \alpha_i}\right)_{\{\alpha_k\neq\alpha_i\}} \tag{A21}$$

is used. The expansion coefficients $a_{ij}$ are given by the background values of thermodynamic derivatives,

$$a_{ij}=\frac{1}{k_BT}\left(\frac{\partial^2 e}{\partial\alpha_i\partial\alpha_j}\right)=\frac{1}{k_BT}\left(\frac{\partial F_i}{\partial\alpha_j}\right)_{\{\alpha_{k\neq j}\}}=\frac{1}{k_BT}\left(\frac{\partial F_j}{\partial\alpha_i}\right)_{\{\alpha_{k\neq i}\}}. \tag{A22}$$

The response of the densities to variations of the conjugated fields determines the correlation functions of the densities. Expanding the local probability function (A1) to first order in $\delta\mathcal{H}$, one gets

$$w_{\rm loc}=\frac{e^{-\mathcal{H}}}{\mathcal{N}_0}(1+\delta\mathcal{H}-\langle\delta\mathcal{H}\rangle). \tag{A23}$$

The corresponding nonequilibrium expectation values are then

$$\langle\alpha_i(x)\rangle_{\rm loc}=\langle\alpha_i(x)\rangle+\langle\alpha_i(x)\delta\mathcal{H}\rangle_c, \tag{A24}$$

where the index $c$ denotes the cummulant,

$$\langle\alpha_i\alpha_j\rangle_c=\langle\alpha_i\alpha_j\rangle-\langle\alpha_i\rangle\langle\alpha_j\rangle. \tag{A25}$$

Assuming that the deviation between the local nonequilibrium mean value and the local equilibrium mean value is small, we may write

$$\langle\alpha_i(x)\rangle_{\rm loc}-\langle\alpha_i(x)\rangle=\sum_{j=0}^{N}\left(\frac{\partial\alpha_i}{\partial F_j}\right)_{\{F_{k\neq j}\}}\delta F_j(x). \tag{A26}$$

Inserting Eq. (A16) into Eq. (A24) we obtain on the other side

$$\langle\alpha_i(x)\rangle_{\rm loc}-\langle\alpha_i(x)\rangle=\frac{1}{k_BT}\sum_{j=0}^{N}\int d^3x'\langle\alpha_i(x-x')\alpha_j(0)\rangle_c\delta F_j(0). \tag{A27}$$

Comparing Eq. (A27) with Eq. (A26) the correlation functions may be related to the thermodynamic derivatives. Introducing the Fourier transformed correlation functions,

$$\langle\alpha_i\alpha_j\rangle_c\equiv\langle\alpha_i\alpha_j\rangle_c(k=0)=\int d^3x'\langle\alpha_i(x-x')\alpha_j(0)\rangle_c, \tag{A28}$$

we obtain

$$\langle\alpha_i\alpha_j\rangle_c=k_BT\left(\frac{\partial\alpha_i}{\partial F_j}\right)_{\{F_{k\neq j}\}}. \tag{A29}$$

Because Eq. (A26) is a local relation, the above thermodynamic derivatives are also local functions, and contain contributions from fluctuations. The static functional (A20) together with Eq. (A29) constitutes the basis for the application of renormalization group theory. Within the scope of a critical theory it remains to establish an order parameter, which of course depends on the particular system. In the same way, relevant and irrelevant couplings are determined and the number of terms in Eq. (A20) is reduced considerably.

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