

Criticality of the “critical state” of granular media: Dilatancy angle in the Tetris model

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The dilatancy angle describes the propensity of a granular medium to dilate under an applied shear. Using a simple spin model (the Tetris model) which accounts for geometrical “frustration” effects, we study such a dilatancy angle as a function of density. An exact mapping can be drawn with a directed percolation process which proves that there exists a critical density ρ_c above which the system expands and below which it contracts under shear. When applied to packings constructed by a random deposition under gravity, the dilatancy angle is shown to be strongly anisotropic, and it constitutes an efficient tool to characterize the texture of the medium.

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I. INTRODUCTION

Granular materials [1] give rise to a number of original phenomena, which mostly result from their peculiar rheological behavior. Even using the most simple description of the grains (rigid equal-sized spherical particles) a granular system displays a rather complex behavior which shows that the origin of this rheology has to be found at the level of the geometrical arrangement of the grains.

Guided by these considerations, models have been proposed to account for the geometrical constraints of assemblies of hard-core particles [2–8]. The motivation of these models is not to reproduce faithfully the local details of granular media, but rather to show that simple geometrical constraints can reproduce under coarse-graining some features observed in real granular media. Along these lines, one of the most impressive examples is the “Tetris” model [2,3] which, in its simplest version, is basically a spin model with only hard core repulsion interactions. This model has been introduced in order to discuss the slow kinetics of the compaction of granular media under vibrations. In spite of the simplicity of the definition of the model, the kinetics of compaction has been shown to display a very close resemblance to most of the experimentally observed features of compaction [9] and segregation [10].

Our aim here is to consider again the Tetris model and to focus on a basic property of the quasistatic shearing of a granular assembly. It is well known since Reynolds that dense granular media have to dilate in order to accommodate a shear [11], whereas loose systems contract. This observation is important since it gives access to one of the basic ingredients (the direction of the plastic strain rate) necessary to describe the mechanical behavior in continuum modeling. The dilatancy angle is defined as the ratio of the rate of volume increase to the rate of shearing. Denoting with ε_{xy} the component xy of the strain tensor ε , Fig. 1 illustrates an experiment where a shear rate $\dot{\varepsilon}_{xy}$ is imposed together with a

zero longitudinal strain rate $\dot{\varepsilon}_{xx} = 0$, and the volumetric strain rate (here vertical expansion) $\dot{\varepsilon}_{yy}$ is measured. The direction of the velocity of the upper wall makes an angle ψ with respect to the horizontal direction. This angle is called the dilatancy angle ψ . In this particular geometry we have

$$\tan(\psi) = \frac{\dot{\varepsilon}_{yy}}{\dot{\varepsilon}_{xy}}. \quad (1)$$

More generally, the tangent of the dilation angle is the ratio of the volumetric strain rate [$\text{tr}(\dot{\varepsilon})$] to the deviatoric part of the strain rate.

Numerous experimental studies have confirmed the validity of such a behavior, and have lead to extensions such as what is known in soil mechanics as the “critical state” concept [12]. Assuming that the incremental (tangent) mechanical behavior can be parametrized using only the density of the medium, ρ , a loose medium will tend under continuous shear towards a state such that no more contraction takes place, i.e., it will assume asymptotically a density ρ_c such that $\psi(\rho_c) = 0$. This state is by definition the “critical state.” Conversely, if the strain were homogeneous, a dense granular media would dilate until it reached the critical state density ρ_c . However, for dense media, the strain may be localized in a narrow shear band which may allow a further

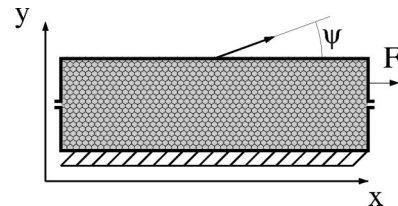


FIG. 1. Schematic view of shearing of granular media in a shear cell. The upper part of the cell moves only if the medium dilates so that the direction of the motion forms an angle ψ , the dilatancy angle, with the horizontal direction.

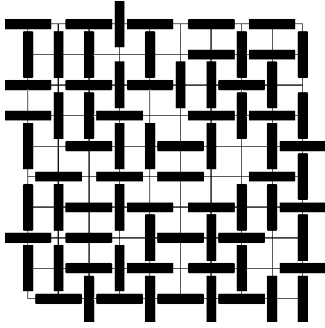


FIG. 2. Illustration of the Tetris model. The sites of a square lattice can host elongated particles shown as rectangles. The width and length of the particles induce geometrical frustrations.

shearing without any more volume change so that the mean density may remain at a value somewhat higher than the critical value. Recent triaxial tests [13] in a scanner apparatus have however shown that in the shear bands the density of the medium was quite comparable to the critical density, thus providing further evidence for the validity of the critical state concept.

The word “critical” used in this context has become the classical terminology, but it has no *a priori* relation with any kind of *critical* phenomenon in the statistical physics vocabulary [14]. One of the results presented in this article is to show that indeed the critical state of soil mechanics is also a critical point in the sense of phase transitions, for the Tetris model considered here.

II. MODEL AND DEFINITION OF DILATANCY

A group of lattice gas models in which the main ingredient is the geometrical frustration has been introduced recently under the name Tetris [2,3]. The Tetris model is a simple lattice model in which the sites of a square lattice can be occupied by (in its simplest version) a single type of rectangular shaped particle with only two possible orientations along the principal axis of the underlying lattice. A hard-core repulsion between particles is considered so that two particles cannot overlap. This forbids in particular that two nearest neighbor sites could be both occupied by particles aligned with the inter-site vector. An illustration of a typical admissible configuration is shown schematically in Fig. 2. More generally one can consider particles that move on a lattice and present randomly chosen shapes and sizes [3]. The interactions in the system obey to the general rule that one cannot have particle overlaps. The interactions are not spatially quenched but are determined in a self-consistent way by the local arrangements of the particles. The definition of the dilation angle as sketched in Fig. 1 is difficult to implement in practice in the Tetris model due to the underlying lattice structure which defines the geometric constraints only for particles on the lattice sites, and not in the continuum.

We may, however, circumvent this difficulty through the following construction illustrated in Fig. 3. We consider a semi-infinite line starting at the origin and oriented along one of the four cardinal directions. This line is (and all the sites attached to it are) pushed in one of the principal directions of the underlying square lattice by one lattice constant. In the following, we will consider only a displacement perpendicu-

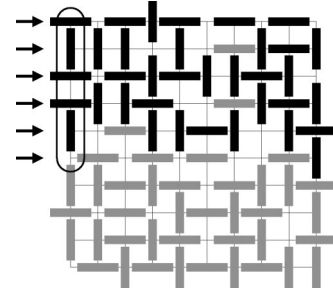


FIG. 3. Procedure used to define the dilation angle. All particles located on a semi-infinite line (the particles enclosed in the rounded rectangle on the left-hand side of the lattice) are moved by one lattice unit in the horizontal direction (shown by the arrows). Using the hard-core repulsion between particles, we determine the particles which are pushed (shown in black) and those which may stay in place (gray). For each column we consider the lowest (in general the most external) black site. (The gray particles within the cluster of black particles do not play any role in the determination of the dilation angle). The curve connecting all these points defines the profile of the pushed region. The line connecting the first and the last points of this profile determines the angle ψ with respect to the direction of motion. This angle provides the value of the dilation angle for the particular realization considered. The dilation angle is actually measured performing an average over a large number of realizations.

lar to the line, although a parallel displacement may also be considered. As this set of particles is moved, all other particles which may overlap with them are also translated with the same displacement. In this way, we determine the set \mathcal{D} of particles which moves. In the sequel, we will show that this domain is nothing but a directed percolation cluster [15] grown from the line. Anticipating on the following, the mean shape of \mathcal{D} will be shown to be a wedge limited by a generally rough boundary whose mean orientation forms an angle ψ with the direction of motion. The angle ψ can be shown to be exactly equal to the dilatancy angle as defined previously.

Exploiting the nonoverlap constraint, we may simply determine the rule for constructing the domain \mathcal{D} . Let us choose the particular case of a displacement in the direction $(1,0)$, and consider a nonempty site (i,j) which is displaced. The particles which may have to be displaced together with site (i,j) can be identified easily.

If the particle in (i,j) is horizontal, $(i+1,j)$ if the site is occupied by a particle with any orientation, $(i+2,j)$ if the site is occupied by a horizontal particle. If the particle in (i,j) is vertical, $(i+1,j)$ if the site is occupied by a particle with any orientation, $(i+1,j \pm 1)$ if the site is occupied by a vertical particle.

Using these rules, it is straightforward to identify the cluster of particles \mathcal{D} . The model thus appears to be a directed percolation problem with a mixed site/bond local formulation. Thus unless long range correlations are induced by the construction of the packing, the resulting problem will belong to the universality class of directed percolation. The density of particles, $p \in [0,1]$, in the lattice plays the role of the site or bond presence probability, i.e., the control parameter of the transition.

Let us recall, for sake of clarity, some properties of the two-dimensional directed percolation. For $p < p_c$ (where p_c is the directed percolation threshold), a typical connected

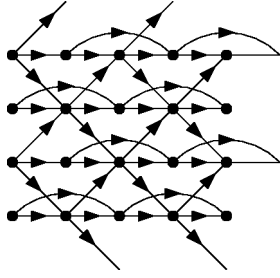


FIG. 4. Lattice over which directed site percolation is taking place. The arc bonds connect second neighbors along the x axis (horizontal).

cluster extends over a distance of the order of ξ_{\parallel} in the parallel direction (the preferential direction) and a distance ξ_{\perp} in the perpendicular direction. For $p > p_c$ there appears a directed percolating cluster which extends over the whole system. This cluster possesses a network of nodes and compartments. Each compartment has an anisotropic shape similar to the connected cluster below p_c , characterized by ξ_{\parallel} in the parallel direction and ξ_{\perp} in the perpendicular direction. On both sides of the percolation transition, the two lengths present the power-law behavior $\xi_{\parallel} \sim |p - p_c|^{-\nu_{\parallel}}$ and $\xi_{\perp} \sim |p - p_c|^{-\nu_{\perp}}$.

III. MONOCRYSTAL

Let us first examine a simple geometrical packing. There exist (two) special ordered configurations of particles such that the density can reach unity (one particle per site). This corresponds to a perfect staggered distribution of particle orientations. Thus a simple way of continuously tuning the density is to randomly dilute one of these perfectly ordered states. In this case, if a site is occupied by a particle, its orientation is prescribed. Therefore the above rules can be easily reformulated as a simple directed site percolation problem in a lattice having a particular distribution of bonds (up to second neighbors). Figure 4 illustrates the specific distribution of bonds corresponding to such an ordered state.

For $p = 1$, suppose that the initial seed is $(0, j)$ for $j \geq 0$ and this line is pushed in the x direction. Then the infinite cluster is the set of sites (i, j) such that $j \geq -i$, for a vertical spin at the origin. Thus moving the semi-infinite line (seed) introduces vacancies in the lattice which was initially fully occupied. The system dilates and its dilatancy angle is $\psi_1 = \pi/4$.

As p is reduced, the orientation of the boundary changes up to the stage where it becomes parallel to the x axis for $p = p_R$. At this point the dilatancy is zero. A motion is possible without changing the volume. This point corresponds precisely to the directed percolation threshold (using the precise rules defined above). From the theory of directed percolation, we can directly conclude that the behavior of the dilatancy angle ψ in the vicinity of p_R obeys

$$\tan(\psi) \propto (p - p_R)^{\nu_{\parallel} - \nu_{\perp}}, \quad (2)$$

where the correlation length exponents are $\nu_{\parallel} \approx 1.732$ and $\nu_{\perp} \approx 1.096$ independently of the lattice used.

A further decrease of p leads to a subcritical regime where only a finite cluster is connected to the initial seed. Only a

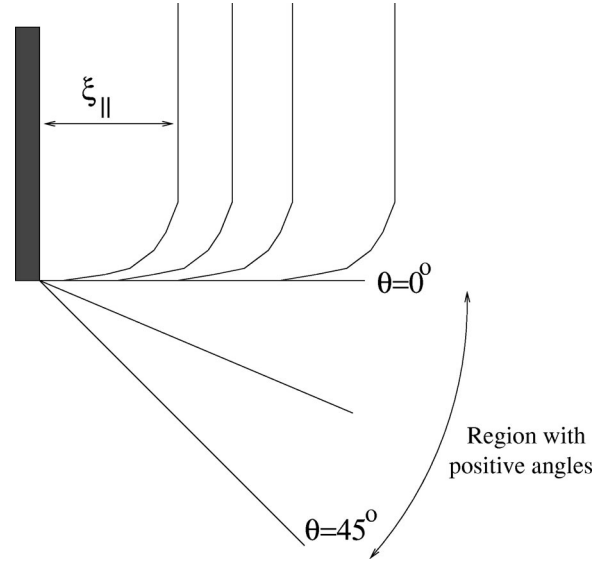


FIG. 5. Schematic representation of the mobilized region in the shearing procedure. Starting from $p = 1$, where one has a dilation with an angle of $\pi/4$, the dilation angle reduces until 0 (for $p = p_R$). A further reduction of p brings the system in a subcritical regime where only a finite layer of thickness $\xi_{\parallel} \propto (p_R - p)^{-\nu_{\parallel}}$ along the y axis is mobilized and the system compactifies.

finite layer of thickness $\xi_{\parallel} \propto (p_R - p)^{-\nu_{\parallel}}$ along the y axis is mobilized. This means that it not possible to define in the same way the dilatancy angle for $p < p_R$ (negative angles). What happens in practice is that for $p < p_R$ the shearing produces a compaction of the system in front of the semi-infinite line pushing the system. Figure 5 summarizes schematically the situation for all the values of p . The horizontal line corresponds to $p = p_R$ and a zero dilatancy angle.

We performed numerical simulations of this problem using a transfer matrix algorithm which allowed to generate system of size up to $10^4 \times (3 \times 10^4)$. These large system sizes allowed for a very accurate determination of the dilatancy angle as a function of the occupation probability (density) p . Figure 6 shows the boundaries of the domains \mathcal{D} for $p = 0.58$, close to the directed percolation threshold p_R , and $p = 0.7$.

Figure 7 shows the estimated dilatancy angle as a function of the density of particles. Angles are evaluated on lattice of size $10^4 \times (3 \times 10^4)$ and are averaged over 100 different realizations. The onset of dilatancy is thus estimated to be

$$p_R = 0.583 \pm 0.001. \quad (3)$$

The singular variation of ψ close to the onset of dilatancy Eq. (2) has been checked to be consistent with our numerically determined values as shown by the dotted curve in Fig. 7 which corresponds to the expected critical behavior.

IV. RANDOM SEQUENTIAL DEPOSITION

It is worth emphasizing that the directed percolation problem associated with the dilatancy angle determination is simply a site percolation problem in the above special case where each site is assigned only one possible orientation for the particle. In the more general case, the way the cluster is

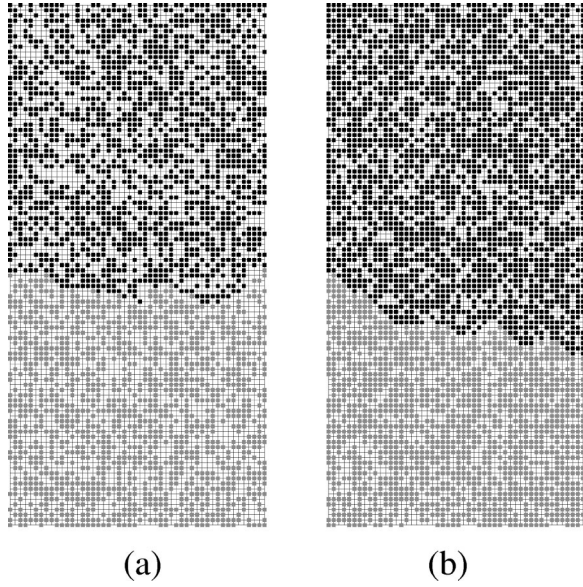


FIG. 6. Shapes of the boundaries of two clusters for (a) $p = 0.58$ (close to the threshold for a vanishing dilatancy) and (b) $p = 0.7$. The clusters mobilized are above and in both cases the line interpolating linearly between the first and the last point the boundaries defines the dilation angle.

grown locally depends on the specific orientation of the particle. Thus it is a mixed site/bond percolation problem. Therefore, depending on the way the system has been built, the onset for dilatancy p_R will vary.

This is illustrated by constructing the system through a random deposition process, i.e., differently from the above procedure. The algorithm used to construct the system is the following. At each time step, an empty site and a particle orientation are chosen at random. If the particle can fit on this site (without overlap with other particles), then the site is occupied, otherwise a new random trial is made. This is similar to the “random sequential” problem often studied in the literature [16], here adapted to the Tetris model.

This procedure leads to a maximum density of particles around $p_{\max} \approx 0.75$ above which it becomes impossible to

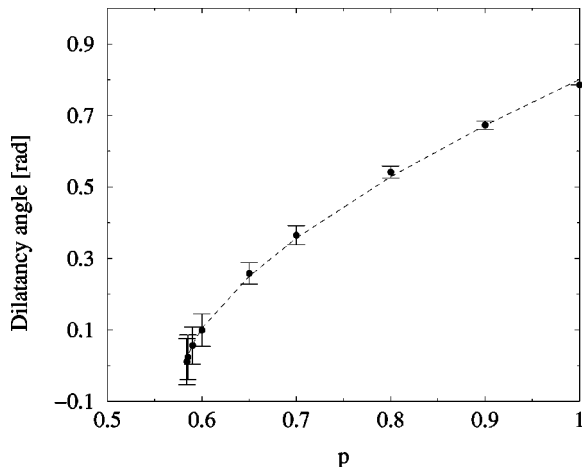


FIG. 7. Dilatancy angle as a function of the density p in the case of a random dilution of the perfectly ordered Tetris model. The dashed line represents a fit obtained using Eq. (2) with $p_R = 0.583 \pm 0.001$. The relative errors diverge at the transition.

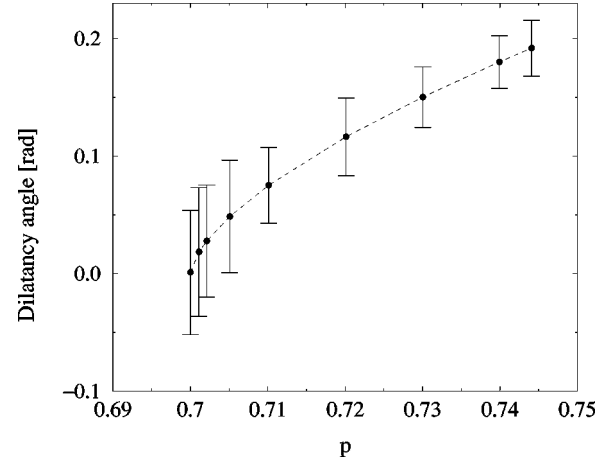


FIG. 8. Dilatancy angle as a function of the density p in the case of a random sequential deposition. The dashed line represents a fit obtained using Eq. (2) with $p_R = 0.70 \pm 0.01$. The relative errors diverge at the transition.

add new particles. Differently from the previous case, in the random sequential deposition simulations could not have been performed using the transfer matrix algorithm and thus we generated systems of size up to 500×1500 . We studied the dilatancy angle in such systems stopping the construction at different p values, averaging for each p over 100 realizations. Figure 8 shows the estimated dilatancy angle which is definitely different from the data of Fig. 7. In particular the onset of dilatancy is determined to be

$$p_R = 0.70 \pm 0.01. \quad (4)$$

However, this procedure is not expected to induce long range correlations in the particle density or orientation, and thus, we expect that the universality class of the model remains unchanged. In particular, the critical behavior Eq. (2) is expected to hold with the same exponents. Although the system sizes are much smaller in the present case, our data are consistent with such a law.

V. BALLISTIC DEPOSITION UNDER GRAVITY

Finally we would like to point out another property related to the texture of the medium. Up to now the two pro-

TABLE I. Results for the dilatancy angles obtained using differently prepared samples and different displacement orientations at $p \approx 0.8$. Dilution indicates samples obtained by diluting a perfect monocrystal (see text) to the desired density; BDG indicates samples obtained with a ballistic deposition procedure under gravity. We cannot compare directly in this table the results obtained with the random sequential deposition procedure (RSD) because, as mentioned in the text, this procedure leads to a maximum density of particles around $p_{\max} \approx 0.75$ above which it becomes impossible to add new particles.

Method	Orientation	ψ
Dilution	$\pm x \pm y$	31.0 ± 0.1
BDG	$-y$	6.6 ± 0.5
BDG	$\pm x$	23.8 ± 0.5
BDG	$+y$	5.8 ± 0.5

cedures followed to generate the packing of particles did not single out any privileged direction.

We now construct the packing by random deposition under gravity. Particles with a random orientation are placed at a random x position, and large y . Then the particle falls (along $-y$) down to the first site where it hits an overlap constraint. In this way, the packing assumes a well defined bulk density $p \approx 0.8$.

We used this construction procedure to generate lattices of size 500×1500 (averaged over 500 samples) cutting out the top part of the lattice which is characterized by a very wide interface and a nonconstant density profile. On this configuration (and thus at a fixed density) we measured the dilatancy angle for different orientations of the imposed displacement on the wall with respect to “gravity.”

Table I shows the resulting dilatancy angles obtained for the same density $p = 0.81$ using the following different constructions. The dilution of the ordered state and the sequential deposition (in both of these cases the dilatancy angle does not depend on the orientation of the motion). It is worth noticing how a direct comparison between this case and the others is not possible because with the random sequential deposition one cannot obtain densities larger than ≈ 0.75 . The ballistic deposition using a displacement along $-y$ (against gravity), y (along gravity), and x (perpendicular to gravity). In the latter case, we could study the problem for two orientations of the semi-infinite line ($x = 0$ and $y > 0$ or $y < 0$). We checked that the dilatancy angle was not dependent on this orientation.

The data reported in Table I indeed show that the dilatancy angle can be dependent on the direction of the imposed displacement. This measurement is thus sensitive to texture effects. As a side result, we note that the usual characterization of the dilatancy in terms of a single scalar (angle), albeit useful, is generally an oversimplification for textured media. Indeed, a number of studies have revealed [17] that granular

media (even consisting of perfect spheres) easily develop a non isotropic texture as can be shown by studying the distribution of contact normal orientations. This remark is almost obvious from a theoretical point of view, however, few attempts have been made to incorporate these texture effects in the dilatancy angle or even more generally in the rheology of granular media.

VI. CONCLUSION

We have shown that dilatancy can be precisely defined in the Tetris model, and that it is a function of the density as it is well known for granular media. The onset of dilatancy, i.e., the “critical state” of soil mechanics, has been shown to correspond to a directed percolation threshold density, hence justifying the term “critical” in this expression. From this point of view it is important to stress how any comparison of our approach with real granular materials should be done in the neighborhood of the critical point where we expect a largely universal (in the sense of critical phenomena in the statistical physics vocabulary) behavior. Using different lattices we expect, for instance, to recover the same critical behavior (same exponents) but not the same values for the critical density. To our knowledge this is the first time that such a mapping has been proposed. We have also shown that the dilatancy angle was not only determined by the density but also by the packing history. Finally, we have shown from a simple anisotropic construction that texture affects the dilatancy angle, even for a fixed density.

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