Ballistic versus diffusive pair dispersion in the Richardson regime

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Two-particle dispersion in fully developed turbulence is modeled within an asymmetric Lévy-walk framework which yields the Richardson t^3 law and which recovers recently observed results on the probability distribution function of pair separations in two-dimensional turbulence [M.C. Jullien, J. Paret, and P. Tabeling, Phys. Rev. Lett. **82**, 2872 (1999)]. The ballistic nature of the Richardson dispersion is discussed in terms of the persistence parameter and compared with the Richardson diffusive behavior.

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I. INTRODUCTION

The mechanism behind the Richardson t^3 pair separation in turbulent flows has been a puzzle since it was reported in 1926 [1], and has led to a large number of works on enhanced diffusion [2–9]; see Ref. [10] for a review. It is only recently that emphasis has been put on characterizing the trajectories of single particles and of two-particle separations in flows, both experimentally and theoretically. Analyzing and understanding the trajectories could shed light on the type of stochastic process underlying this strongly enhanced diffusion, and could help to distinguish between the various proposed models.

In general, models for the anomalous t^3 law can be classified in terms of local and nonlocal equations for the probability distribution functions (PDFs) of the two-particle separations. Richardson's equation, which is a modified diffusion equation with a space-dependent diffusion coefficient, is an example of a local description which is diffusive in nature [1]. A very different approach based on a nonlocal integro-differential equation has been formulated in terms of random walks in continuous time and the Lévy-walk idea [11–13]. The Lévy-walk approach is a probabilistic description of anomalous diffusion, based on Lévy-stable distributions with slowly decaying tails. In contrast to the diffusive character of the Richardson equation, characteristic for Lévy walks is the persistence of ballistic motion on all scales.

One is confronted therefore with two very different mechanisms, which lead to the t^3 law. These mechanisms differ, however, in their PDFs. Recent experiments on Richarson dispersion in two-dimensional turbulent flow have shown that the results of the PDF corresponding to the Richardson t^3 behavior could not be fitted by the Richardson expression, and have not been able to find indications for Lévy walks [14].

Here we introduce a model based on asymmetric Lévy walks which yields the Richardson t^3 dispersion and recovers the experimentally observed PDF. The current model extends a previous work on symmetric Lévy walks [13] and demonstrates the relevance of the persistence parameter introduced in Ref. [15] to describe diffusion under the influence of flow velocity fields. The importance of measuring the trajectories, and the possibility to obtain from them the persistence parameter is discussed.

II. SCALING REGIMES FOR TURBULENT TRANSPORT

Well-developed turbulent flows, when neglecting intermittency, are an example of scaling velocity fields, in which the two-time correlation function of the relative velocities at points separated by the distance r behaves as $\langle v(\mathbf{r},t_1)v(\mathbf{r},t_2)\rangle \propto \langle v^2(r)\rangle g[(t_2-t_1)/\tau(r)]$, where $\tau(r)$ is a distance-dependent correlation time. Under Kolmogorov scaling, the mean square relative velocity and the correlation time scale as

$$\langle v^2(r)\rangle \propto v_0^2\left(\frac{r}{r_0}\right)^{\alpha} \tag{1}$$

and

$$\tau(r)\propto \tau_0\left(\frac{r}{r_0}\right)^{\beta}, \tag{2}$$

where r_0, τ_0, and v_0 are some characteristic lengths, times, and velocities. Kolmogorov's considerations lead to the values of $\alpha=2/3$ and $\beta=2/3$ which fulfill the equality $\beta=1-\alpha/2$. This equality relies on the existence of a single dimensional parameter, which characterizes the kinematic similarity of the flow [16]. Thus, if a unique kinematic parameter Ξ of dimension $[L^a/T^b]$ exists, which determines the flow's behavior in a certain range, then from scaling considerations it follows immediately that any velocity (if only scaling and coordinate-dependent) behaves as $v^2(r) \propto(\Xi r^{b-a})^{2/b}$, so that $\alpha=2(1-a/b)$ in Eq. (1). Furthermore, a characteristic correlation time exists which scales with r as $\tau(r)\propto(\Xi^{-1}r^a)^{1/b}$, so that $\beta=a/b$ in Eq. (2), and the equality $\beta=1-\alpha/2$ follows. Under the above assumption the behavior of the flow is characterized by the exponent α and by the value of the parameter Ξ. The value of the dimensionless combination $P_s=v_0\tau_0/r_0$ remains, however, unspecified. It is referred to as a persistence parameter of the flow and plays a central role in describing single particle diffusion and pair separation [15]. In the Kolmogorov case the unique parameter Ξ is the energy dissipation rate ε, whose dimension is $[L^3/T^2]$, from which the values of $\alpha=2/3$ and $\beta=2/3$ follow.

Let us now recall some general properties of particle dispersion in such scaling fields. The mean square separation of particles is connected with the two-time correlation function of the relative velocities [10]:

$$\frac{d\langle r^2(t)\rangle}{dt}=2\int_0^t\langle \mathbf{v}(\mathbf{r}(t),t)\mathbf{v}(\mathbf{r}(t'),t')\rangle dt'. \tag{3}$$

If the correlation time $\tau(r)$ is small enough, the mean free path $l(r)=v(r)\tau(r)$ is small compared to r, so that changes in $\mathbf{r}$ during the correlation time can be neglected, and both velocities can be evaluated at the same space point. Moreover, the lower boundary of time integration can be shifted to $-\infty$. Thus, using Eqs. (1) and (2):

$$\frac{d\langle r^2(t)\rangle}{dt}=2\langle v^2(r)\rangle\int_{-\infty}^t g\left(\frac{t-t'}{\tau(r)}\right)dt'$$

$$=2\langle v^2(r)\rangle\tau(r)\propto v_0^2\tau_0\left(\frac{r}{r_0}\right)^{\alpha+\beta}. \tag{4}$$

This corresponds to diffusive behavior with a position-dependent diffusion coefficient, $K(r)\propto r^{\alpha+\beta}=r^{4/3}$, as proposed by Richardson [1]. Taking, as a scaling assumption $r\propto\langle r^2(t)\rangle^{1/2}=R$ and integrating Eq. (4), we find

$$R^2=\langle r^2(t)\rangle\propto\left(\frac{v_0^2\tau}{r_0^{\alpha+\beta}}t\right)^{[2/2-(\alpha+\beta)]}, \tag{5}$$

so that the mean square separation R grows as $R^2\propto t^\gamma$ with $\gamma=2/[2-(\alpha+\beta)]$. For the Kolmogorov values $\alpha=\beta=2/3$, the Richardson $\gamma=3$ behavior follows.

Another picture emerges when one focuses on the question of whether the turbulent motion is dominated by a diffusive process, or mainly by persistent ballistic events in the two-particle separation. The parameter for the validity of the either of the limiting behaviors in the local picture is the dimensionless perturbative expansion parameter $l(r)/r=v(r)\tau(r)/r=v_0\tau_0/r_0$, namely, the persistence parameter of the flow P_s [15]. Small values of P_s correspond to erratic, diffusive motion, while large values of P_s imply that the motion is strongly persistent with high weight of ballistic events. This immediately suggests the applicability of the Lévy-walk scheme. In the latter case the integration of the ballistic equation of motion

$$\frac{d}{dt}R=v(R) \tag{6}$$

yields

$$R_{\max}^2(t)=\left(\frac{2-\alpha}{2}\right)^{4/(2-\alpha)}(v_0r_0^{\alpha/2})^{4/(2-\alpha)}t^\gamma, \tag{7}$$

where we took $v(R)=\langle v^2(r)\rangle^{1/2}=v_0(r/r_0)^{\alpha/2}$. This behavior corresponds to

$$R^2\propto\left(\frac{v_0}{r_0^{\alpha/2}}t\right)^{4/(2-\alpha)}. \tag{8}$$

Remarkably, the exponent is now $\gamma=4/(2-\alpha)$, which with $\alpha=\beta=2/3$ again has the value 3.

Although leading to the same Richardson type behavior, $R^2\propto t^3$, Eqs. (5) and (8) depend differently on the parameters of the flow. Equation (8) corresponds to the fastest possible separation, in which the direction of the relative velocity points outwards the whole time, while the diffusive separation given by Eq. (5) is slower, due to multiple changes in the directions of the velocities. Assuming that P_s is the only relevant parameter in the problem, we arrive at

$$R^2(t)=f(P_s)R_{\max}^2(t), \tag{9}$$

where the function $f(P_s)$ behaves as $f(P_s)\propto P_s^\gamma$ for $P_s\ll 1$ and $f(P_s)\to\text{const}$ for $P_s\gg 1$. Namely, the proportionality coefficient between Eqs. (5) and (8) is P_s^3 in the Kolmogorov case.

Since, depending on the persistence parameter P_s, the dispersion can be either diffusive ($P_s\ll 1$) or ballistic ($P_s\gg 1$) in nature, one expects different probability density functions (PDFs) in these two limits. In the diffusive regime the PDF $p(\mathbf{r},t)$ obeys Richardson's diffusion equation, which, assuming spherical symmetry reads

$$\frac{\partial p}{\partial t}=\frac{1}{r^{d-1}}\frac{\partial}{\partial r}(K_0r^{d-1+\alpha+\beta})\frac{\partial}{\partial r}p. \tag{10}$$

The Green's function solution of Eq. (10) has a stretched-Gaussian form [12]:

$$p(r,t)=\left[2\pi^{d/2}z^{d/z-1}\frac{\Gamma(2/z)}{\Gamma(d/2)}\right]^{-1}(K_0t)^{-d/z}\exp\left(-\frac{r^z}{z^2K_0t}\right). \tag{11}$$

In Eq. (11) d is the spatial dimension and the parameter z obeys

$$z=2-(\alpha+\beta)=2/\gamma. \tag{12}$$

In the Kolmogorov case $z=2/3$. In the strongly correlated regime $P_s\gg 1$, a simple closed form for $p(r,t)$ is not known in general. Under certain conditions $p(r,t)$ has been obtained in a closed form for a Lévy distribution of ballistic persistences which in our model corresponds to $P_s\gg 1$. This PDF is characterized by a peak at the wing of the distribution which, as we later see, is typical for large persistence parameters [17], $P_s\gg 1$, range dominated by Lévy walks. When averaged over the distribution of velocities, Eq. (11) has been recovered [13].

The experimental findings in Ref. [14] suggest that $p(r,t)$ in the Richardson dispersion regime in two-dimensional turbulence is still a stretched Gaussian, Eq. (11), but with z being close to 0.5, a value significantly lower than the one which follows from Richardson's equation.

III. THE MODEL

In order to understand how the transition from diffusive to ballistic motion depends on the relevant parameters and what the range of validity of Eq. (11) is, we consider a one-dimensional model, proposed in Ref. [15] which is a stochastic model that enables the derivation of the t^3 law under a broad range of system parameters. The approach is heuristic:

the model behaves diffusively for small τ_0 and changes to being ballistic for τ_0 large. This model is akin to that of Ref. [13] in that it also relies on Lévy walks, which provide an extension to Drude's approach to transport. Thus, we consider the motion of particles on a line, whose velocities are position dependent. We take the magnitude of the velocity to be a function of r only, $v(r)=v_0(r/r_0)^{\alpha/2}$. Moreover, we account for the temporal changes of the flow by letting the particle change its velocity direction from time to time, while keeping the velocity's magnitude constant. We also let the probability of such a change depend on the particle's position. This modifies the model of Ref. [13], which assumed a symmetric Lévy-walk process. Within the model the differential probability dp of a change during dt is

$$dp=\frac{dt}{\tau(r)}=\tau_0^{-1}(r/r_0)^{-\beta}dt. \tag{13}$$

Here different scattering events are considered to be independent (they are viewed to stem, say, from different eddies).

The model proposed here accounts for two important properties of dispersion in real turbulent flows: (a) The fact that the relative velocity scales (as in the Kolmogorov-Obukhov energy spectrum form), and (b) that the velocity correlation time grows with the interparticle distance. The model is, of course, simple when compared to real flows, but nevertheless, as shown later, the predictions of our model compare favorably with the experimental results.

Within a Drude scattering picture we now calculate from Eq. (13) the probability of being scattered while moving a distance dr:

$$dp=\frac{dr}{v(r)\tau(r)}=\frac{1}{v_0\tau_0}(r/r_0)^{-(\beta+\alpha/2)}dr. \tag{14}$$

The probability not to be scattered on the way from r_1 to r_2 can be obtained by the procedure used for the Hertz distribution [18,19],

$$\begin{aligned}P(r_2|r_1)&=\exp\left(-\int_{r_1}^{r_2}\frac{1}{v_0\tau_0}(r/r_0)^{-(\beta+\alpha/2)}dr\right)\\&=\exp\left(-P_s^{-1}r_0^{(\beta+\alpha/2)-1}\int_{r_1}^{r_2}r^{-(\beta+\alpha/2)}dr\right).\end{aligned} \tag{15}$$

Performing the integration, we get for $\beta<1-\alpha/2$:

$$\begin{aligned}P(r_2|r_1)=&\exp\left[-\frac{P_s^{-1}}{1-(\beta+\alpha/2)}\left(\frac{r_2}{r_0}\right)^{1-(\beta+\alpha/2)}\right]\\&\times\exp\left[\frac{P_s^{-1}}{1-(\beta+\alpha/2)}\left(\frac{r_1}{r_0}\right)^{1-(\beta+\alpha/2)}\right].\end{aligned} \tag{16}$$

For $\beta=1-\alpha/2$ this probability corresponds to

$$P(r_2|r_1)=\left(\frac{r_2}{r_1}\right)^{-1/P_s} \tag{17}$$

for $r_2>r_1$, and to

$$P(r_2|r_1)=\left(\frac{r_2}{r_1}\right)^{1/P_s} \tag{18}$$

for $r_2<r_1$. The step length distribution, which directly relates to the turning points in the scattering process, $\Psi(r_2|r_1)$, follows from Eqs. (17) and (18). For $r_2>r_1$ we have

$$\Psi(r_2|r_1)=\frac{1}{P_s r_1}\left(\frac{r_2}{r_1}\right)^{-1/P_s-1}, \tag{19}$$

whereas for $r_2<r_1$ we obtain

$$\Psi(r_2|r_1)=\frac{1}{P_s r_1}\left(\frac{r_2}{r_1}\right)^{1/P_s-1}. \tag{20}$$

By arriving at the turning points, the direction of motion is chosen anew. The time necessary to travel from r_1 to r_2 is given by the integration of the ballistic equation of motion, $dr/dt=v(r)$, whose solution, for $v(r)=v_0(r/r_0)^{\alpha/2}$, is

$$\Delta t=\frac{2}{2-\alpha}v_0^{-1}r_0^{-\alpha/2}|r_2^{1-\alpha/2}-r_1^{1-\alpha/2}|. \tag{21}$$

The random process considered here is very similar to the genuine Lévy-walk process of Ref. [17], which considers a particle moving between turning points whose relative distances are distributed according to a PDF with a power-law tail. The models differ in that (i) here the velocity between turning points is not constant, but depends on the actual distance (a fact which was also taken into account when considering the Richardson dispersion [11,13]), and (ii) that here the distribution of possible turning points is strongly asymmetric and depends also on the initial position. Note that here the power-law distributions appear quite naturally as a result of the model assumptions, with exponents that depend on the persistence parameter P_s. As in earlier works on Lévy walks, the diffusional properties depend on the power-law exponent (on P_s in our case) which dictates if the moments of the distributions in Eqs. (19) and (20) exist or diverge.

IV. SIMULATION RESULTS

Let us now turn to the results of simulations. The algorithm used is quite straightforward: we generate a sequence of possible turning points r_i according to the probability distributions, Eqs. (19) and (20), and also the corresponding time intervals Δt_i. The position of the particle at a given moment of time T is obtained by determining its position at its last turning point before T (i.e., at the time $t_i=\Sigma_i\Delta t_i$, so that $t_i<T<t_i+\Delta t_{i+1}$), and by calculating the position at T through $r(T)=r_i+\Delta r(T-t_i)$, with $\Delta r(t)$ being

$$\Delta r=\left[r_1^{1-\alpha/2}\pm\frac{2-\alpha}{2}v_0 r_0^{\alpha/2}t\right]^{2/(2-\alpha)}, \tag{22}$$

where the sign in Eq. (22) depends on the direction of the velocity. In what follows we fix the values of α, v_0, and r_0 to $\alpha=2/3$ and $v_0=r_0=1$ and leave τ_0 to be a free parameter of simulations. Figure 1 presents typical trajectories obtained by the algorithm for persistence parameters $P_s=0.4$ and $P_s=0.1$. The initial separation is chosen to be 0.1. The trajectories display persistences on many scales, emphasizing the

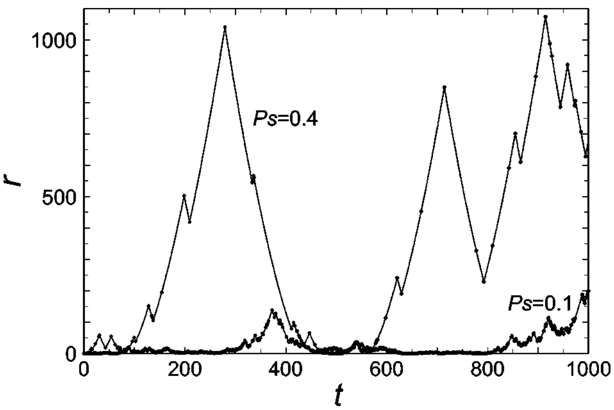

FIG. 1. Typical trajectories $r(t)$ for two different values of the persistence parameter $P_s=0.4$ and $P_s=0.1$. The dots along the trajectories denote the turning points, namely, the points of possible changes in direction.

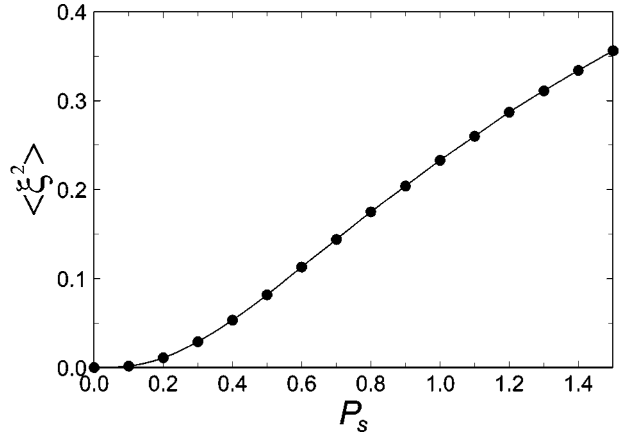

FIG. 3. The mean squared dimensionless distance $\langle\xi^2\rangle$ vs P_s which corresponds to the function $f(P_s)$ in Eq. (9). The dots are the calculated values and the line is a guide to the eye.

contribution of long Lévy-walk segments in the $P_s=0.4$ case. The figure shows clearly that for $P_s=0.1$ the trajectory has a much larger number of scattering events than in the case $P_s=0.4$ and that it corresponds therefore to the diffusive Richardson limit. The results presented below are determined by performing some 3×10^5 to 3×10^6 realizations of the process.

The simulations confirm that the model leads to a power-law dependence $R^2(t)=R_0 t^\gamma$; see Fig. 2. The slopes of all lines of the figure on double logarithmic scales are consistent with the value of $\gamma=3$. The value of the prefactor R_0 grows with P_s, and for large P_s values approaches the prefactor obtained for a purely ballistic process, for which $R^2(t)=R_{\max}^2(t)$. This relation is shown in Fig. 2 as a thick solid line. At this point it is convenient to introduce the dimensionless distance $\xi=R(t)/R_{\max}(t)$. In order to elucidate scaling with P_s we plotted in Fig. 3 $\langle\xi^2(t)\rangle$ at $t=1000$ as a function of P_s, which basically follows Eq. (9).

In order to assess the importance of correlations we have calculated the correlation function of the radial separation velocities taken backwards in time (BCF), as introduced in Ref. [14]. This function is defined as $C(\tau)=\langle v_r(t-\tau)v_r(t)\rangle/\langle v_r^2(t)\rangle$ and shows what part of its history is remembered by a particle in motion. We plot in Fig. 4 $C(\tau)$ calculated at $t=1000$. The function is plotted for values of P_s ranging from $P_s=0.1$ to $P_s=10.0$ and indicates a considerable increase in correlation with growing P_s. For moderate values of P_s ($P_s=0.4$) the behavior of $C(\tau)$ strongly resembles the experimental findings of Ref. [14].

Let us now turn to the form of the PDF of distances as follows from our model. This distribution $P(\xi)$ is plotted in Fig. 5 as a function of the dimensionless distance $\xi=r/R_{\max}(t)$. The time t is taken to be $t=1000$ and P_s varies from $P_s=0.1$ (diffusive regime) through an intermediate regime with $P_s=1.0$ to $P_s=10$ (ballistically dominated regime). Strong differences between these three regimes are clearly evident. Thus, in the diffusive regime the distribution is strongly peaked at small values of ξ, and shows a stretched-Gaussian decay for larger ξ values. On the contrary, for $P_s=10$ the distribution is peaked near $\xi=1$, indicating that most particles have not been scattered and move ballistically. The transition between the two regimes occurs at $P_s=1$, where the forward step-length distribution, Eq. (19), ceases to be normalizable, which implies that a finite amount of particles can achieve the maximum possible sepa-

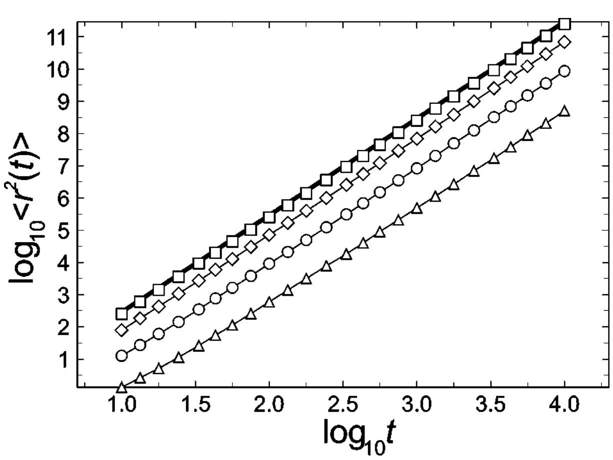

FIG. 2. A double logarithmic representation of $R^2(t)$ vs t for different values of the parameter P_s: $P_s=0.1$ (triangles), $P_s=0.3$ (circles), $P_s=1.0$ (diamonds), and $P_s=10$ (squares). The data is consistent with the slope $\gamma=3$ expected for the Richardson behavior. The thick line indicates the purely ballistic behavior of Eq. (7).

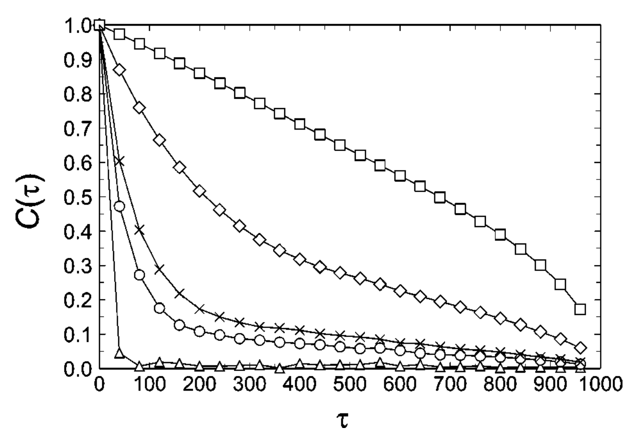

FIG. 4. The backward correlation function $C(\tau)$ for $t=1000$ for different values of P_s: $P_s=0.1$ (triangles), $P_s=0.3$ (circles), $P_s=0.4$ (crosses), $P_s=1.0$ (diamonds), and $P_s=10$ (squares).

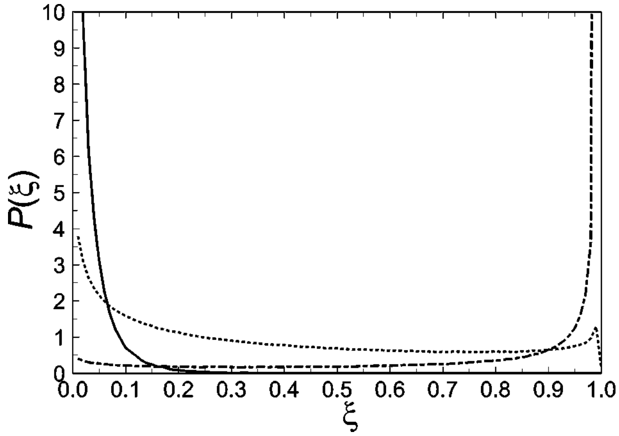

FIG. 5. The PDF as a function of the dimensionless distance ξ for $P_s=0.1$ (solid line), $P_s=1$ (dotted line), and $P_s=10.0$ (dashed-dotted line). Note the peaks at $\xi=0$ for $P_s=0.1$ and at $\xi=1$ for $P_s=10$. See text for details.

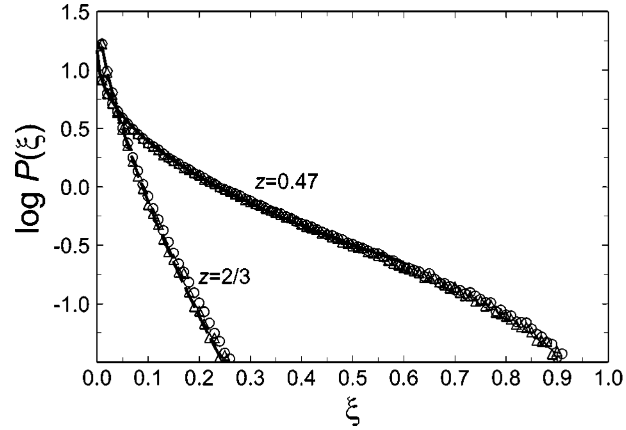

FIG. 6. The PDFs $P(\xi)$ vs ξ for $P_s=0.1$ and $P_s=0.4$ plotted on a decimal log-linear scale. The PDFs obtained for $t=100$ (triangles) and $t=1000$ (circles) display a scaling behavior. The calculated PDF for $P_s=0.1$ follows the Richardson stretched-Gaussian PDF with $z=2/3$ shown as a dashed line. The PDF for $P_s=0.4$ is shown to fit a stretched Gaussian with $z=0.47$ (full line).

ration $R_{\max}(t)$ without being scattered at all. The distribution $P(\xi)$ obtained numerically for this case is shown by a dotted line. For $t=1000$ this distribution shows a rather low central peak and a flat tail ending with a small local maximum. This maximum corresponds to a finite-time effect and is due to the fact that although the majority of particles perform diffusive motion, some of them can reach the distance $\xi=1$ without being scattered. The amount of such particles decreases with time. The distributions $P(\xi)$ are time independent, as can be readily inferred from Fig. 6, where two such distributions, for $P_s=0.1$ and $P_s=0.4$, are plotted for $t=100$ and $t=1000$. Note that for $P_s=0.1$ the distribution follows closely Richardson's solution, Eq. (11), also shown in Fig. 6 by a dashed line.

The results following from our Lévy-walk model can be now compared with recent experimental observations in two-dimensional turbulent flows. According to Refs. [14] and [20] the mean square separation of the pair at time t, which can be read out of the value of the prefactor $R_0\approx 0.0025$ in the expression $\langle r^2(t)\rangle = R_0 t^3$, is approximately 0.06 of the maximal ballistic separation $R_{\max}$, which can be obtained from the mean square velocity of the flow, $\langle v^2(r)\rangle = A r^{2/3}$ with $A\approx 0.26$. From Fig. 3 one readily finds that the value of $\langle r^2(t)\rangle / R_{\max}^2 = 0.06$ corresponds to $P_s\approx 0.4$. Note that the form of the BCF calculated for this value of P_s agrees qualitatively well with the experimentally determined scaling form, although the experimental curve is somewhat more rounded near its maximum at $\tau=0$; compare our Fig. 4 with Fig. 5 of Ref. [14]. Moreover, the probability distribution of the interparticle distances in its initial part is well fitted by a stretched-Gaussian expression $p(\xi)=A\exp(-a\xi^z)$, as in Eq. (11), but with $z=0.47$ and $a=2.31$, i.e., parameters close to those obtained experimentally $z=0.5$ and $a=2.6$ [14]. These findings confirm that our stochastic model captures reasonably well the most important features of particles' dispersion by turbulent fields. For the value $P_s=0.4$ the probability distribution in Eq. (19) has finite both first and second moments. Nevertheless, the pair separation is not purely diffusive and its PDF differs considerably from the Richardson PDF. This stems from the fact that the mean free path of the relative particle motion is approximately 0.4 of the interparticle distance, i.e., the motion is highly weighted by a ballistic component [20].

The persistence parameter which corresponds to the experimental realization in Ref. [14] has been shown to be $P_s \simeq 0.4$. This value has been inferred from the analysis of the data, as discussed above. For consistency, the P_s could also be obtained from the distribution of ballistic events. This distribution can be found directly from the pair separation trajectories. Fitting the distribution to the power law in Eq. (19) gives the power exponent, and therefore P_s. We believe that such measurements will further clarify the nature of the Richardson t^3 law.

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[1] L.F. Richardson, Proc. R. Soc. London, Ser. A **110**, 709 (1926).
[2] G.K. Batchelor, Q.J.R. Meteorol. Soc. **76**, 133 (1950).
[3] G.K. Batchelor, Proc. Cambridge Philos. Soc. **48**, 345 (1952).
[4] A. Okubo, J. Oceanol. Soc. Japan **20**, 286 (1962).
[5] R.H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[6] H.G.E. Hentschel and I. Procaccia, Phys. Rev. A **29**, 1461 (1984).
[7] F.W. Elliot and A.J. Majda, Phys. Fluids **8**, 1052 (1996).
[8] F. Fung and J.C. Vassilicos, Phys. Rev. E **57**, 1677 (1998).
[9] G. Boffetta, A. Celani, A. Crisanti, and A. Vulpiani, Europhys. Lett. **42**, 177 (1999).
[10] A.J. Majda and P.R. Kramer, Phys. Rep. **314**, 237 (1999).
[11] M.F. Shlesinger, B. West, and J. Klafter, Phys. Rev. Lett. **58**,

1100 (1987).
[12] J. Klafter, A. Blumen, and M.F. Shlesinger, Phys. Rev. A **35**, 3081 (1987).
[13] I.M. Sokolov, A. Blumen, and J. Klafter, Europhys. Lett. **47**, 152 (1999).
[14] M.C. Jullien, J. Paret, and P. Tabeling, Phys. Rev. Lett. **82**, 2872 (1999).
[15] I. M. Sokolov, Phys. Rev. E **66**, 5528 (1999).
[16] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995).
[17] G. Zumofen, J. Klafter, and A. Blumen, Phys. Rev. E **47**, 2183 (1993).
[18] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[19] A. Blumen, G. Zumofen, and J. Klafter, Chem. Phys. **26**, 287 (1977).
[20] I.M. Sokolov and R. Reigada, Phys. Rev. E **59**, 5412 (1999).