Passive scalar transport in a random flow with a finite renewal time: Mean-field equations

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A mean-field equation for a passive scalar (e.g., for a mean number density of particles) in a random velocity field (incompressible and compressible) with a finite constant renewal time is derived. The finite renewal time of a random velocity field results in the appearance of high-order spatial derivatives in the mean-field equation for a passive scalar. We considered three models of a random velocity field: (i) a velocity field with a small renewal time; (ii) the Gaussian approximation for Lagrangian trajectories; and (iii) a small inhomogeneity of the velocity and mean passive scalar fields. For a small renewal time we recovered results obtained using the δ-function-correlated in time random velocity field. The finite renewal time and compressibility of the velocity field can cause a depletion of turbulent diffusion and a modification of an effective drift velocity. For a compressible velocity field the form of the mean-field equation for a passive scalar depends on the details of the velocity field, i.e., the universality is lost. For an incompressible velocity field the universality exists in spite of the finite renewal time. Results by Saffman [J. Fluid Mech. **8**, 273 (1960)] for the effect of molecular diffusivity in turbulent diffusion are generalized for the case of a compressible and anisotropic random velocity field. The obtained results may be of relevance in some atmospheric phenomena (e.g., atmospheric aerosols and smog formation).

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I. INTRODUCTION

Turbulent transport of passive scalar (e.g., the number density of particles) in a fluid flow was studied in a large number of publications (see, e.g., Refs. [1–6]). The equation for a passive scalar $n(t,\mathbf{r})$, advected by an incompressible velocity field $\mathbf{v}$, is given by

$$\frac{\partial n}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})n=D\Delta n, \tag{1}$$

where D is the coefficient of molecular diffusion. Averaging this equation over a turbulent velocity field, Taylor [1] derived a mean-field equation for a passive scalar $N=\langle n\rangle$,

$$\frac{\partial N}{\partial t}+(\overline{\mathbf{V}}\cdot\boldsymbol{\nabla})N=D_T\Delta N, \tag{2}$$

where $\overline{\mathbf{V}}=\langle\mathbf{v}\rangle$ is the mean fluid velocity, $D_T\simeq\langle(\delta\mathbf{x})^2\rangle/\tau$ is the coefficient of turbulent diffusion, and $\langle(\delta\mathbf{x})^2\rangle$ is a mean-square displacement or dispersion of infinitesimal fluid particles from their original positions for a correlation time τ of a turbulent velocity field. The mean-field equation (2) was used in a large number of applications, e.g., to study a turbulent transport in atmosphere and ocean, in astrophysics, and in industrial applications.

However, a range of validity and applicability and rigorous justification of Eq. (2) still remain a subject of research. In particular, it is not clear why the mean-field equation for a passive scalar does not contain high-order spatial derivatives. What is the role of the molecular diffusion? Note that Taylor in his derivation of Eq. (2) (see Ref. [1]) neglected the molecular diffusion. It was found by Saffman [7] that there are small and subtle interactions between turbulent diffusion and molecular diffusion for the mean concentration field. In the first approximation the two processes are additive, but at a more detailed level of description the local smoothing by molecular diffusion acts to reduce the total dispersion.

Later, using different approximate approaches like closure procedures (see e.g., Refs. [2,5]) the mean-field equation for a passive scalar was derived. However, all these methods do not use an exact solution of Eq. (1) for a derivation of the mean-field equation. This shortcoming was overcome using the δ-function-correlated in time velocity field approximation [8] for incompressible (see e.g., Refs. [3,4]) and compressible (see e.g., Refs. [9,10]) flows. An exact solution of Eq. (1) in the form of the Feynman-Kac formula was used in this approach. Notably, the derived mean-field equation for a passive scalar comprises an additional mean effective drift velocity. The latter is associated either with the compressibility of a low-Mach-number compressible fluid flow or with particle inertia [9,10]. The mean-field equation for a passive scalar derived for the δ-function-correlated in time incompressible velocity field is in agreement with that derived by other methods. However, it is not clear to what extent it is possible to extrapolate the results obtained using the δ-function-correlated in time velocity field approximation to the velocity field with a finite correlation time. Note that different methods of the derivation of the mean-field equation yield quantitatively different turbulent transport coefficients (see, e.g., Refs. [11,12]).

In the present study we consider a random velocity field with a finite renewal time. We derived a mean-field equation for a passive scalar advected by a random incompressible and compressible velocity fields. We showed that the mean-field equation has a more complicated form than the mean-field equation (2). In particular, the finite renewal time of the random velocity field results in the appearance of high-order spatial derivatives in the mean-field equation for a passive scalar. The criterion of validity of the δ-function-correlated in time velocity field approximation is found. In all known limiting cases the derived equation recovers previously obtained results for turbulent diffusion.

Note that in Ref. [13] an exact solution of Eq. (1) in the form of path integrals was used for calculating a coefficient of turbulent diffusion for incompressible homogeneous and isotropic fluid flow. In the present paper we used the other form of an exact solution for the passive scalar equation for compressible anisotropic fluid flow. In addition, a case of inhomogeneous velocity field is also analyzed. Our results are in agreement with those obtained in Ref. [13].

II. GOVERNING EQUATIONS

The number density $n(t,\mathbf{r})$ of small particles advected by a turbulent compressible fluid flow is given by

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot(n\mathbf{v})=D\Delta n, \tag{3}$$

where $\mathbf{v}$ is a random velocity field of the particles which they acquire in a turbulent fluid velocity field. Note that $\boldsymbol{\nabla}\cdot\mathbf{v}\neq 0$, which is due to either the compressibility of the fluid or to particle inertia (see, e.g., Refs. [9,10,14–16]). Equation (3) corresponds to the conservation of the total number of particles.

The goal of the present study is to derive a mean-field equation for a passive scalar advected by a random velocity field with a finite renewal time. The procedure of the derivation of this equation is as follows.

(a) We use an exact solution of Eq. (3) in the form of a functional integral for an arbitrary velocity field, taking into account a small yet finite molecular diffusion. This functional integral implies an averaging over random Brownian motions of a particle.

(b) The form of the exact solution used in the present paper allows us to separate the averaging over both a random Brownian motions of a particle and a random velocity field.

(c) The final result by means of a change of variables is rewritten in a form which at zero molecular diffusion ($D=0$) contains only the Lagrangian displacement of a fluid particles. This allows us to recover a classical result by Taylor for the coefficient of turbulent diffusion.

(d) The derived mean-field equation for a passive scalar generally is an integrodifferential equation. However, when the characteristic scale of variation of the mean passive scalar field is much larger than the correlation length of a random velocity field, the mean-field equation (2) for passive scalar field for incompressible flow [or Eq. (4) in [9] for compressible flow] is recovered.

Now we use an exact solution of the Cauchy problem for Eq. (3) with an initial condition $n(t=s,\mathbf{x})=n(s,\mathbf{x})$ in the form

$$n(t,\mathbf{x})=M_{\boldsymbol{\zeta}}\{J(t,s,\boldsymbol{\zeta})\tilde{G}(t,s,\boldsymbol{\zeta})n(s,\boldsymbol{\zeta}(t,s))\} \tag{4}$$

(see Appendix A), where

$$\tilde{G}(t,s,\boldsymbol{\zeta})=\exp\left[-\int_s^t b(\sigma,\boldsymbol{\zeta}(t,\sigma))d\sigma\right], \tag{5}$$

$$J(t,s,\boldsymbol{\zeta})=\exp\left[-(2D)^{-1/2}\int_0^{t-s}\mathbf{v}(t-\eta,\boldsymbol{\zeta}(t,\eta))\cdot d\mathbf{w}(\eta)\right.$$
$$\left.-(4D)^{-1}\int_0^{t-s}\mathbf{v}^2(t-\eta,\boldsymbol{\zeta}(t,\eta))d\eta\right], \tag{6}$$

$$\boldsymbol{\zeta}(t,s)=\mathbf{x}+(2D)^{1/2}[\mathbf{w}(t)-\mathbf{w}(s)], \tag{7}$$

and $\mathbf{w}(t)$ is a Wiener process, $M_{\boldsymbol{\zeta}}\{\cdot\}$ denotes the mathematical expectation over the Wiener paths $\boldsymbol{\zeta}$, and $b=\boldsymbol{\nabla}\cdot\mathbf{v}$. The first integral $\int_0^{t-s}\mathbf{v}(t-\eta,\boldsymbol{\zeta}(t,\eta))\cdot d\mathbf{w}(\eta)$ in Eq. (6) is the Ito stochastic integral (see, e.g., Ref. [17]). Solution (4) was first found in Ref. [18] for a passive vector (magnetic field) which is determined by the induction equation in an incompressible fluid flow. Equations (4)–(6) generalize the solution obtained in Ref. [18] for a passive scalar advected by a compressible random velocity field. Note that there is a singularity in Eq. (6) at $D\to 0$. However, this singularity in the final result can be eliminated by a change of variables [see Eq. (18) below]. The path-integral representation for the effective diffusion function of a passive scalar field for $b=0$ was suggested by Drummond in Ref. [13].

We compare solution (4) with that determined by the Feynman-Kac formula:

$$n(t,\mathbf{x})=M_{\boldsymbol{\xi}}\{G(t,s,\boldsymbol{\xi}(t,s))n(s,\boldsymbol{\xi}(t,s))\}, \tag{8}$$

$$G(t,s,\boldsymbol{\xi})=\exp\left[-\int_s^t b(\sigma,\boldsymbol{\xi}(t,\sigma))d\sigma\right], \tag{9}$$

where $M_{\boldsymbol{\xi}}\{\cdot\}$ denotes the mathematical expectation over the Wiener paths $\boldsymbol{\xi}(t,s)$:

$$\boldsymbol{\xi}(t,s)=\mathbf{x}-\int_0^{t-s}\mathbf{v}[t-\sigma,\boldsymbol{\xi}(t,\sigma)]d\sigma+(2D)^{1/2}\mathbf{w}(t-s). \tag{10}$$

Equation (10) describes a set of random trajectories which pass through the point $\mathbf{x}$ at time t. The Wiener process in Eq. (10) describes the molecular diffusion (i.e., it describes the Brownian motion; see, e.g., Refs. [3,4]). Equation (8) allowed us to derive equations for the mean passive scalar field and its higher moments for a δ-function-correlated in time random velocity field (see, e.g., Refs. [9,10,15,16]).

The main difference between solutions (4) and (8) is as follows. The function $n(s,\boldsymbol{\xi}(t,s))$ in Eq. (8) explicitly depends on the random velocity field $\mathbf{v}$ via the Wiener path $\boldsymbol{\xi}$, while the function $n(s,\boldsymbol{\zeta}(t,s))$ in Eq. (4) is independent of the velocity $\mathbf{v}$. It is difficult to use the Feynman-Kac formula (8) for a derivation of equations for the mean passive scalar field and its higher moments in a random velocity field with a finite renewal time. Trajectories in the Feynman-Kac formula (8) are determined by both a random velocity field and Brownian motion. On the other hand, trajectories in Eq. (4) are determined only by Brownian motion. As follows from the Cameron-Martin-Girsanov theorem, the transformation from Eq. (8) to Eq. (4) can be considered as a change of variables $\boldsymbol{\xi}\to\boldsymbol{\zeta}$ in the integral (8) (see, e.g., Ref. [19]).

Due to the Markovian property of the Wiener process, solution (4) can be rewritten in the form

$$n(t,\mathbf{x})=E\{S(t,s,\mathbf{x},\mathbf{Y})n(s,\mathbf{Y})\}=\int Q(t,s,\mathbf{x},\mathbf{y})n(s,\mathbf{y})d\mathbf{y}, \tag{11}$$

where

$$Q(t,s,\mathbf{x},\mathbf{y})=(4\pi D(t-s))^{3/2}\exp\left(-\frac{(\mathbf{y}-\mathbf{x})^2}{4D(t-s)}\right)S(t,s,\mathbf{x},\mathbf{y}), \tag{12}$$

$$S(t,s,\mathbf{x},\mathbf{y})=M_{\boldsymbol{\mu}}\{J(t,s,\boldsymbol{\mu})\tilde{G}(t,s,\boldsymbol{\mu})\}, \tag{13}$$

and $M_{\boldsymbol{\mu}}\{\cdot\}$ is the path integral taken over the set of Wiener trajectories $\boldsymbol{\mu}$ which connect points $(t,\mathbf{x})$ and $(s,\mathbf{y})$. The mathematical expectation $E\{\cdot\}$ in Eq. (11) denotes the averaging over the set of random points $\mathbf{Y}$ which have a Gaussian statistics (see, e.g., Ref. [20]). Here we used the following property of the averaging over the Wiener process:

$$E\{M_{\boldsymbol{\mu}}\{\cdot\}\}=M_{\boldsymbol{\zeta}}\{\cdot\}. \tag{14}$$

III. PASSIVE SCALAR IN A RANDOM VELOCITY FIELD

Consider a random velocity field with a finite constant renewal time. Assume that in the intervals $\ldots(-\tau,0];(0,\tau];(\tau,2\tau];\ldots$ the velocity fields are statistically independent and have the same statistics. This implies that the velocity field loses memory at the prescribed instants $t=k\tau$, where $k=0,\pm1,\pm2,\ldots$. This velocity field cannot be considered as a stationary velocity field for small times $\sim\tau$; however, it behaves like a stationary field for $t\gg\tau$.

In Eq. (11) we specify instants $t=(m+1)\tau$ and $s=m\tau$. Note that the fields $n(m\tau,\mathbf{y})$ and $Q((m+1)\tau,m\tau,\mathbf{x},\mathbf{y})$ are statistically independent because the field $n(m\tau,\mathbf{y})$ is determined in the time interval $(-\infty,m\tau]$, whereas the function $Q((m+1)\tau,m\tau,\mathbf{x},\mathbf{y})$ is defined on the interval $(m\tau,(m+1)\tau)$. Due to a renewal, the velocity field as well as its functionals $n(m\tau,\mathbf{y})$ and $Q((m+1)\tau,m\tau,\mathbf{x},\mathbf{y})$ in these two time intervals are statistically independent. Averaging Eq. (11) over the random velocity field yields the equation for the mean passive scalar field,

$$N((m+1)\tau,\mathbf{x})=(2\pi)^{-3}\int P(\tau,\mathbf{x},\mathbf{y})N(m\tau,\mathbf{y})d\mathbf{y}, \tag{15}$$

where $N(t,\mathbf{x})=\langle n(t,\mathbf{x})\rangle$ is the mean passive scalar field, the angular brackets $\langle\cdot\rangle$ denote the ensemble average over the random velocity field, and

$$P(\tau,\mathbf{x},\mathbf{y})=(2\pi)^3\langle Q((m+1)\tau,m\tau,\mathbf{x},\mathbf{y})\rangle. \tag{16}$$

The function $P(\tau,\mathbf{x},\mathbf{y})$ is independent of m because all time intervals $\ldots(-\tau,0];(0,\tau];(\tau,2\tau];\ldots$ are statistically equivalent.

In the case of a homogeneous random velocity field $P(\tau,\mathbf{x},\mathbf{y})=P(\tau,\mathbf{y}-\mathbf{x})$. In $\mathbf{k}$ space, we obtain

$$N((m+1)\tau,\mathbf{k})=P(\tau,-\mathbf{k})N(m\tau,\mathbf{k}), \tag{17}$$

$$P(\tau,-\mathbf{k})=M_{\boldsymbol{\xi}}\{\langle\exp(i\mathbf{k}\cdot\boldsymbol{\xi}^{(\tau)})G(t,s,\boldsymbol{\xi})\rangle\} \tag{18}$$

(see Appendix B), where $\boldsymbol{\xi}^{(\tau)}=\boldsymbol{\xi}((m+1)\tau,s)-\boldsymbol{\xi}(m\tau,s)=-\int_{m\tau}^{(m+1)\tau}\mathbf{v}[t-\sigma,\boldsymbol{\xi}(t,\sigma)]d\sigma+(2D)^{1/2}\mathbf{w}(\tau)$, and we used Feynman-Kac formula (8). Note that Eqs. (17) and (18) are valid also for $D=0$.

In the case of inhomogeneous random velocity field, Eqs. (17) and (18) in $\mathbf{r}$ space are given by

$$N((m+1)\tau,\mathbf{x})=P(\tau,\mathbf{x},i\boldsymbol{\nabla})N(m\tau,\mathbf{x}), \tag{19}$$

$$P(\tau,\mathbf{x},i\boldsymbol{\nabla})=M_{\boldsymbol{\xi}}\{\langle G(\tau,\boldsymbol{\xi}(\mathbf{x}))\exp[\boldsymbol{\xi}^{(\tau)}(\mathbf{x})\cdot\boldsymbol{\nabla}]\rangle\} \tag{20}$$

(see Appendix B), where $\boldsymbol{\nabla}=\partial/\partial\mathbf{x}$ and the operator $\exp[\boldsymbol{\xi}^{(\tau)}(\mathbf{x})\cdot(\partial/\partial\mathbf{x})]$ is determined by

$$\exp[\boldsymbol{\xi}^{(\tau)}(\mathbf{x})\cdot\boldsymbol{\nabla}]=1+\boldsymbol{\xi}^{(\tau)}(\mathbf{x})\cdot\boldsymbol{\nabla}+(1/2!)[\boldsymbol{\xi}^{(\tau)}(\mathbf{x})\cdot\boldsymbol{\nabla}]^2+\cdots \tag{21}$$

$$+(1/m!)[\boldsymbol{\xi}^{(\tau)}(\mathbf{x})\cdot\boldsymbol{\nabla}]^m+\cdots, \tag{22}$$

where the operator $\boldsymbol{\nabla}$ acts only on the function $N(m\tau,\mathbf{x})$. Equations (17) and (19) for the mean number density of particles are generally integral equations. In order to use these equations we need to specify the explicit form of the operator $P(\tau,\mathbf{x},i\boldsymbol{\nabla})$ (see Sec. IV).

IV. MEAN PASSIVE SCALAR FIELD EQUATION

In this section we consider three types of a random velocity field for which an explicit form of the function P can be found.

A. Random velocity field with a small renewal time

In the model of a velocity field with a small renewal time we expand the functions $\boldsymbol{\xi}(t,s)$ and $G(\tau,\boldsymbol{\xi})$ in a Taylor series of small renewal time τ (see Appendix C). Using Eqs. (20), (22), and (C6), we obtain

$$P(\tau,\mathbf{x},i\boldsymbol{\nabla})=1-\tau\mathbf{V}_{\rm eff}\cdot\boldsymbol{\nabla}+\tau D_{mn}\nabla_m\nabla_n+\cdots, \tag{23}$$

where

$$D_{mn}=(2\tau)^{-1}M_{\boldsymbol{\xi}}\{\langle G\xi_m^{(\tau)}\xi_n^{(\tau)}\rangle\}, \tag{24}$$

$$\mathbf{V}_{\rm eff}=\tau^{-1}M_{\boldsymbol{\xi}}\{\langle G\boldsymbol{\xi}^{(\tau)}\rangle\}, \tag{25}$$

and we considered a statistically homogeneous random velocity field with $\langle\mathbf{v}\rangle=0$ and $\langle\mathbf{b}\rangle=0$. Thus an equation for the mean passive scalar field is given by

$$\frac{\partial N}{\partial t}+(\mathbf{V}_{\rm eff}\cdot\boldsymbol{\nabla})N=D_{mn}\nabla_m\nabla_n N, \tag{26}$$

where

$$D_{mn}=D\delta_{mn}+(1/2)\langle v_m v_n\rangle\tau-(1/2)\langle b v_m v_n\rangle\tau^2+(D\tau^2/6)\times(\Delta f_{mn}+\nabla_p\nabla_n f_{mp}+\nabla_p\nabla_m f_{np})_{r=0}+O(\tau^3), \tag{27}$$

$$\mathbf{V}_{\rm eff}=-(1/2)\langle\mathbf{v}b\rangle\tau+(D/3)\langle(\nabla_p\mathbf{v})(\nabla_p b)\rangle\tau^2+(1/2)\times\langle b(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\rangle\tau^2, \tag{28}$$

where v_m are the components of the vector $\mathbf{v}$ in a Cartesian system of coordinates, $f_{mn}=\langle v_m(t,\mathbf{x})v_n(t,\mathbf{y})\rangle$, $\mathbf{r}=\mathbf{y}-\mathbf{x}$, $\boldsymbol{\nabla}=\partial/\partial\mathbf{r}$, and we used Eqs. (C4)–(C6). Note that for incompressible velocity field $(b=\boldsymbol{\nabla}\cdot\mathbf{v}=0)$ the effective velocity $\mathbf{V}_{\rm eff}$ vanishes and the diffusion tensor D_{mn} is given by

$$D_{mn}=D\delta_{mn}+(1/2)\langle v_m v_n\rangle\tau+(D\tau^2/6)(\Delta f_{mn})_{r=0}+O(\tau^3). \tag{29}$$

The third term in Eq. (29) describes interactions between turbulent diffusion and molecular diffusion for the mean concentration field. This result was predicted by Saffman [7] for an isotropic and incompressible random velocity field. In order to compare Eq. (29) with the result obtained by Saffman [7], we consider an isotropic and homogeneous random velocity field. The two point correlation function for the velocity field is given by

$$f_{ij}(r)=(u_0^2/3)\{[F(r)+F_c(r)]\delta_{ij}+(rF'/2)P_{ij}(r)+rF_c' r_{ij}\}, \tag{30}$$

where $F'=dF/dr$, $P_{ij}(r)=\delta_{ij}-r_{ij}$, and $r_{ij}=r_i r_j/r^2$. Hereafter r is the dimensionless distance which is measured in the units of the integral length scale l_0, and $F(r)$ and $F_c(r)$ are the incompressible and compressible components of the correlation function for the velocity field, i.e., $\langle(\boldsymbol{\nabla}\cdot\mathbf{v})^2\rangle=-5\tau_0^{-2}(F_c'/r)_{r=0}$, and $\langle(\boldsymbol{\nabla}\times\mathbf{v})^2\rangle=-5\tau_0^{-2}(F'/r)_{r=0}$ and $\tau_0=l_0/u_0$. Using an identity

$$\left(\frac{\partial f_{ij}}{\partial r_m\partial r_n}\right)_{r=0}=(1/6\tau_0^2)\{\delta_{ij}\delta_{mn}[(4F'+2F_c')/r]_{r=0}+(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm})[(2F_c'-F')/r]_{r=0}\},$$

we obtain

$$D_{mn}=\delta_{mn}\{D+(1/2)\tau u_0^2-(D\tau^2/18)\times[\langle(\boldsymbol{\nabla}\cdot\mathbf{v})^2\rangle+\langle(\boldsymbol{\nabla}\times\mathbf{v})^2\rangle]+O(\tau^3)\}. \tag{31}$$

The last term $D_s=-(D\tau^2/18)\langle(\boldsymbol{\nabla}\times\mathbf{v})^2\rangle$ in Eq. (31) coincides with that obtained by Saffman [7] (see also Ref. [13]). Compressibility of a random velocity field causes an additional contribution to the diffusion tensor D_{mn}. Note that the last term in Eq. (27) generalizes the result by Saffman [7] to the case of compressible and anisotropic random velocity field.

Since in a homogeneous random flow $\boldsymbol{\nabla}\langle v_m v_n\rangle=0$ and $\nabla_p\mathbf{V}_{\rm eff}=0$, Eq. (26) reduces to the conservation law for the total number of particles. For inhomogeneous random velocity field Eqs. (C4)–(C6) yield

$$\frac{\partial N}{\partial t}+\nabla_m(\mathbf{V}_{\rm eff}N-D_{mn}\nabla_n N)=0. \tag{32}$$

Equation (32) coincides with that derived for the δ-function-correlated in time random velocity field [9,10] after the change $\tau\to 2\tau_c$, where τ_c is the correlation time. Note that in the derivation of the mean-field equations for the δ-function-correlated in time random velocity field model we assumed that the values $\langle\tau_c\mathbf{v}b\rangle$ and $\langle\tau_c v_m v_n\rangle$ do not vanish.

B. Random velocity field with Gaussian statistics for the Lagrangian trajectories

We assume here that the Lagrangian trajectories $\boldsymbol{\xi}^{(\tau)}$ and the random function $\boldsymbol{\xi}^{(\tau)}G((m+1)\tau,m\tau,\boldsymbol{\xi})$ have Gaussian statistics at the some instants $m\tau$ of the renewal. This model allows us to derive the equation for mean passive scalar field in a closed form. For a homogeneous random velocity field we assume that

$$M_{\boldsymbol{\xi}}\{\langle\boldsymbol{\xi}^{(\tau)}\rangle\}=0, \tag{33}$$

$$M_{\boldsymbol{\xi}}\{\langle\xi_m^{(\tau)}\xi_n^{(\tau)}\rangle\}=2\tau W\delta_{mn}, \tag{34}$$

$$M_{\boldsymbol{\xi}}\{\langle\xi_m^{(\tau)}G\rangle\}=-\tau V_m, \tag{35}$$

$$M_{\boldsymbol{\xi}}\{\langle G\rangle\}=1. \tag{36}$$

Equation (33) implies that there is no a mean drift, and Eq. (36) implies the conservation of the total number of particles (see Appendix D). An equation for the mean passive scalar field in the model for the random velocity field with Gaussian statistics for the Lagrangian trajectories is given by

$$\partial N/\partial t=\hat{L}N(t,\mathbf{x}), \tag{37}$$

$$\hat{L}=W\Delta+\tau^{-1}\ln(1-\tau\mathbf{V}\cdot\boldsymbol{\nabla}) \tag{38}$$

(see Appendix D). For small τV the operator $\hat{L}$ can be expanded in a series

$$\hat{L}=-\mathbf{V}\cdot\boldsymbol{\nabla}+(W\delta_{ij}-\tau V_i V_j/2)\nabla_i\nabla_j+\cdots. \tag{39}$$

For an inhomogeneous random velocity field, the operator $\hat{L}$ is given by

$$\hat{L}=\nabla_m W\nabla_m+\tau^{-1}\ln[1-\tau\boldsymbol{\nabla}\cdot(\mathbf{V}\cdots)] \tag{40}$$

(see Appendix D), where $M_{\boldsymbol{\xi}}\{\langle\eta_m\eta_n\rangle\}=2\tau W\delta_{mn}$, and $M_{\boldsymbol{\xi}}\{\langle\eta_m g\rangle\}=-\tau V_m$, and $M_{\boldsymbol{\xi}}\{\langle\boldsymbol{\xi}^{(\tau)}\rangle\}=\tau\mathbf{U}$, and $G=\overline{G}+g$, and $\boldsymbol{\xi}=\overline{\boldsymbol{\xi}}+\boldsymbol{\eta}$, and $\overline{G}=M_{\boldsymbol{\xi}}\{\langle G\rangle\}=\exp(\alpha\tau)$, and $\overline{\boldsymbol{\xi}}=M_{\boldsymbol{\xi}}\{\langle\boldsymbol{\xi}\rangle\}$, and $M_{\boldsymbol{\xi}}\{\langle g\rangle\}=0$, and $M_{\boldsymbol{\xi}}\{\langle\boldsymbol{\eta}\rangle\}=0$. Note that $\boldsymbol{\nabla}$ in Eqs. (37) and (40) is applied to both fields $\mathbf{V}$ and N. Equations (37) and (40) imply the conservation law for the total number of particles when $\mathbf{U}=\boldsymbol{\nabla}W$ and $\alpha=\tau^{-1}\ln(1-\tau\boldsymbol{\nabla}\cdot\mathbf{V})$. In the case of a homogeneous random velocity field, Eq. (40) coincides with Eq. (38). The drift velocity $\mathbf{V}$ is caused by the effect of compressibility.

C. Weakly inhomogeneous random velocity field

We present the operator $P(\tau,\mathbf{x},i\boldsymbol{\nabla})$ in the following form:

$$P(\tau,\mathbf{x},i\boldsymbol{\nabla})=\exp[-\tau(\boldsymbol{\nabla}\cdot\mathbf{V}^{(\rm eff)})+\tau(\nabla_m D_{mn}-V_n^{(\rm eff)})\nabla_n+\tau D_{mn}\nabla_m\nabla_n+\cdots]. \tag{41}$$

Equation (41) implies the conservation law for the total number of particles. On the other hand, Eq. (20) yields

$$P(\tau,\mathbf{x},i\boldsymbol{\nabla})=M_{\xi}\{\langle G(\tau,\boldsymbol{\xi}(\mathbf{x}))\exp[\boldsymbol{\xi}^{(\tau)}(\mathbf{x})\cdot\boldsymbol{\nabla}]\rangle\}$$
$$=M_{\xi}\{\langle G(\tau,\boldsymbol{\xi})\times[1+\xi_m^{(\tau)}\nabla_m+(1/2)\xi_m^{(\tau)}\xi_n^{(\tau)}\nabla_m\nabla_n+\cdots]\rangle\}$$
$$=\overline{G}(1-\tau V_m\nabla_m+\tau W_{mn}\nabla_m\nabla_n+\cdots), \qquad (42)$$

where $\mathbf{V}=-\tau^{-1}\overline{G\boldsymbol{\xi}^{(\tau)}}/\overline{G}$, $W_{mn}=(2\tau)^{-1}\overline{G\xi_m^{(\tau)}\xi_n^{(\tau)}}/\overline{G}$, and $\overline{F}=M_{\xi}\{\langle F\rangle\}$, and $\overline{G}=\exp[-\tau\boldsymbol{\nabla}\cdot\mathbf{V}^{(\mathrm{eff})}]$. An equation for the mean passive scalar field in the model for the weakly inhomogeneous random velocity field is given by Eq. (32), where the effective velocity $\mathbf{V}^{(\mathrm{eff})}$ and the tensor D_{mn} are determined by means of Eqs. (41) and (42), i.e.,

$$V_n^{(\mathrm{eff})}=V_n+\nabla_m D_{mn}, \qquad (43)$$

$$D_{mn}=W_{mn}-\tau V_m V_n/2. \qquad (44)$$

It is seen from Eq. (44) that compressibility and finite renewal time of the random velocity field cause a depletion of turbulent diffusion. These equations for small molecular diffusion and incompressible fluid flow (in which $\mathbf{V}=0$) coincide with those derived by Taylor [1].

V. DISCUSSION

In the present paper we derived the mean-field equation for a passive scalar (e.g., for a mean number density of particles) advected by a random incompressible and compressible velocity field with a finite renewal time. Generally, the mean-field equation is an integral equation. We used three models of a random velocity field: (i) a velocity field with a small renewal time; (ii) Gaussian approximation for Lagrangian trajectories; (iii) a small inhomogeneity of the velocity and mean passive scalar fields. For these models an explicit form of the mean-field equation for a passive scalar is found. The finite renewal time of the random velocity field results in the appearance of higher than second-order spatial derivatives in the mean-field equation.

The finite renewal time and compressibility of the velocity field result in a decrease of turbulent diffusion and a modification of an effective drift velocity. For a compressible velocity field the form of the mean-field equation for a passive scalar depends on details of the velocity field model, i.e., universality is lost. In particular, in the model of the random velocity field with a small inhomogeneity of the velocity and mean passive scalar fields, the mean-field equation for a passive scalar cannot be written as an equation of convective diffusion [Eq. (32)], because the effective drift velocity and gradient of the turbulent diffusion are not separated. On the other hand, in the model of a random velocity field in the Gaussian approximation for Lagrangian trajectories, they are separated. For an incompressible velocity field the universality exists in spite of the finite renewal time. For a small renewal time we recovered results obtained using a model with the δ-function-correlated in time velocity field. The criterion of the applicability of the approximation of the δ-function-correlated in time velocity field is given by $\tau\boldsymbol{\nabla}\cdot(\mathbf{V}^{(\mathrm{eff})}N)/N\ll 1$. This implies that the approximation of the δ-function-correlated in time velocity field is valid either for a weak inhomogeneity of the random velocity and mean passive scalar fields or a weak compressibility of the effective drift velocity $\mathbf{V}^{(\mathrm{eff})}$.

The obtained results may be of relevance in some atmospheric phenomena (e.g., atmospheric aerosols, cloud formation, and smog formation) and turbulent industrial flows. We considered a random velocity field with $\boldsymbol{\nabla}\cdot\mathbf{v}\neq 0$, which is due to, e.g., particle inertia (see, e.g., Refs. [9,14,16]). The velocity of particles $\mathbf{v}$ depends on the velocity of the surrounding fluid, and it can be determined from the equation of motion for a particle. This equation represents a balance of particle inertia with the fluid drag force produced by the motion of the particle relative to the surrounding fluid and gravity force. A solution of the equation of motion for small particles with $\rho_p\gg\rho$ yields $\mathbf{v}=\mathbf{u}+\mathbf{W}-\tau_p\{\partial\mathbf{u}/\partial t+[(\mathbf{u}+\mathbf{W})\cdot\boldsymbol{\nabla}]\mathbf{u}\}+O(\tau_p^2)$, where $\mathbf{u}$ is the velocity of the surrounding fluid, $\mathbf{W}=\tau_p\mathbf{g}$ is the terminal fall velocity, $\mathbf{g}$ is the acceleration due to gravity, τ_p is the characteristic time of coupling between the particle and atmospheric fluid (Stokes time), ρ_p is the material density of particles, and ρ is the density of the fluid. For instance, for spherical particles of radius a_* the Stokes time is $\tau_p=m_p/(6\pi a_*\rho\nu)$, where m_p is the particle mass and ν is the kinematic viscosity. The velocity field of particles is compressible, i.e., $\boldsymbol{\nabla}\cdot\mathbf{v}\neq 0$. Indeed, the equation for the velocity of particles and the Navier-Stokes equation for atmospheric fluid yield $\boldsymbol{\nabla}\cdot\mathbf{v}=\tau_p\Delta P_f/\rho+O(\tau_p^2)$, where P_f is atmospheric fluid pressure and we neglected small $\boldsymbol{\nabla}\cdot\mathbf{u}$. The degree of compressibility $\sigma=\langle(\boldsymbol{\nabla}\cdot\mathbf{v})^2\rangle/\langle(\boldsymbol{\nabla}\times\mathbf{v})^2\rangle$ of inertial particles velocity field is given by $\sigma=12\,\mathrm{Re}(\tau_p/\tau_0)^2$, where Re is the Reynolds number.

The inertia of particles results in that particles inside the turbulent eddy are carried out to the boundary regions between the eddies by inertial forces. On the other hand, the inertia effect causes $\boldsymbol{\nabla}\cdot\mathbf{v}\propto\tau_p\Delta P_f\neq 0$. In addition, for large Peclet numbers $\boldsymbol{\nabla}\cdot\mathbf{v}\propto -dn/dt$ [see Eq. (3)]. Therefore, $dn/dt\propto-\tau_p\Delta P_f$. This means that in regions where $\Delta P_f<0$ there is an accumulation of inertial particles (i.e., $dn/dt>0$). Similarly, there is an outflow of inertial particles from the regions with $\Delta P_f>0$. When there is a large-scale inhomogeneity of the temperature of the turbulent flow, the mean heat flux $\langle\mathbf{u}\theta\rangle\neq 0$. Therefore, fluctuations of both temperature θ and velocity $\mathbf{u}$ of fluid are correlated. Fluctuations of temperature cause fluctuations of pressure of fluid, and vice versa. The pressure fluctuations result in fluctuations of the number density of inertial particles. Indeed, an increase (decrease) of the pressure of atmospheric fluid is accompanied by an accumulation (outflow) of the particles. Therefore, the direction of mean flux of particles coincides with that of heat flux, i.e., $\langle\mathbf{v}n\rangle\propto\langle\mathbf{u}\theta\rangle\propto-\boldsymbol{\nabla}T$, where $T=\langle T_f\rangle$ is the mean temperature of an atmospheric fluid with the characteristic value T_*, and $T_f=T+\theta$. Therefore, the mean flux of the inertial particles is directed to the minimum of the mean temperature and the inertial particles are accumulated in this region, e.g., in the vicinity of the temperature inversion layer (for details see Refs. [9,16]).

The equation for the mean number density of particles $N=\langle n\rangle$ has the form of Eq. (32) after the change $\mathbf{V}_{\mathrm{eff}}\to\mathbf{V}_{\mathrm{eff}}+\mathbf{W}$, where

$$D_{mn}=D_T[\delta_{mn}-(3/2)(V_{\mathrm{eff}}/u_0)^2 e_m e_n], \qquad (45)$$

$$\mathbf{V}_{\mathrm{eff}}\sim-(1/2)\tau\langle\mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{v})\rangle\sim -W\Lambda_P\ln(\mathrm{Re})(\boldsymbol{\nabla}T)/T, \qquad (46)$$

where $\Lambda_P = |\nabla P_f / P_f|^{-1}$, $\mathrm{Re} = l_0 u_0 / \nu$ is the Reynolds number, and e_m is the unit vector in the direction opposite to the gravity $\mathbf{g}$. Equation (46) was derived in Refs. [9,16]. The last term in Eq. (45) describes a depletion of the turbulent diffusion coefficient due to the finite correlation time of a random velocity field. The effective velocity $\mathbf{V}_{\rm eff}$ of particles determines a turbulent contribution to particle velocity due to both the effect of inertia and the mean temperature gradient. The ratio $|\mathbf{V}_{\rm eff}/W|$ is of the order of

$$|\mathbf{V}_{\rm eff}/W| \sim (\Lambda_P/\Lambda_T)(\delta T/T_*) \ln \mathrm{Re}$$

(for details, see Refs. [9,16]), where δT is the temperature difference in the scale Λ_T, and T_* is the characteristic temperature. Using the characteristic parameters of the atmospheric turbulent boundary layer (see, e.g., Refs. [21,22])—the maximum scale of turbulent flow $l_0 \sim 10^3$–10^4 cm, the velocity in the scale l_0, $u_0 \sim 30$–100 cm/s, and the Reynolds number $\mathrm{Re} \sim 10^6$–10^7—we estimate the ratio $|\mathbf{V}_{\rm eff}/W|$ and the depletion of the turbulent diffusion coefficient. For particles with material density $\rho_p \sim 1$–2 g/cm^3 and radius $a_* = 30$ μm, the ratio $|\mathbf{V}_{\rm eff}/W| \approx 0.9$ for the temperature gradient 1 $K/200$ m, where $W \sim 10^{-2} a_*^2$ cm/s, and a_* is measured in microns. For these parameters the coefficient of turbulent diffusion in the vertical direction can be depleted by 25% due to the finite correlation time of a turbulent velocity field. The latter result is in compliance with the known anisotropy of the coefficient of turbulent diffusion in the atmosphere (see, e.g., Ref. [23]). Thus two competitive mechanisms of particle transport, i.e., the mixing by the decreased turbulent diffusion and accumulation of particles due to the effective velocity act simultaneously together with the effect of gravitational settling of particles.

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APPENDIX A: SOLUTION OF EQ. (3)

Here we show that Eq. (4) is a solution of Eq. (3). We calculate

$$\partial n/\partial t = \{[n(t+\Delta t, \mathbf{x}) - n(t, \mathbf{x})]/\Delta t\}_{\Delta t \to 0}. \tag{A1}$$

We consider $n(t,\mathbf{x})$ as an initial condition for solution (4) for the field $n(t+\Delta t,\mathbf{x})$. If the total field $n(t,\mathbf{x})$ is specified at an instant t, then we can determine the total field $n(t+\Delta t,\mathbf{x})$ at a near instant $t+\Delta t$ by means of substitutions $t \to t+\Delta t$ and $s \to t$ in Eq. (4). The result is given by

$$n(t+\Delta t, \mathbf{x}) = M_{\boldsymbol{\zeta}}\{J(t+\Delta t, t, \boldsymbol{\zeta}) \tilde{G}(t+\Delta t, t, \boldsymbol{\zeta}) n(t, \boldsymbol{\zeta}(\Delta t))\}, \tag{A2}$$

where

$$J(t+\Delta t, t, \boldsymbol{\zeta}) = \exp\left[-(2D)^{-1/2} \int_0^{\Delta t} v_p(t+\Delta t - \eta, \boldsymbol{\zeta}) dw_p(\eta) - (4D)^{-1} \int_0^{\Delta t} \mathbf{v}^2(t+\Delta t - \eta, \boldsymbol{\zeta}) d\eta \right], \tag{A3}$$

$$\tilde{G}(t+\Delta t, t, \boldsymbol{\zeta}) = \exp\left[-\int_t^{t+\Delta t} b(\sigma, \boldsymbol{\zeta}) d\sigma \right], \tag{A4}$$

where v_p and w_p are the components of the vectors $\mathbf{v}$ and $\mathbf{w}$, respectively, in a Cartesian system of coordinates, $\boldsymbol{\zeta}(t_2,t_1) = \mathbf{x} + \sqrt{2D}[\mathbf{w}(t_2 - t_1)]$. Now we expand the expansions of the functions $J(t+\Delta t, t, \boldsymbol{\zeta})$ and $\tilde{G}(t+\Delta t, t, \boldsymbol{\zeta})$ in a Taylor series for small Δt:

$$J = 1 - (I^{(1)} + I^{(2)}) + (1/2)(I^{(1)} + I^{(2)})^2 + \cdots,$$

$$G = 1 - I^{(3)} + (1/2)(I^{(3)})^2 + \cdots,$$

where $I^{(1)} = (2D)^{-1/2} \int_0^{\Delta t} v_p(t+\Delta t - \eta, \boldsymbol{\zeta}) dw_p(\eta)$, $I^{(2)} = (4D)^{-1} \int_0^{\Delta t} \mathbf{v}^2(t+\Delta t - \eta, \boldsymbol{\zeta}) d\eta$, and $I^{(3)} = \int_t^{t+\Delta t} b(\sigma, \boldsymbol{\zeta}) d\sigma$. The integrals $I^{(1)}$, $I^{(2)}$, and $I^{(3)}$ can be evaluated by means of the "mean value" theorem. This yields $I^{(1)} = (2D)^{-1/2} v_p w_p(\Delta t)$, $I^{(2)} = (4D)^{-1} v^2 \Delta t$, and $I^{(3)} = b \Delta t$, where the functions v_p, v^2, and b are calculated at the instants which are inside the interval $(t, t+\Delta t)$. Thus the expansions of the functions $J(t+\Delta t, t, \boldsymbol{\zeta})$ and $\tilde{G}(t+\Delta t, t, \boldsymbol{\zeta})$ in a Taylor series are given by

$$J(t+\Delta t, t, \boldsymbol{\zeta}) = 1 - (2D)^{-1/2} v_p w_p + (4D)^{-1} v_p v_j w_p w_j - (4D)^{-1} v^2 \Delta t + O((\Delta t)^{3/2}), \tag{A5}$$

$$\tilde{G}(t+\Delta t, t, \boldsymbol{\zeta}) = 1 - b \Delta t + O((\Delta t)^2). \tag{A6}$$

Here we took into account the definition of the Wiener process $M\{w_m(t) w_n(t)\} = t \delta_{mn}$. This implies that $|\mathbf{w}| \sim t^{1/2}$.

Now we expand the functions $n(t, \boldsymbol{\zeta}(\Delta t))$ in a Taylor series in the vicinity of the point $\mathbf{x}$:

$$n(t, \boldsymbol{\zeta}(\Delta t)) = n(t, \mathbf{x}) + (\nabla_p n)(\boldsymbol{\zeta} - \mathbf{x})_p + (1/2)(\nabla_p \nabla_s n)(\boldsymbol{\zeta} - \mathbf{x})_p (\boldsymbol{\zeta} - \mathbf{x})_s + \cdots, \tag{A7}$$

where $\boldsymbol{\zeta}(\Delta t) - \mathbf{x} = \sqrt{2D}\mathbf{w}(\Delta t)$. Using the definition of $\boldsymbol{\zeta}(\Delta t)$, we obtain

$$n(t, \boldsymbol{\zeta}(\Delta t)) = n(t, \mathbf{x}) + (2D)^{1/2} w_p (\nabla_p n) + D w_p w_s (\nabla_p \nabla_s n) + O((\Delta t)^{3/2}). \tag{A8}$$

Combination of Eqs. (A2), (A5), (A6), and (A8), and averaging over the Wiener paths $\boldsymbol{\zeta}$, yield the expression for the passive scalar field $n(t+\Delta t, \mathbf{x})$. Using Eq. (A1) we obtain Eq. (3). Thus it is shown that for small Δt Eq. (4) is the solution of Eq. (3) with the initial condition $n(t=s, \mathbf{x}) = n(s, \mathbf{x})$. Now we use the following property of the function $J(t, s, \boldsymbol{\zeta})$:

$$J(t, s, \boldsymbol{\zeta}(t,s)) = J(t, s', \boldsymbol{\zeta}(t, s')) J(s', s, \boldsymbol{\zeta}(s', s)) \tag{A9}$$

[see Eq. (6)]. This property allows us to calculate the right-hand side of Eq. (A1) and to show that the above proof is valid for an arbitrary time t.

APPENDIX B: HOMOGENEOUS AND INHOMOGENEOUS RANDOM VELOCITY FIELDS. DERIVATION OF EQS. (18), (19), AND (20)

The Fourier transformation of Eq. (15) yields

$$N((m+1)\tau,\mathbf{k})=P(\tau,-\mathbf{k})N(m\tau,\mathbf{k}), \tag{B1}$$

$$P(\tau,-\mathbf{k})=(2\pi)^{-3}\int P(\tau,\mathbf{z})\exp(i\mathbf{k}\cdot\mathbf{z})d\mathbf{z}, \tag{B2}$$

where $\mathbf{z}=\mathbf{y}-\mathbf{x}$. Note that a particular solution [Eq. (11)], averaged over the ensemble of random velocity fields with the initial condition $N(s,\mathbf{z})=(2\pi)^{-3}\exp(i\mathbf{k}\cdot\mathbf{z})$, coincides in form with integral (B2). Now we use Eqs. (11)–(14) to evaluate integral (B2):

$$\begin{aligned}P(\tau,-\mathbf{k})&=E\{\langle S((m+1)\tau,m\tau,\mathbf{x},\mathbf{x}+\mathbf{z})\\&\quad\times\exp[i\sqrt{2D}\mathbf{k}\cdot\mathbf{w}(t-s)]\rangle\}\\&=E\{M_{\boldsymbol{\mu}}\{\langle J(t,s,\boldsymbol{\mu})\tilde{G}(t,s,\boldsymbol{\mu})\exp[i\sqrt{2D}\mathbf{k}\cdot\mathbf{w}]\rangle\}\}\\&=M_{\boldsymbol{\zeta}}\{\langle J(t,s,\boldsymbol{\zeta})\tilde{G}(t,s,\boldsymbol{\zeta})\exp[i\sqrt{2D}\mathbf{k}\cdot\mathbf{w}]\rangle\}.\end{aligned} \tag{B3}$$

Note that Eq. (B3) can be obtained directly from the solution given by Eq. (4) with the initial condition $n(s,\mathbf{z})=\exp(i\mathbf{k}\cdot\mathbf{z})$ at $\mathbf{x}=0$. Now we rewrite equation for the function $P(\tau,-\mathbf{k})$ using the Feynman-Kac formula (8). This yields Eq. (18). The solution $n((m+1)\tau,\mathbf{k})$ also can be rewritten using the Feynman-Kac formula:

$$\begin{aligned}n((m+1)\tau,\mathbf{x})=&\int\exp(i\mathbf{k}\cdot\mathbf{x})\\&\times M_{\boldsymbol{\xi}}\{\exp(i\mathbf{k}\cdot\boldsymbol{\xi}^{(\tau)})G(\tau,\boldsymbol{\xi})n(m\tau,\mathbf{k})\}d\mathbf{k}.\end{aligned} \tag{B4}$$

Note that Eqs. (B1), (B2), (18), and (B4) are also valid for $D=0$.

In the case of inhomogeneous random velocity field we make a change of variables $(\mathbf{x},\mathbf{y})\rightarrow(\mathbf{x},\mathbf{y}=\mathbf{z}+\mathbf{x})$ in Eqs. (15), and use that $P(\tau,\mathbf{x},\mathbf{y})=P(\tau,\mathbf{x},\mathbf{z}+\mathbf{x})\equiv P(\tau,\mathbf{x},\mathbf{z})$. The Fourier transformation in Eq. (15) yields

$$\begin{aligned}N((m+1)\tau,\mathbf{x})=(2\pi)^{-3}&\int\int P(\tau,\mathbf{x},\mathbf{k})\\&\times\exp(i\mathbf{k}\cdot\mathbf{z})d\mathbf{k}\int N(m\tau,\mathbf{q})\\&\times\exp[i\mathbf{q}\cdot(\mathbf{z}+\mathbf{x})]d\mathbf{q}d\mathbf{z}.\end{aligned}$$

Since $\delta(\mathbf{k}+\mathbf{q})=(2\pi)^{-3}\int\exp[i(\mathbf{k}+\mathbf{q})\cdot\mathbf{z}]d\mathbf{z}$, we obtain that

$$N((m+1)\tau,\mathbf{x})=\int P(\tau,\mathbf{x},-\mathbf{q})N(m\tau,\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{x})d\mathbf{q}. \tag{B5}$$

In Eq. (B5) we expand the function $P(\tau,\mathbf{x},-\mathbf{q})$ in Taylor series at $\mathbf{q}=0$, and after Fourier transformation we arrive at an equation

$$N((m+1)\tau,\mathbf{x})=P(\tau,\mathbf{x},i\boldsymbol{\nabla})N(m\tau,\mathbf{x}).$$

Using Eq. (18), we obtain

$$P(\tau,\mathbf{x},-\mathbf{q})=M_{\boldsymbol{\xi}}\{\langle\exp[i\boldsymbol{\xi}^{(\tau)}(\mathbf{x})\cdot\mathbf{q}]G(\tau,\boldsymbol{\xi}(\mathbf{x}))\rangle\},$$

$$P(\tau,\mathbf{x},i\boldsymbol{\nabla})=M_{\boldsymbol{\xi}}\{\langle G(\tau,\boldsymbol{\xi}(\mathbf{x}))\exp[\boldsymbol{\xi}^{(\tau)}(\mathbf{x})\cdot\boldsymbol{\nabla}]\rangle\}.$$

Thus we obtained Eqs. (19) and (20).

APPENDIX C: RANDOM VELOCITY FIELD WITH A SMALL RENEWAL TIME. DERIVATION OF EQ. (26)

We expand the velocity field $\mathbf{v}(\sigma,\boldsymbol{\xi})$ in a Taylor series in the vicinity of the point $\mathbf{x}$ for small renewal time τ,

$$\begin{aligned}\mathbf{v}(\sigma,\boldsymbol{\xi})=&\mathbf{v}(t,\mathbf{x})+(\nabla_p\mathbf{v})(\boldsymbol{\xi}-\mathbf{x})_p\\&+(1/2)(\nabla_p\nabla_n\mathbf{v})(\boldsymbol{\xi}-\mathbf{x})_p(\boldsymbol{\xi}-\mathbf{x})_n+\cdots,\end{aligned} \tag{C1}$$

where

$$\boldsymbol{\xi}(t_2,t_1)-\mathbf{x}=-\int_0^{t_2-t_1}\mathbf{v}(t_s,\boldsymbol{\xi}_s)ds+\sqrt{2D}\mathbf{w}(t_2-t_1), \tag{C2}$$

where $t_s=t_2-s$ and $\boldsymbol{\xi}_s=\boldsymbol{\xi}(t_2,t_s)$. Using the iteration procedure we obtain the expansion of the velocity field $\mathbf{v}(\sigma,\boldsymbol{\xi})$ in a Taylor series in the vicinity of the point $\mathbf{x}$ for small renewal time τ,

$$\begin{aligned}\mathbf{v}(\sigma,\boldsymbol{\xi})=&\mathbf{v}(t,\mathbf{x})+(\nabla_p\mathbf{v})\Bigg[\sqrt{2D}w_p-v_p\sigma\\&-\sqrt{2D}(\nabla_p v_l)\int_0^\sigma w_l d\sigma'\Bigg]+(1/2)(\nabla_p\nabla_n\mathbf{v})\\&\times[2Dw_pw_n-\sqrt{2D}\sigma(v_pw_n+v_nw_p)]+O(\sigma^2),\end{aligned} \tag{C3}$$

and similarly for the function $b(\sigma,\boldsymbol{\xi})$. Thus the expansions in a Taylor series of the functions $\xi^{(\tau)}$, $\int_t^{t+\tau}b(\sigma,\boldsymbol{\xi}_\sigma)d\sigma$, and $G(\tau,\boldsymbol{\xi})$ in the vicinity of the point $\mathbf{x}$ for small renewal time τ are given by

$$\begin{aligned}\xi_m^{(\tau)}=&\sqrt{2D}w_m-v_m(t,\mathbf{x})\tau-\sqrt{2D}(\nabla_pv_m)\int_0^\tau w_pd\sigma\\&+\frac{1}{2}v_p(\nabla_pv_m)\tau^2-(\nabla_p\nabla_nv_m)\Bigg[D\int_0^\tau w_pw_nd\sigma\\&-(1/2)\sqrt{2D}\bigg(v_p\int_0^\tau\sigma w_nd\sigma+v_n\int_0^\tau\sigma w_pd\sigma\bigg)\Bigg]\\&+\sqrt{2D}(\nabla_pv_m)(\nabla_lv_p)\int_0^\tau\bigg(\int_0^\sigma w_lds\bigg)d\sigma+O(\tau^3),\end{aligned} \tag{C4}$$

$$\int_t^{t+\tau} b(\sigma,\boldsymbol{\xi}_\sigma)d\sigma = b(t,\mathbf{x})\tau - (1/2)(v_p\nabla_p)b\tau^2 + \sqrt{2D}(\nabla_p b)\int_0^\tau w_p d\sigma + D(\nabla_p\nabla_n b)\int_0^\tau w_p w_n d\sigma + O(\tau^{5/2}), \tag{C5}$$

$$G(\tau,\boldsymbol{\xi}) = 1 - b(t,\mathbf{x})\tau + (1/2)\nabla_p(v_p b)\tau^2 - \sqrt{2D}(\nabla_p b)\int_0^\tau w_p d\sigma - D(\nabla_p\nabla_n b)\int_0^\tau w_p w_n d\sigma + O(\tau^{5/2}). \tag{C6}$$

We will take into account that for a homogeneous random flow, $\nabla_j\langle v_p b\rangle = 0$.

APPENDIX D: RANDOM VELOCITY FIELD WITH GAUSSIAN STATISTICS FOR THE LAGRANGIAN TRAJECTORIES. DERIVATION OF EQS. (37) AND (40)

Equation (36) implies the conservation of the total number of particles. Indeed, for $\mathbf{k}=0$, Eq. (17) yields $P(\tau,\mathbf{k}=0)=1$, because $N(m\tau,\mathbf{k}=0)=(2\pi)^{-3}\int N(m\tau,\mathbf{x})d\mathbf{x}=\text{const}$. On the other hand, Eq. (18) for $\mathbf{k}=0$ yields $M_{\xi}\{\langle G\rangle\} = P(\tau,\mathbf{k}=0)=1$. Equations (18) and (33)–(36) yield

$$P(\tau,-\mathbf{k}) = (1 - i\tau\mathbf{k}\cdot\mathbf{V})\exp(-\tau k^2 W). \tag{D1}$$

In derivation of Eq. (D1), we used the identities

$$E\{\exp(a\eta)\} = \exp(a^2\sigma^2/2), \tag{D2}$$

$$\left(\frac{\partial}{\partial\lambda}\exp(c+\lambda g)\right)_{\lambda=0} = g\exp(c), \tag{D3}$$

where η is a Gaussian random variable with zero mean value, and the dispersion σ^2. Using Eq. (D1) we rewrite Eq. (17) in $\mathbf{r}$ space:

$$N((m+1)\tau,\mathbf{x}) = (1 - \tau\mathbf{V}\cdot\boldsymbol{\nabla})\exp(\tau W\Delta)N(m\tau,\mathbf{x}). \tag{D4}$$

Introducing the operator $\hat{L} = W\Delta + \tau^{-1}\ln(1-\tau\mathbf{V}\cdot\boldsymbol{\nabla})$, we obtain

$$N((m+1)\tau,\mathbf{x}) = \exp(\tau\hat{L})N(m\tau,\mathbf{x}). \tag{D5}$$

Note that Eq. (D5) can be presented in the form of a differential equation (37). In order to do this we will use the identity

$$N((m+1)\tau,\mathbf{x}) = \exp(\tau\partial/\partial t)N(m\tau,\mathbf{x}), \tag{D6}$$

which follows from the Taylor expansion

$$f(t+\tau) = \sum_{m=1}^{\infty}\left(\tau\frac{\partial}{\partial t}\right)^m \frac{f(t)}{m!} = \exp\left(\tau\frac{\partial}{\partial t}\right)f(t).$$

Comparing Eqs. (D6) and (D5), we obtain

$$\exp\left(\tau\frac{\partial}{\partial t}\right)N(m\tau,\mathbf{x}) = \exp(\tau\hat{L})N(m\tau,\mathbf{x}).$$

For the sake of simplicity we assume that operator $\hat{L}$ has a complete set of eigenfunctions. Expanding the function $N(m\tau,\mathbf{x})$ in a series of the eigenfunctions, we obtain Eq. (37).

For an inhomogeneous random velocity field, Eqs. (33)–(36) are modified:

$$M_{\xi}\{\langle\boldsymbol{\xi}^{(\tau)}\rangle\} = \tau\mathbf{U}, \tag{D7}$$

$$M_{\xi}\{\langle\eta_m\eta_n\rangle\} = 2\tau W\delta_{mn}, \tag{D8}$$

$$M_{\xi}\{\langle\eta_m g\rangle\} = -\tau V_m, \tag{D9}$$

$$\overline{G} = M_{\xi}\{\langle G\rangle\} = \exp(\alpha\tau), \tag{D10}$$

where $G=\overline{G}+g$, $\boldsymbol{\xi}^{(\tau)}=\overline{\boldsymbol{\xi}}+\boldsymbol{\eta}$, $\overline{\boldsymbol{\xi}}=M_{\xi}\{\langle\boldsymbol{\xi}^{(\tau)}\rangle\}$, and $M_{\xi}\{\langle g\rangle\}=0$, and $M_{\xi}\{\langle\boldsymbol{\eta}\rangle\}=0$, and we represented the functions G and $\boldsymbol{\xi}$ as a sum of the mean value and fluctuations. Equations (D7)–(D10) contain four functions ($\mathbf{U},\mathbf{V},W$, and α), and two of them are independent. Therefore we have to find two additional equations for these two parameters. To this purpose we use that the mean-field equation for number density of particles implies the conservation law for the total number of particles. We also use that

$$P(\tau,\mathbf{x},-\mathbf{q}) = \exp[i\tau(\mathbf{U}\cdot\mathbf{q})][\exp(\alpha\tau)M_{\xi}\{\langle\exp(i\boldsymbol{\eta}\cdot\mathbf{q})\rangle\} + M_{\xi}\{\langle g\exp(i\boldsymbol{\eta}\cdot\mathbf{q})\rangle\}]. \tag{D11}$$

By means of identities (D2) and (D3), we rewrite Eq. (D11) in $\mathbf{r}$ space,

$$P(\tau,\mathbf{x},i\boldsymbol{\nabla}) = [\exp(\alpha\tau) - \tau(\mathbf{V}\cdot\boldsymbol{\nabla})]\exp[\tau(\mathbf{U}\cdot\boldsymbol{\nabla}) + \tau W\Delta], \tag{D12}$$

where the operators $\boldsymbol{\nabla}$ and Δ act only on the function N. Therefore, the operator $\hat{L}$ for Eq. (D5) for an inhomogeneous random velocity field is given by Eq. (40). Equation (37) implies the conservation law for the total number of particles when $\mathbf{U}=\boldsymbol{\nabla}W$ and $\alpha=\tau^{-1}\ln(1-\tau\boldsymbol{\nabla}\cdot\mathbf{V})$.

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