Recurrence time statistics in deterministic and stochastic dynamical systems in continuous time: A comparison

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The dynamics of transitions between the cells of a finite phase-space partition is analyzed for deterministic and stochastic dynamical systems in continuous time. Special emphasis is placed on the dependence of mean recurrence time on the resolution τ between successive observations, in the limit $\tau \rightarrow 0$. In deterministic systems the limit is found to exist, and to depend on only the intrinsic parameters of the underlying dynamics. In stochastic systems two different cases are identified, leading to a τ-independent behavior and a $\tau^{1/2}$ behavior, depending on whether a finite speed of propagation of the signals exists or not. An extension of the results to the second moment of the recurrence time is outlined.

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I. INTRODUCTION

Recurrence time statistics [1] provides useful information on the nature of the processes going on in a dynamical system in a variety of contexts, from the foundations of statistical mechanics [2] to the classification of atmospheric ''analogs'' and the prediction of short-term weather fluctuations [3]. It has a long history in the theory of stochastic processes and, more recently, it has been the subject of investigations in low-dimensional deterministic systems [4,5] giving rise to complex behavior.

The advent of nonlinear dynamics and chaos theories has highlighted a number of unexpected connections between deterministic dynamics and stochastic processes. Thus information-theoretic concepts are at the basis of the definition of the Kolmogorov-Sinai entropy [6], one of the principal indicators of dynamical complexity. Furthermore, upon projection on a suitable phase space partition, deterministic dynamics can be mapped in a rigorous manner to a Markov process or even a Bernoulli process [7]. What is more, classic and time-honored examples of random processes such as Brownian motion are now realized to be manifestations of deterministic chaos in a high-dimensional phase space [8].

On the other hand, at least certain stochastic processes, taken in the strict sense, do present marked differences from deterministic behavior. One of the most prominent of these is the lack of uniqueness of a realization (the analog of a phase trajectory in the deterministic case), even when initial conditions are specified. The objective of the present paper is a comparative study of continuous time deterministic and stochastic processes on the basis of their recurrence properties. We shall focus specifically on the mean recurrence time in the limit in which the sampling or resolution time τ tends to zero.

We shall be dealing with a one-parameter family T_t of transformations, where t denotes time. It will be assumed that T_t is ergodic and that it induces in the phase space Γ a completely additive measure μ, such that $\mu(\Gamma)=1$. Let C be a subset of Γ such that $\mu(C)>0$. Supposing that the dynamical system is probed after every interval τ, the following expression may be derived [1] for the mean time of recurrence to C:

$$\langle \theta_\tau \rangle = \tau/\mu(C). \tag{1.1}$$

Now, in the limit $\tau \rightarrow 0$ of continuous sampling, a naive application of Eq. (1.1) yields the trivial (and incorrect) result $\langle \theta_\tau \rangle \rightarrow 0$. This happens because, in deriving Eq. (1.1), no account has been taken of the fact that the representative point in phase space must definitely leave the cell C first before returning to it, in order for the event to qualify as a genuine recurrence to C. A reformulation of the recurrence problem that rectifies this drawback was given by Smoluchowski [9] (also see Ref. [2]), leading to the modified expression

$$\langle \theta_\tau \rangle = \frac{\tau[1-\mu(C)]}{\mu(C)-\mu(C_1)} \tag{1.2}$$

for the mean time of recurrence to C. Here C_1 denotes the set of points $\mathbf{x} \in \Gamma$ such that

$$C_1 = \{\mathbf{x} \in C, T_\tau \mathbf{x} \in C\}. \tag{1.3}$$

In other words, $\mu(C_1)$ is the measure of the set of points that start in C and remain in that cell without leaving it at time τ. What is interesting and noteworthy is that Eq. (1.2) permits the possibility of a nonvanishing limiting value for the mean recurrence time when one passes to the limit of continuous sampling, $\tau \downarrow 0$.

In Sec. II we recapitulate in brief the derivation and import of Eqs. (1.2) and (1.3), starting from the corresponding recurrence time distributions. In Sec. III, we consider deterministic dynamics and show that $\langle\theta_\tau\rangle$ attains a well-defined limiting value as $\tau\to 0$, that depends entirely on the intrinsic parameters of the dynamics, provided certain ergodic properties are satisfied. Section IV is devoted to a class of stochastic processes, comprising both jump processes and continuous processes, for which a non-vanishing $\tau=0$ limit of $\langle\theta_\tau\rangle$ exists. It turns out that the essential property required for this is the existence of a finite transition rate between states, satisfying suitable integrability conditions. In particular, we elaborate on the specific example of flows (linear and nonlinear) driven by dichotomous noise. In Sec. V, the diffusive counterparts of the foregoing flows are considered. We show that $\langle\theta_\tau\rangle$, as given by Eq. (1.2), retains in this case a characteristic proportionality to $\tau^{1/2}$ for small τ, over and above its dependence on the intrinsic parameters of the process. The origin of this behavior is elucidated. In Sec. VI, we extend the Smoluchowski formulation to higher moments of the recurrence time. Specifically, a formal expression is derived for the second moment in the $\tau\to 0$ limit. Section VII is devoted to concluding remarks.

II. RECAPITULATION OF THE SMOLUCHOWSKI FORMULA

It is helpful, for what follows, to recapitulate briefly how Eq. (1.2) is arrived at [2]. We begin with a deterministic system in discrete time $n\tau(n=0,1,\ldots)$ with a time evolution operator T_τ and invariant measure $d\mu=\rho(\mathbf{x})d\mathbf{x}$. Define the sets C_n by

$$C_1=\{\mathbf{x}\in C, T_\tau\mathbf{x}\in C\}$$

$$C_n=\{\mathbf{x}\in C, T_\tau\mathbf{x}\in\bar{C},\ldots,T_\tau^{n-1}\mathbf{x}\in\bar{C},T_\tau^{n}\mathbf{x}\in C\},\quad n\geqslant 2. \tag{2.1}$$

Let $W_n(n\geqslant 1)$ be the measure of the set $\{\mathbf{x}\in\bar{C},T_\tau\mathbf{x}\in\bar{C},\ldots,T_\tau^{n-1}\mathbf{x}\in\bar{C}\}$. In terms of $\chi(\mathbf{x})$, the indicator function of C,

$$W_n=\int_\Gamma\prod_{k=0}^{n-1}[1-\chi(T_\tau^k\mathbf{x})]d\mu. \tag{2.2}$$

Further, let $W_0=\int_\Gamma d\mu=1$. Then, using the fact that $\mu(\mathbf{x})=\mu(T_\tau\mathbf{x})$, it can be shown that

$$\mu(C_n)=W_{n-1}-2W_n+W_{n+1},\quad n\geqslant 1. \tag{2.3}$$

Since the sequence $\{W_n\}$ is nonincreasing and bounded from below (by 0), $\lim_{n\to\infty}W_n$ exists. If follows that

$$\sum_{n=1}^{\infty}\mu(C_n)=1-W_1=\mu(C), \tag{2.4}$$

so that recurrence to C is assured for almost all initial conditions. Moreover, the mean recurrence time to C is, by definition,

$$\langle\theta_\tau\rangle=\tau\sum_{n=1}^{\infty}n\mu(C_n)\Big/\sum_{n=1}^{\infty}\mu(C_n), \tag{2.5}$$

which simplifies to

$$\langle\theta_\tau\rangle=\tau(1-\lim_{N\to\infty}W_N)/\mu(C). \tag{2.6}$$

As the system is ergodic and $\mu(C)>0$, $\lim_{N\to\infty}W_N$ must in fact vanish. Therefore,

$$\langle\theta_\tau\rangle=\frac{\tau}{\mu(C)}=\frac{\tau}{1-W_1}. \tag{2.7}$$

As mentioned in Sec. I, formula (2.7) suffers from the defect that $\langle\theta_\tau\rangle$ vanishes as $\tau\to 0$, essentially because a *stay* of the representative point in C has been counted as a *recurrence*. Smoluchowski's modification [9] consists of the replacement of Eq. (2.5) by the alternative definition

$$\langle\theta_\tau\rangle=\tau\sum_{n=1}^{\infty}n\mu(C_{n+1})\Big/\sum_{n=1}^{\infty}\mu(C_{n+1}), \tag{2.8}$$

which simplifies, on using Eq. (2.3), to

$$\langle\theta_\tau\rangle=\frac{\tau[1-\mu(C)]}{\mu(C)-\mu(C_1)}=\frac{\tau W_1}{W_1-W_2}. \tag{2.9}$$

For a better understanding of this modification, and also because we shall be dealing with stochastic systems as well, let us express the foregoing in terms of the corresponding probability measures. Thus $\mu(C_n)$ is simply the joint probability $P(C,0;\bar{C},\tau;\ldots;\bar{C},(n-1)\tau;C,n\tau)$, where the coarse-grained probabilities are defined in terms of the pointwise probability densities according to

$$P(C)=\int_C\rho(\mathbf{x})d\mathbf{x}, \tag{2.10}$$

$$\begin{aligned}P(C,0;\bar{C},\tau)&=\int_C d\mathbf{x}_0\int_{\bar{C}}d\mathbf{x}\,\rho(\mathbf{x}_0,0;\mathbf{x},\tau)\\&=\int_C d\mathbf{x}_0\int_{\bar{C}}d\mathbf{x}\,\rho(\mathbf{x}_0)\rho(\mathbf{x}_0,0|\mathbf{x},\tau),\end{aligned} \tag{2.11}$$

and so on, where $\rho(\mathbf{x})$ is the invariant probability density. The original formula for $\langle\theta_\tau\rangle$, Eq. (2.7), follows if we define the probability $F(n)$ of a first return to C as the *conditional* probability

$$F(n)=P(C,0|\bar{C},\tau;\ldots;\bar{C},(n-1)\tau;C,n\tau), \tag{2.12}$$

and re-express it in terms of the joint probability, i.e.,

$$\begin{aligned}F(n)&=P(C,0;\bar{C},\tau;\ldots;\bar{C},(n-1)\tau;C,n\tau)/P(C)\\&=\mu(C_n)/\mu(C),\end{aligned} \tag{2.13}$$

on using Eq. (2.4). On the other hand, the modified formula (2.9) corresponds to defining $F(n)$ as the conditional probability

$$F(n)=P(C,0;\overline{C},\tau|\overline{C},2\tau;\ldots;\overline{C},n\tau;C,(n+1)\tau). \tag{2.14}$$

Once again, in terms of joint probabilities, this becomes

$$F(n)=P(C,0;\overline{C},\tau;\ldots;\overline{C},n\tau;C,(n+1)\tau)/P(C,0;\overline{C},\tau). \tag{2.15}$$

The numerator is just $\mu(C_{n+1})$, while the denominator can be rewritten as

$$P(C,0;\overline{C},\tau)=P(C)-P(C,0;C,\tau). \tag{2.16}$$

Therefore,

$$F(n)=\mu(C_{n+1})/[\mu(C)-\mu(C_1)] \tag{2.17}$$

and Eqs. (2.8) and (2.9) follow at once. In handling probabilities (measures) in coarse-grained or cell dynamics, it is important to note that while the joint probability $P(C,0;C',\tau)$ can be written as in Eq. (2.11), the conditional probability $P(C,0|C',\tau)$ *cannot* be written directly as $\int_C d\mathbf{x}_0\int_{C'}d\mathbf{x}_1\rho(\mathbf{x}_0,0|\mathbf{x}_1,\tau)$.

Returning to Eq. (2.9), the mean recurrence time in the continuous time limit is given by

$$\langle t\rangle\equiv\lim_{\tau\to 0}\langle\theta_\tau\rangle=\lim_{\tau\to 0}\frac{\tau[1-\mu(C)]}{\mu(C)-\mu(\mathbf{x}\in C;T_\tau\mathbf{x}\in C)}. \tag{2.18}$$

Equivalently, in terms of probability measures,

$$\langle t\rangle=\lim_{\tau\to 0}\frac{[1-P(C)]}{Q_\tau}, \tag{2.19}$$

where

$$Q_\tau=\frac{1}{\tau}[P(C)-P(C,0;C,\tau)]. \tag{2.20}$$

Thus the leading small-τ behavior of $P(C)-P(C,0;C,\tau)$ determines $\langle t\rangle$. In particular if this difference turns out to be regular in the neighborhood of $\tau=0$ and is $O(\tau)$ as $\tau\to 0$, we obtain a finite, nonzero $\langle t\rangle$. In what follows, we shall examine a variety of dynamical systems, both deterministic and stochastic, to study the existence or otherwise of a mean recurrence time in the $\tau=0$ limit.

III. DETERMINISTIC DYNAMICS

Deterministic dynamical systems are characterized by the property that the instantaneous state $\mathbf{x}_t$ in phase space is uniquely determined from the initial state $\mathbf{x}_0$ according to

$$\mathbf{x}_t=T_t\mathbf{x}_0, \tag{3.1}$$

where T_t is the time evolution operator. If the dynamics runs continuously in time (t), the evolution can further be cast in the form of a set of first-order differential equations

$$\frac{dx}{dt}=f(\mathbf{x}), \tag{3.2}$$

where f is the evolution operator.

As we have seen in Sec. II, in order to evaluate $\langle\theta_\tau\rangle$ [Eqs. (2.9) and (2.17)] one needs the two-time probability density $\rho(\mathbf{x}_0,0;\mathbf{x},\tau)$. Owing to the constraint of Eq. (3.1), in a deterministic system this quantity is given by

$$\rho(\mathbf{x}_0,0;\mathbf{x},\tau)=\rho(\mathbf{x}_0)\,\delta(\mathbf{x}-T_\tau\mathbf{x}_0), \tag{3.3}$$

yielding

$$\langle\theta_\tau\rangle=[1-P(C)]/Q_\tau, \tag{3.4}$$

where

$$Q_\tau\equiv\frac{1}{\tau}\left[\int_C d\mathbf{x}_0\rho(\mathbf{x}_0)-\int_C d\mathbf{x}_0\rho(\mathbf{x}_0)\int_C d\mathbf{x}\,\delta(\mathbf{x}-T_\tau\mathbf{x}_0)\right]. \tag{3.5}$$

This may be reduced to

$$Q_\tau=\frac{1}{\tau}\int_C d\mathbf{x}\left[\rho(\mathbf{x})-\sum_\alpha\left|\frac{\partial\mathbf{x}^\alpha_{-\tau}}{\partial\mathbf{x}}\right|^{-1}\rho(\mathbf{x}^\alpha_{-\tau})\right], \tag{3.6}$$

where the sum runs over all the preimages $\mathbf{x}^\alpha_{-\tau}$ of $\mathbf{x}$ that lie in C. Precisely at $\tau=0,\rho(\mathbf{x}^\alpha_{-\tau})=\rho(\mathbf{x})$, and the Jacobian determinant is equal to unity. As a result, the integrand in Eq. (3.6) vanishes. As τ increases from zero, the $\mathbf{x}^\alpha_{-\tau}$ are progressively moved out of C, and the integration effectively bears on the complement of $C\cap C_{-\tau}$ in C, i.e.,

$$Q_\tau=\frac{1}{\tau}\int_{C/C\cap C_{-\tau}}d\mathbf{x}\,\rho(\mathbf{x}). \tag{3.7}$$

The existence of a finite, intrinsic, resolution-independent mean recurrence time therefore amounts to the condition that $\lim_{\tau\downarrow 0}Q_\tau$ exists and is finite, i.e.,

$$\begin{aligned}Q_0&=\lim_{\tau\downarrow 0}\frac{1}{\tau}\int_{C/C\cap C_{-\tau}}d\mathbf{x}\,\rho(\mathbf{x})\\&=-\left[\frac{d}{d\tau}\int_{C\cap C_{-\tau}}d\mathbf{x}\,\rho(\mathbf{x})\right]_{\tau=0^+}=\text{finite}.\end{aligned} \tag{3.8}$$

For unstable systems, it is convenient to decompose formally the integration in Eq. (3.8) to an integration over the coordinates s and u along the stable and unstable manifolds respectively. For a given C of sufficient smallness and for any small, but nonvanishing, positive τ, $C_{-\tau}$ is deformed with respect to C in such a way that it is squeezed along u and stretched along s (see Fig. 1). Equation (3.9) can then be written in the more transparent form

$$Q_0=-\left[\frac{d}{d\tau}\int_{x_{s0}}^{x_{s0}+b}ds\int_{\max(\mathbf{x}_{u0}-v\tau,\mathbf{x}_{u0})}^{\min(\mathbf{x}_{u0}+a-v\tau,\mathbf{x}_{u0}+a)}du\,\rho(s,u)\right]_{\tau=0^+}, \tag{3.9}$$

where v is a characteristic phase space velocity along the unstable manifold within C. Evaluating the time derivative, we obtain

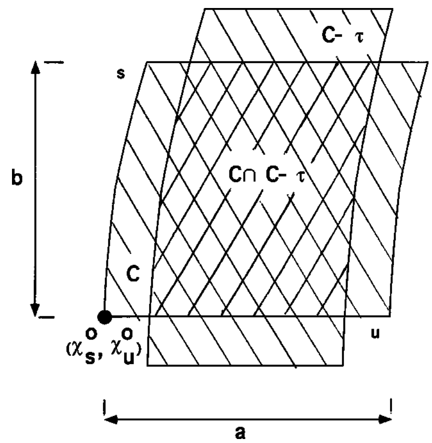

FIG. 1. Schematic representation of the integration domain in Eq. (3.8), in the presence of unstable dynamics. *s* and *u* stand for directions along the stable and unstable manifolds, respectively.

$$Q_0=\int_{\mathbf{x}_{s0}}^{\mathbf{x}_{s0}+b} ds\, v[\rho(s,\mathbf{x}_{u0}+a)-\rho(s,\mathbf{x}_{u0})]. \qquad (3.10)$$

This expression is well behaved for systems admitting a Sinai-Ruelle-Bowen (SRB) measure [10]: as a function of x_{u0}, since such measures are smooth along the unstable direction; and as a consequence of the integration over the stable direction, even though ρ may be singular with respect to *s*, provided it remains integrable. It is important to note that this property does *not* guarantee that the integral in Eq. (3.9), or the expression in square brackets in Eq. (3.6), can be expanded in powers of τ with only the linear term contributing to the final result — the expressions involved are generally nonanalytic in τ.

For integrable, stable dynamical systems, such as those exhibiting periodic or quasiperiodic behavior, the arguments above need to be suitably adapted. Since recurrence implies ergodicity, the natural representation of a phase space point **x** within *C* is now in terms of coordinates along which the Lyapunov exponent is zero, such as angle variables obtained by a canonical transformation. In the case of uniform motion a typical term contributing to Q_τ in Eq. (3.5) would then be of the form

$$Q_\tau=\frac{1}{(2\pi)^n\tau}\int_{a_1}^{b_1}d\varphi_{10}\cdots\int_{a_n}^{b_n}d\varphi_{n0} \times\left[1-\int_{a_1}^{b_1}d\varphi_1\cdots\int_{a_n}^{b_n}d\varphi_n\prod_{j=1}^{n}\delta(\varphi_j-\varphi_{j0}-\omega_j\tau)\right]. \qquad (3.11)$$

This expression can be evaluated analytically in a variety of ways, and produces a finite, nonvanishing result in the limit $\tau\to 0$. Nevertheless, it is not differentiable or analytic at $\tau=0$. For the purpose of demonstrating this explicitly, it suffices to consider the case of a single angular coordinate. The integral

$$I_\tau=\int_a^b d\varphi_0\int_a^b d\varphi\;\delta(\varphi-\varphi_0-\omega\tau) \qquad (3.12)$$

can be evaluated directly, or by using the representation of the δ function as an exponential integral, to obtain

$$I_\tau=\begin{cases}(b-a)-|\omega\tau|, & |\omega\tau|<(b-a)\\ 0, & |\omega\tau|>(b-a)\end{cases}. \qquad (3.13)$$

Therefore the leading behavior of Q_τ comes from

$$Q_\tau\approx\frac{1}{(2\pi)^n\tau}\left[\prod_{j=1}^{n}(b_j-a_j)^n-\prod_{j=1}^{n}\{(b_j-a_j)-\omega_j|\tau|\}\right], \qquad (3.14)$$

where $|\tau|<\min_j(b_j-a_j)/\omega_j$. Thus, as $\tau\to 0$,

$$Q_\tau=\frac{|\tau|}{(2\pi)^n\tau}\sum_{j=1}^{n}\omega_j\prod_{k\neq j}(b_k-a_k). \qquad (3.15)$$

This is not differentiable at $\tau=0$, although $\lim_{\tau\downarrow 0}Q_\tau$ exists, and is given by

$$Q_0=\frac{1}{(2\pi)^n}\sum_{j=1}^{n}\omega_j\prod_{k\neq j}(b_k-a_k). \qquad (3.16)$$

$\langle\theta_\tau\rangle$ thus tends to a finite, nonvanishing limit in this case too, as $\tau\to 0^+$.

IV. STOCHASTIC DYNAMICS WITH A FINITE TRANSITION RATE

We now turn to the application of Eq. (2.18) to stochastic dynamical systems. Two broad classes can be distinguished here, depending upon the small-τ behavior of the conditional probability $P(C,0|C,\tau)$. In the first case, we find $P(C,0|C,\tau)=1+O(\tau)$, so that a finite (i.e., nonzero) value of $\langle t\rangle$ emerges. In the second case, typified by the presence of white noise, the leading correction is $O(\tau^{1/2})$, and $\langle t\rangle$ vanishes. We take these up in turn in this section and in Sec. V. In the former case, it is convenient to deal with jump processes first, and then continuous processes.

A. Jump processes

When a finite transition rate can be defined in the state space (which can be discrete or continuous), the behavior of $P(C,0|C,\tau)$ becomes analytic in the vicinity of $\tau=0$. The simplest illustration is provided by a stationary dichotomous Markov process (DMP) $\xi(t)$ which switches between the values $\pm c$ at a mean rate λ. Identifying *C* with the state $+c$ (and hence $\overline{C}$ with the state $-c$), it is trivially seen that $P(C)=\frac{1}{2}$ and

$$\langle t\rangle=\lim_{\tau\to 0}\tau/(1-e^{-\lambda\tau}\cosh\lambda\tau)=\lambda^{-1}. \qquad (4.1)$$

Note that $\langle t\rangle$ is not $2\lambda^{-1}$, as one might naively expect.

Next, consider a Markov jump process $x(\in[\mathbb{R}])$ driven by a Poisson sequence of pulses with a mean rate λ (the Kubo-Anderson process [11]). Assuming that a normalized stationary density $\rho(x)$ exists, in this case we have

$$P(C)=\int_C \rho(x)dx \tag{4.2}$$

and

$$\rho(x_0,0|x,t)=\delta(x-x_0)e^{-\lambda t}+\rho(x)(1-e^{-\lambda t}). \tag{4.3}$$

This leads to

$$P(C,0;C,\tau)=P(C)e^{-\lambda\tau}+P^2(C)(1-e^{-\lambda\tau}) \tag{4.4}$$

and thus, in the limit $\tau\to 0$, to

$$\langle t\rangle=[\lambda P(C)]^{-1}. \tag{4.5}$$

At the next stage of generalization, we may consider a state-dependent jump rate $\lambda(x)$. This leads to a stationary Markov process (the so-called kangaroo process [11]), with the transition probability density

$$\rho(x_0|x,\tau)=\delta(x-x_0)[1-\lambda(x_0)\tau]+\lambda(x_0)W(x)\tau+O(\tau^2), \tag{4.6}$$

where $W(x)$ is related to the stationary density $\rho(x)$. Using Eq. (4.6) to write down the master equation for the process, we find

$$W(x)=\lambda(x)\rho(x)/\langle\lambda\rangle, \tag{4.7}$$

where

$$\langle\lambda\rangle=\int_\Gamma \lambda(x)\rho(x)dx. \tag{4.8}$$

Also defining the restricted average rate

$$\langle\lambda\rangle_C=\int_C \lambda(x)\rho(x)dx, \tag{4.9}$$

in this case we find

$$\langle t\rangle=\frac{\langle\lambda\rangle[1-P(C)]}{\langle\lambda\rangle_C(\langle\lambda\rangle-\langle\lambda\rangle_C)}. \tag{4.10}$$

Another direction of generalization from Eq. (4.3) is to an equilibrium renewal process governed by an arbitrary (nonexponential) waiting-time distribution function $\phi(t)$: a so-called ''continuous time random walk'' or renewal process [12]. In this case $x(t)$ is non-Markovian. The density of the interval between successive jumps is $\psi(t)=-\dot{\phi}(t)$. The mean time between jumps is given by

$$\lambda^{-1}=\int_0^\infty t\psi(t)dt=\int_0^\infty \phi(t)dt. \tag{4.11}$$

To find the transition probability of the jump process itself, however, we need the first waiting-time distribution, $\phi_0(t)$, for the first jump starting from an arbitrary origin of time. For an ongoing equilibrium renewal process, this is found from the relationship

$$-\dot{\phi}_0(t)=\lambda\phi(t), \tag{4.12}$$

provided $\lambda\neq 0$. The first waiting-time distribution is therefore given by

$$\phi_0(t)=\lambda\int_t^\infty \phi(t')dt'. \tag{4.13}$$

Returning to the jump process governed by this renewal process, we can show that Eq. (4.3) is replaced by

$$\rho(x_0,0|x,t)=\delta(x-x_0)\phi_0(t)+\rho(x)[1-\phi_0(t)]. \tag{4.14}$$

This gives, using Eq. (4.12) and the fact that $\phi_0(0)=1$, by definition,

$$\left(\frac{\partial\rho(x_0,0|x,t)}{\partial t}\right)_{t=0}=\rho(x)-\delta(x-x_0). \tag{4.15}$$

Equation (2.19) then leads to

$$\langle t\rangle=[\lambda P(C)]^{-1}, \tag{4.16}$$

exactly as in the case of the corresponding Markovian jump process [Eq. (4.5)].

B. Continuous processes: dichotomous flows

Turning from jump processes to continuous processes, we see that $\langle t\rangle$ is nonvanishing whenever we can write the small-τ expansion

$$\rho(x_0,0|x,\tau)=\delta(x-x_0)+\tau\left(\frac{\partial\rho(x_0,0|x,t)}{\partial t}\right)_{t=0}+O(\tau^2), \tag{4.17}$$

where the time derivative can be read off from the master equation

$$\frac{\partial}{\partial t}\rho(x_0,0|x,t)=\int_\Gamma dx'[\rho(x_0,0|x',t)w(x'\to x) - \rho(x_0,0|x,t)w(x\to x')]. \tag{4.18}$$

Here $w(x\to x')$ is the transition probability per unit time. Further, assuming stationarity and the existence of a normalized invariant density $\rho(x)=\lim_{t\to\infty}\rho(x_0,0|x,t)$, we have

$$P(C)=\int_C \rho(x)dx. \tag{4.19}$$

Equation (2.19) then yields the following expression for the continuous time limit of the mean time of recurrence to C:

$$\langle t\rangle=\frac{\int_{\bar{C}}\rho(x)dx}{\int_{\bar{C}}dx\int_C dx'\rho(x')w(x'\to x)}. \tag{4.20}$$

A very instructive illustration of the behavior of the mean recurrence time for a continuous process in the limit $\tau\to 0$ is provided by the linear dichotomous flow, given by the stochastic differential equation

$$\dot{x}=-\gamma x+\xi(t), \tag{4.21}$$

where $x\in\mathbb{R},\gamma>0$ and the noise $\xi(t)$ is the stationary DMP defined in the beginning of this section. The process $x(t)$ describes the overdamped motion of a particle in a parabolic potential, with a (stable) fixed point at the origin. The interesting case is therefore the two-cell partition $C=[0,\infty)$ and $\overline{C}=(-\infty,0)$. The probability densities $\rho_\pm(x_0,0|x,t)$, corresponding to $\xi=\pm c$, satisfy the coupled equations

$$\frac{\partial}{\partial t}\rho_\pm=\frac{\partial}{\partial x}[(\gamma x\mp c)\rho_\pm]-\lambda(\rho_\pm-\rho_\mp). \tag{4.22}$$

The normalized stationary solution [13–15] for the total probability density,

$$\rho(x_0,0|x,t)\equiv\rho_+(x_0,0|x,t)+\rho_-(x_0,0|x,t), \tag{4.23}$$

has the compact support $-c/\gamma\leqslant x\leqslant c/\gamma$, in which region it is given by

$$\rho(x)=\frac{1}{B\left(\frac{1}{2},\frac{\lambda}{\gamma}\right)}\left(\frac{c}{\gamma}\right)^{1-2\lambda/\gamma}\left(\frac{c^2}{\gamma^2}-x^2\right)^{\lambda/\gamma-1}. \tag{4.24}$$

The system has a ''phase transition'' at $\lambda=\gamma$. For $\lambda>\gamma$ (respectively, $\lambda<\gamma$), $\rho(x)$ vanishes (respectively, diverges) at the end points $x=\pm c/\gamma$. As $\rho(x)$ is an even function of $x, P(C)=\frac{1}{2}$, where $C=[0,c/\gamma]$ and $\overline{C}=[-c/\gamma,0]$.

It is evident from the general formula (2.18) that $\langle t\rangle$ depends on both the stationary distribution $\rho(x)$ as well as the small-t behavior of $\rho(x_0,0|x,t)$. To highlight this point, we consider the general initial conditions

$$\rho_+=w_+\delta(x-x_0),\rho_-=w_-\delta(x-x_0), \tag{4.25}$$

where $x_0\in C$ and $0<w_\pm<1,w_++w_-=1$. The general solution of Eqs. (4.22) with the initial conditions (4.25) can be found, and is quite complicated. However for the purpose at hand it suffices to know the short-time behavior of the solution. This is determined by the motion of the characteristics of the first order partial differential equation (4.22), and is found to be

$$\rho_\pm(x_0,0|x,t)\approx w_\pm e^{-\lambda t}\delta\left[\left(x\mp\frac{c}{\gamma}\right)-\left(x_0\mp\frac{c}{\gamma}\right)e^{-\gamma t}\right]. \tag{4.26}$$

Substituting this in

$$P(C,O;C,\tau)=\int_0^{c/\gamma}\mathbf{d}x_0\rho(x_0)\int_0^{c/\gamma}\mathbf{d}x[\rho_+(x_0,\mathbf{0}|\mathbf{x},\tau)+\rho_-(x_0,0|x,\tau)], \tag{4.27}$$

we find that the support of the δ function in ρ_+ lies on the line

$$x=x_0e^{-\gamma\tau}+\frac{c}{\gamma}(1-e^{-\gamma\tau}), \tag{4.28}$$

which lies above x_0 in the entire range of the latter variable, and reaches the value c/γ at the upper limit $x_0=c/\gamma$. Hence the entire range of x_0 contributes to this term, which becomes

$$w_+e^{-\lambda\tau}\int_0^{c/\gamma}dx_0\rho(x_0)=w_+P(C)e^{-\lambda\tau}. \tag{4.29}$$

On the other hand, the support of the δ function in ρ_- lies on the line

$$x=x_0e^{-\gamma\tau}-\frac{c}{\gamma}(1-e^{-\gamma\tau}), \tag{4.30}$$

which lies below x_0. Hence the range of x_0 is restricted to run from $c(e^{\gamma\tau}-1)/\gamma$ to c/γ. (As we are concerned with $\gamma\tau\ll1$, the lower limit of integration does lie below c/γ). Using the fact that $P(C)=\frac{1}{2}$ and $w_++w_-=1$, we obtain

$$P(C)-P(C,0;C,\tau)\simeq\frac{1}{2}(1-e^{-\lambda\tau})+w_-e^{-\lambda\tau}\int_0^{c(e^{\gamma\tau}-1)/\gamma}dx_0\rho(x_0). \tag{4.31}$$

Substituting for $\rho(x_0)$ from Eq. (4.24), the integral can be evaluated exactly in terms of an incomplete β function. However, only the leading small-τ behavior is required in the foregoing. Since $\rho(x_0)$ is regular in the neighborhood of $x_0=0$, the latter is proportional to τ itself. Passing to the limit $\tau=0$ we obtain, using Eqs. (2.19) and (2.20),

$$\langle t\rangle=[\lambda+2w_-c\rho(0)]^{-1}. \tag{4.32}$$

Substituting for $\rho(0)$, we arrive at the following finite, nonvanishing mean recurrence time in the continuous-time limit:

$$\langle t\rangle=(\lambda^{-1})\bigg/\left[1+\left(\frac{2w_-}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\lambda}{\gamma}+\frac{1}{2}\right)}{\Gamma\left(\frac{\lambda}{\gamma}+1\right)}\right)\right]. \tag{4.33}$$

Thus $\langle t\rangle$ is diminished as w_- is increased, as may be expected on physical grounds.

Equation (4.33) also helps us see how $\langle t\rangle$ varies as a function of the damping parameter γ for a given value of λ. The modulating factor multiplying λ^{-1} in the expression for $\langle t\rangle$ increases from $(1+2w_-)^{-1}$ (which is obtained for $\gamma\gg\lambda$) toward unity as γ decreases. For $\gamma\ll\lambda$, this factor is $[1+(\text{const.})\gamma^{1/2}]^{-1}$. In the limit $\gamma=0$, which corresponds to pure dichotomous (or ''persistent'') diffusion [16,17], we have $\langle t\rangle=\lambda^{-1}$.

Finally, these considerations may be extended to the case of a general dichotomous flow [13–16]. Here we are interested in particular in the case of motion in the bistable potential:

$$V(x)=\tfrac{1}{4}x^4-\tfrac{1}{2}\gamma x^2. \tag{4.34}$$

The counterpart of Eq. (4.21) is now the stochastic differential equation

$$\dot{x} = f(x) + \xi(t), \tag{4.35}$$

where

$$f(x) = \gamma x - x^3.$$

Assuming that c is sufficiently large, the stationary probability density is again a symmetric function,

$$\rho(x) = \text{const} \times [c^2 - f^2(x)]^{-1} \left[-2\lambda \int_0^x dy\, f(y) \times [c^2 - f^2(y)]^{-1} \right], \tag{4.36}$$

with the compact support $|x| < x_{\max}$, where $x_{\max}$ is the real root of $f(x) + c = 0$. We can show that, as in the linear case, the short-time behavior of $\rho(x_0,0|x,t)$ essentially corresponds to ballistic motion, in the sense that $\rho(x_0,0|x,t)$ is, apart from a modulating factor $e^{-\lambda t}$, the sum of two δ functions whose peaks are located at the solutions x_+ and x_- of

$$\dot{x} = f(x) \pm c, \tag{4.37}$$

respectively. The arguments made in the linear case can then be adapted suitably to arrive at the conclusion that $P(C) - (P(C,0;C,\tau)$ is again proportional to τ itself as $\tau \to 0$, so that Q_0, and hence $\lim_{\tau \downarrow 0} \langle \theta_\tau \rangle$, is finite and nonzero.

V. DIFFUSIVE PROCESSES

A. Ornstein-Uhlenbeck process

We turn now to the important case of stochastic processes for which no finite transition probability per unit time can be defined—typically, the situation when white noise components are present. For a ready comparison with the case of dichotomous noise just considered, we again look at both linear drift as well as that in a bistable potential, but with a Gaussian white noise rather than dichotomous noise. We begin with the case of a linear drift, i.e., the familiar Ornstein-Uhlenbeck (OU) process, given by the Langevin equation

$$\dot{x} = -\gamma x + (2D)^{1/2} \eta(t) \quad (\gamma > 0), \tag{5.1}$$

with $\langle \eta(t) \rangle = 0, \langle \eta(t) \eta(t') \rangle = \delta(t-t')$. As before, let $C = (0,\infty)$ and $\overline{C} = (-\infty,0)$. The solution for the conditional probability density obeying the initial condition $\delta(x-x_0)$ is given by the well-known solution

$$\rho(x_0,0|x,t) = \left[\frac{\gamma}{2\pi D(1-e^{-2\gamma t})} \right]^{1/2} \exp\left[-\frac{\gamma(x - x_0 e^{-\gamma t})^2}{2D(1-e^{-2\gamma t})} \right]. \tag{5.2}$$

The stationary density is

$$\rho(x) = \left(\frac{\gamma}{2\pi D} \right)^{1/2} \exp\left(\frac{-\gamma x^2}{2D} \right). \tag{5.3}$$

By symmetry, $P(C) = \frac{1}{2}$. Further, the integrals involved in

$$P(C,0;C,\tau) = \int_0^\infty dx_0 \int_0^\infty dx\, \rho(x_0) \rho(x_0,0|x,\tau) \tag{5.4}$$

can be evaluated in closed form, and we find

$$Q_\tau = \frac{1}{\tau} [P(C) - P(C,0;C,\tau)] = \frac{1}{2\pi\tau} \tan^{-1}[(e^{2\gamma\tau} - 1)^{1/2}]. \tag{5.5}$$

It is noteworthy that D does not appear in this expression (recall that we have chosen $x=0$ as the boundary of C). Thus, as $\tau \to 0, Q_\tau$ scales like $(\gamma/\tau)^{1/2}$, so that $\langle t \rangle$ tends to zero like $(\tau/\gamma)^{1/2}$ in the limit $\tau = 0$, essentially because of the infinite velocity of the diffusion process. If the left boundary of C is at some point $a \neq 0$, we find

$$Q_\tau = \frac{1}{2\tau} \left(\frac{\alpha}{\pi} \right)^{1/2} \int_0^\infty du\, e^{-\alpha(u+1)^2} \operatorname{erfc}\left[\frac{\sqrt{\alpha}(u-\varepsilon)}{\sqrt{\varepsilon}(2+\varepsilon)} \right], \tag{5.6}$$

where $\alpha = \gamma a^2/(2D)$ and $\varepsilon = e^{\gamma\tau} - 1$. Once again, in the limit $\tau \to 0$, this quantity has a leading behavior proportional to $\varepsilon^{1/2}/\tau$, so that $\langle t \rangle$ vanishes like $\tau^{1/2}$. Notice that $\langle t \rangle$ depends now on D as well as on γ, the dominant dependence being given by the factor $e^{-\gamma a^2}/2D$. In the limit of small D this implies that, for any given τ, the recurrence process is dramatically accelerated.

B. Diffusion in a bistable potential

Next we consider the case in which diffusion occurs in a bistable potential [18], so that the Langevin equation (5.1) is augmented by a cubic nonlinearity and the sign of the linear term is inverted, to read

$$x = \gamma x - x^3 + (2D)^{1/2} \eta(t). \tag{5.7}$$

The invariant probability density can again be evaluated exactly from the corresponding Fokker-Planck equation subject to no-flux conditions at $\pm\infty$, and is given by

$$\rho(x) = Z^{-1} \exp\left[\frac{1}{D} \left(\frac{1}{2} \gamma x^2 - \frac{1}{4} x^4 \right) \right], \tag{5.8}$$

where

$$Z = e^{\gamma^2/4D} \int_0^\infty du\, u^{-1/2} \exp\left[-\frac{1}{4D} (u-\gamma)^2 \right] \tag{5.9}$$

is the normalization factor. For $\gamma < 0$, the origin $(x=0)$ remains the unique fixed point of the deterministic limit of Eq. (5.7), and $\rho(x)$ in Eq. (5.9) is qualitatively similar to the Gaussian that obtains in the case of the OU process. As γ crosses zero to positive values, the origin becomes unstable and a bifurcation occurs to the stable branches $x_\pm = \pm \gamma^{1/2}$. The corresponding invariant density in Eq. (5.9) is now a bimodal one which, in the limit $D \to 0$, reduces to two Gaussians centered at x_+ and x_-, respectively.

Turning to recurrence time statistics, as before we choose as the reference cell C the interval $[0, \infty\rangle$, so that $\overline{C} = (-\infty,0)$. By the symmetry of $\rho(x)$ about $x=0$, we have $P(C) = \frac{1}{2}$, so that Eq. (2.9) becomes

$$\langle \theta_\tau \rangle = \frac{\frac{1}{2}\tau}{\frac{1}{2} - P(C,0;C,\tau)}, \tag{5.10}$$

with

$$P(C,0;C,\tau) = \int_0^\infty dx_0 \rho(x_0) \int_0^\infty dx\, \rho(x_0,0|x,\tau). \tag{5.11}$$

For the purposes of evaluating $\langle \theta_\tau \rangle$ in the continuous sampling limit $\tau \to 0$, it is sufficient to consider the small-τ behavior of the conditional density $\rho(x_0,0|x,t)$. This is given by the time-dependent solution of the Fokker-Planck equation corresponding to Eq. (5.7), with the drift term linearized around x_+, in view of the fact that $\rho(x_0)$ is weighted predominantly around this point. Setting $\zeta = x - x_+$, the linearized Fokker-Planck equation for $\rho(\zeta_0,0|\zeta,t)$ reads

$$\frac{\partial \rho}{\partial t} = 2\gamma \frac{\partial}{\partial \zeta}(\zeta \rho) + D \frac{\partial^2 \rho}{\partial \zeta^2}. \tag{5.12}$$

For an initial condition $\rho(\zeta_0,0|\zeta,0) = \delta(\zeta - \zeta_0)$, the solution is given by

$$\rho(\zeta_0,0|\zeta,t) = \left[\frac{\gamma}{\pi D(1-e^{-4\gamma t})}\right]^{1/2} \exp\left[-\frac{\gamma(\zeta - \zeta_0 e^{-2\gamma t})^2}{D(1-e^{-4\gamma t})}\right]. \tag{5.13}$$

Passing to the small-τ regime and reverting to the original variable x,

$$\rho(x_0,0|x,\tau) \simeq (4\pi D\tau)^{-1/2} \exp\left[-\frac{(x-x_0)^2}{4D\tau}\right]. \tag{5.14}$$

We note that this approximate expression remains properly normalized. Substituting Eq. (5.14) into Eq. (5.11), we obtain

$$P(C,0;C,\tau) = \frac{1}{2} \int_0^\infty dx_0 \rho(x_0) \operatorname{erfc}\left[\frac{-x_0}{2(D\tau)^{1/2}}\right]. \tag{5.15}$$

Using the fact that $\operatorname{erfc}(-z) = 2 - \operatorname{erfc} z$, this yields

$$P(C) - P(C,0;C,\tau) = \frac{1}{2} \int_0^\infty dx_0 \rho(x_0) \operatorname{erfc}\left[\frac{x_0}{2(D\tau)^{1/2}}\right]. \tag{5.16}$$

As $\rho(x_0)$ is peaked at the stable fixed point x_+, and we are interested in the leading small-τ behavior, we may expand $\rho(x_0)$ around $x_0 = x_+$. The result is a Gaussian of the form $\exp[-\gamma(x_0 - \gamma^{1/2})^2/D]$, which must, however, be normalized so as to maintain the measure $P(C) = \frac{1}{2}$. The appropriate expression is

$$\rho(x_0) \simeq Z'^{-1} \exp\left[-\frac{\gamma}{D}(x_0 - \gamma^{1/2})^2\right], \tag{5.17}$$

where the normalization factor is now given by

$$Z' = \left(\frac{\pi D}{\gamma}\right)^{1/2} [1 + \operatorname{erf}(\gamma / D^{1/2})]. \tag{5.18}$$

The factor in square brackets is not very significant in the light of the approximation involved, and has been retained for the formal consistency of the normalization of $\rho(x_0)$. Inserting Eq. (5.17) into Eq. (5.16) and changing variables of integration to $u = (4D\tau)^{-1/2} x_0$, we find

$$P(C) - P(C,0;C,\tau) \simeq Z'^{-1} (D\tau)^{1/2} \int_0^\infty du\, \operatorname{erfc}(u) \times \exp\left[-\frac{\gamma}{D}[2u(D\tau)^{1/2} - \gamma^{1/2}]^2\right]. \tag{5.19}$$

Since $\operatorname{erfc}(u)$ is integrable, the leading small-τ behavior of this expression may be obtained by simply setting $\tau = 0$ in the integrand. We find, finally,

$$\langle \theta_\tau \rangle \simeq \tfrac{1}{2} \pi (\tau/\gamma)^{1/2} [1 + \operatorname{erf}(\gamma / D^{1/2})] e^{\gamma^2/D}. \tag{5.20}$$

As in the case of linear drift, the mean recurrence time is seen to be resolution dependent, vanishing like $\tau^{1/2}$ in the limit $\tau \to 0$. Once again, this is ultimately a consequence of the infinite velocity associated with diffusion, in marked contrast to stochastic dynamics driven by dichotomous noise. The new feature that is obtained in the case of diffusion in a bistable potential, as opposed to the OU process, is that the smallness of the τ-dependent factor in $\langle \theta_\tau \rangle$ is now counteracted by the factor $\exp(\gamma^2/D)$, which is exponentially large for small values of D. This factor has a γ and D dependence that is similar to that of the mean exit time [or Kramers time [19], $\exp(\Delta U/D)$] from the basin of attraction of the stable fixed point x_+ across the potential barrier ΔU, the difference being that the barrier corresponding to Eq. (5.7) is $\gamma^2/4$ rather than γ^2.

VI. HIGHER MOMENTS OF THE RECURRENCE TIME

Having examined the behavior of the mean recurrence time in the continuous time limit for a variety of systems, let us consider what happens to the higher moments of the recurrence time—more specifically, the second moment.

From Eq. (2.17) for the recurrence time distribution, we obtain

$$\sum_{n=1}^{\infty} n^2 \mu(C_{n+1}) = \lim_{N\to\infty} \left[2 \sum_{n=2}^{N+1} W_n - (N+1)^2 W_{N+1} + N^2 W_{N+2} \right]. \tag{6.1}$$

It is easy to show that the condition

$$\sum^{\infty} W_n < \infty \tag{6.2}$$

is sufficient to make $\langle \theta_\tau^2 \rangle$ finite. On the other hand, if $W_n = O(n^{-1})$ as $n \to \infty$, as happens [4,20] in models of intermittent chaos, then $\langle \theta_\tau^2 \rangle$ diverges. Therefore, provided Eq. (6.2) holds, we have

$$\langle \theta_\tau^2 \rangle = \frac{\tau^2 \left[1 - \mu(C) + 2 \sum_{n=2}^{\infty} W_n \right]}{\mu(C) - \mu(C_1)}. \tag{6.3}$$

As $\tau \to 0$, the denominator tends to zero like τ, at best. However, an additional factor of τ^{-1} emerges in the numerator: as $\tau \to 0$, $\Sigma_2^\infty W_n \to \tau^{-1} \int_0^\infty S_{\bar{C}}(t)dt$, where $S_{\bar{C}}(t)$ is the cumulative probability of survival in $\bar{C}$. The continuous-sampling limit of the second moment of the recurrence time is therefore

$$\lim_{\tau \to 0} \langle \theta_\tau^2 \rangle \equiv \langle t^2 \rangle = \left(2 \int_0^\infty S_{\bar{C}}(t)dt \right) \lim_{\tau \to 0} \frac{\tau}{\mu(C) - \mu(C_1)},$$

or, finally,

$$\langle t^2 \rangle = \frac{2\langle t \rangle}{1 - \mu(C)} \int_0^\infty S_{\bar{C}}(t)dt. \tag{6.4}$$

Applying this to the class of continuous stochastic processes considered in Sec. IV B, this becomes, with the help of Eq. (4.20),

$$\langle t^2 \rangle = \frac{2 \int_0^\infty S_{\bar{C}}(t)dt}{\int_{\bar{C}} dx \int_C dx' \rho(x') w(x' \to x)}. \tag{6.5}$$

We have seen that the existence of a finite mean recurrence time is quite general, and essentially follows from the ergodic nature of the dynamics. On the other hand, the second moment of the recurrence time is finite only under a more restrictive condition. The vanishing as $n \to \infty$ of W_n, which is proportional to the probability of a sojourn in $\bar{C}$, is not sufficient; $\Sigma_n W_n$ must converge as well. The existence of finite higher moments imposes successively more stringent conditions on the decay of W_n for large n. Under ''normal'' circumstances, in which W_n falls off generically exponentially with increasing n, all moments of the recurrence time are finite. In the $\tau = 0$ limit of continuous sampling, these moments may vanish in certain cases, as we have seen. However, there do occur situations (such as intermittency in chaos) in which W_n decays according to a power law [4,20], and the higher moments of the recurrence time (including, possibly, the second moment itself) may diverge. This feature will be carried over, in such instances, to the corresponding continuous sampling limit.

VII. CONCLUSIONS

In this paper we have addressed the recurrence properties of dynamical systems in continuous time and state space, from the standpoint of the dependence of the first few moments of the recurrence time on the sampling time τ. The issue here is whether in the limit of continuous sampling ($\tau \to 0$) one obtains, for a given phase space cell, a finite result depending entirely on the intrinsic parameters or, instead, a resolution-dependent expression suggesting that recurrence in this limit is ill-defined. Our main thesis was that these two types of behavior define two wide, different classes of dynamical systems. In particular, the possibility that they provide a clearcut separation between deterministic and stochastic systems was critically examined.

We have shown that in deterministic systems an intrinsic expression for mean recurrence time exists in the limit $\tau \to 0$ for regular motion as well as chaotic motion, provided the probability distribution fulfills certain smoothness properties which are satisfied by SRB type measures. In the opposite end diffusion processes have been considered, and shown to lead to resolution-dependent mean recurrence times tending to zero in the limit $\tau \to 0$. More unexpected was the result, derived in Sec. IV, that there exist processes which are continuous in *both* space and time such as systems driven by dichotomous noise, for which a finite, resolution-independent mean recurrence time can be defined. The main ingredient at the origin of this result was the existence of a finite speed of propagation of signals in such systems, as opposed to the infinite speed of propagation characteristic of diffusion processes. In this context one may recall that stochastic processes continuous in time but discrete in state space such as birth and death processes generally have well-defined recurrence times.

An appealing aspect of our conclusions is the considerable generality of the processes fitting into the different classes that we have identified. Still, we cannot claim to have achieved an exhaustive classification. It would undoubtedly be worth pursuing this goal in future investigations.

Although not explicitly required in the general formulation, much of our analysis focused on dissipative systems possessing sufficiently strong ergodic properties. It would be appropriate to consider more explicitly the case of Hamiltonian dynamics, in which strong and weak ergodic behaviors are intertwined in phase space in addition to being dependent on the initial conditions. Of particular interest would be the signature, at the level of recurrence time statistics, of the transition to nonintegrability and chaos through different scenarios and of the stickiness of the Cantori in the regime of developed chaos.

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